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MIXED-MEAN INEQUALITY FOR SUBMATRIX

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ABSTRACT. For an $m \times n$ matrix $B = (b_{ij})_{m\times n}$ with nonnegative entries b_{ij}, let $B(k,l)$ denote the set of all $k \times l$ submatrices of B. For each $A \in B(k,l)$, let a_A and g_A denote the arithmetic mean and geometric mean of elements of A respectively. It is proved that if k is an integer in $(\frac{m}{2}, m]$ and l is an integer in $(\frac{n}{2}, n]$ respectively, then

$$\Big(\prod_{A\in B(k,l)} a_A\Big)^{\frac{1}{\binom{m}{k}\binom{n}{l}}} \geq \frac{1}{\binom{m}{k}\binom{n}{l}}\Big(\sum_{A\in B(k,l)} g_A\Big),$$

with equality if and only if b_{ij} is a constant for every i, j.

1. Introduction

Let $x_1, \dots, x_n$ be positive real numbers, then the arithmetic-geometric mean inequality is

$$\frac{x_1 + \dots + x_n}{n} \geq \sqrt[n]{x_1 \dots x_n},$$

with equality if and only if $x_1 = \dots = x_n$.

There are many research articles devoted to the classical arithmetic-geometric mean inequality and its generalizations ([1], [2], [4], [9], [10]). The mixed-type arithmetic-geometric mean inequalities ([3], [5], [6], [7], [8], [11]) are one of the most important branch in these generalizations.

In [6], Kedlaya established the following mixed mean inequality, which was conjectured by F. Holland and whose inductive proof was given by T. Matsuda [8]:

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The arithmetic mean of the numbers

$$x_1,\ \sqrt{x_1x_2},\ \sqrt[3]{x_1x_2x_3},\ \ldots,\ \sqrt[n]{x_1x_2\ldots x_n}$$

does not exceed the geometric mean of the numbers

$$x_1,\ \frac{x_1+x_2}{2},\ \frac{x_1+x_2+x_3}{3},\ \frac{x_1+x_2+\cdots+x_n}{n},$$

with equality if and only if $x_1=\cdots=x_n$.

In [3], Carlson established the following mixed mean inequality:

Let the arithmetic and geometric means of the real nonnegative numbers $x_1,\ldots,x_n$ *taken* $n-1$ *at a time be denoted by*

$$a_i=\frac{x_1+\cdots+x_n-x_i}{n-1},\qquad g_i=\Big(\frac{x_1\ldots x_n}{x_i}\Big)^{\frac{1}{n-1}}.$$

Then for $n\geq 3$,

$$\big(a_1\ldots a_n\big)^{\frac{1}{n}}\geq\frac{g_1+\cdots+g_n}{n},$$

with equality if and only if $x_1=\cdots=x_n$.

In [7], Leng, Si and Zhu generalized the above result to any subsets and established another mixed arithmetic-geometric mean inequality:

Let X *be a set of real nonnegative numbers* $x_1,\ldots,x_n$. *For* $A\subset X$, *let* a_A *and* g_A *denote the arithmetic mean and geometric mean of all elements of* A, *respectively. If* k *is an integer in* $(\frac{n}{2},n]$, *then*

$$\Big(\prod_{\substack{|A|=k\\A\subset X}}a_A\Big)^{\frac{1}{\binom{n}{k}}}\geq\frac{1}{\binom{n}{k}}\Big(\sum_{\substack{|A|=k\\A\subset X}}g_A\Big),\tag{1.1}$$

with equality if and only if $x_1=\cdots=x_n$.

In this note, we established a new mixed arithmetic-geometric mean inequality for submatrix, which was an extension of the Carlson inequality and also an extension of (1.1).

Our main result is the following theorem.

Theorem 1.1. *For an* $m\times n$ *matrix* $B=(b_{ij})_{m\times n}$ *with nonnegative entries* $b_{ij}\geq 0$, *let* $B(k,l)$ *denote the set of all* $k\times l$ *submatrices of* B. *For each* $A\in B(k,l)$, *let* a_A *and* g_A *denote the arithmetic mean and geometric mean of elements of* A *respectively. If* k *is an integer in* $(\frac{m}{2},m]$ *and* l *is an integer in* $(\frac{n}{2},n]$ *respectively, then*

$$\Big(\prod_{A\in B(k,l)}a_A\Big)^{\frac{1}{\binom{m}{k}\binom{n}{l}}}\geq\frac{1}{\binom{m}{k}\binom{n}{l}}\Big(\sum_{A\in B(k,l)}g_A\Big),\tag{1.2}$$

with equality if and only if b_{ij} *is a constant for every* i,j.

Remark 1. If $k \le [\frac{m}{2}]$, for the matrix $B = (b_{ij})_{m\times n}$, taking $b_{11} = b_{12} = \cdots = b_{1n} = b_{21} = b_{22} = \cdots = b_{2n} = \cdots = b_{k1} = b_{k2} = \cdots = b_{kn} = 1$, $b_{ij} = 0$ for $k+1 \le i \le m$, then the right-hand side of (1.2) equals $1/\binom{m}{k}$, but the left-hand side is zero. If $l \le [\frac{n}{2}]$, by the same argument, one can get a contradiction. Hence the statement of Theorem 1.1 fails for $k \le [\frac{m}{2}]$ or $l \le [\frac{n}{2}]$.

Remark 2. Taking $k = m$, $l = n$ in Theorem 1.1, inequality (1.2) is just the classical arithmetic-geometric mean inequality. For $m = 1$ or $n = 1$ in Theorem 1.1, inequality (1.2) is just the inequality (1.1).

Remark 3. The condition of our theorem is weaker than the ones of (1.1), because the infimum of $k \times l$ is $\big([\frac{m}{2}]+1\big) \times \big([\frac{n}{2}]+1\big)$, which is less than $[\frac{mn}{2}]+1$, the half of the element number of the matrix $B = (b_{ij})_{m\times n}$.

2. Proof of main results

Let X denote the finite set with positive real numbers $x_1, x_2, \ldots, x_n$ and let A denote the subset of X with k elements $x_{i_1}, \ldots, x_{i_k}$. The r power mean of the elements of A is denoted by

$$m_r(A) = \left[\frac{1}{k}\big(x_{i_1}^r + \cdots + x_{i_k}^r\big)\right]^{\frac{1}{r}}, \tag{2.1}$$

where $r > 0$ is some real number. If $r = 1$ in the above equality, we get a_A, the arithmetic mean of the elements of A. Let $r \to 0$ in (2.1), we get g_A, the geometric mean of the elements of A.

We first established the following lemma.

Lemma 2.1. *For any $m \times n$ matrix $X = (x_{ij})_{m\times n}$ with nonnegative entries $x_{ij} \ge 0$. Denote by $X_1, \ldots, X_{\binom{m}{k}\binom{n}{l}}$ its all $k \times l$-submatrices of X. If k is an integer in $(\frac{m}{2}, m]$ and l is an integer in $(\frac{n}{2}, n]$ respectively, then for any $1 \le i \le \binom{m}{k}\binom{n}{l}$*

$$\begin{aligned} m_r(X_i) = \Bigg[\frac{1}{\binom{m}{k}\binom{n}{l}}\Big[&\big(m_r(X_i\cap X_1)\big)^r + \big(m_r(X_i \cap X_2)\big)^r + \cdots \\ &\cdots + \Big(m_r\Big(X_i \cap X_{\binom{m}{k}\binom{n}{l}}\Big)\Big)^r\Big]\Bigg]^{\frac{1}{r}}. \end{aligned} \tag{2.2}$$

Here the intersection of $X_i \cap X_j$, $j = 1, \ldots, \binom{m}{k}\binom{n}{l}$, is just the general set intersection.

P r o o f. It suffices to prove the equality

$$(m_r(X_i))^r = \frac{1}{\binom{m}{k}\binom{n}{l}}\Big[(m_r(X_i \cap X_1))^r + (m_r(X_i \cap X_2))^r + \cdots \\ \cdots + \Big(m_r\Big(X_i \cap X_{\binom{m}{k}\binom{n}{l}}\Big)\Big)^r\Big]. \tag{2.3}$$

Assume that $X_i = \{x_{i_1}, \dots, x_{i_{k\times l}}\}$, then the left hand of (2.3) is that

$$(m_r(X_i))^r = \frac{1}{k \times l}\big(x_{i_1}^r + \cdots + x_{i_{k\times l}}^r\big).$$

Now we need show that the right hand of (2.3) is also the mean of $\{x_{i_1}^r, \dots, x_{i_{k\times l}}^r\}$ with the same coefficient $\frac{1}{k\times l}$.

Assume that the right hand of (2.3) is $c_{i_1}x_{i_1}^r + \cdots + c_{i_{k\times l}}x_{i_{k\times l}}^r$. By the arbitrariness of X_i, we get that $c_{i_1} = \cdots = c_{i_{k\times l}}$.

Since $k > \frac{m}{2}$ and $l > \frac{n}{2}$, we have

$$X_i \cap X_j \neq \Phi, \qquad j = 1, 2, \dots, \tbinom{m}{k}\tbinom{n}{l}.$$

Then the sum of all coefficients in $(m_r(X_i \cap X_j))^r$ is 1, $j = 1, 2, \dots, \binom{m}{k}\binom{n}{l}$. As a result, the sum of all coefficients in the right hand of (2.3) is

$$\frac{1}{\binom{m}{k}\binom{n}{l}}(\underbrace{1 + 1 + \cdots + 1}_{\binom{m}{k}\binom{n}{l}}) = 1,$$

i.e., $c_{i_1} + \cdots + c_{i_{k\times l}} = 1$. Therefore $c_{i_1} = \cdots = c_{i_{k\times l}} = \frac{1}{k\times l}$. □

Remark 4. (2.2) is the generalization of related result in [7]. But the condition in Lemma is weaker, because the infimum of $k\times l$ is $\big([\frac{m}{2}]+1\big)\times\big([\frac{n}{2}]+1\big)$, which is less than $[\frac{mn}{2}] + 1$, the half of the element number of the matrix $X = (x_{ij})_{m\times n}$.

Remark 5. In (2.2), if $r = 1$, we have

$$a_{X_i} = \frac{1}{\binom{m}{k}\binom{n}{l}}\Big(a_{X_i\cap X_1} + a_{X_i\cap X_2} + \cdots + a_{X_i\cap X_{\binom{m}{k}\binom{n}{l}}}\Big). \tag{2.4}$$

In (2.2), if $r \to 0$, we have

$$g_{X_i} = \Big(g_{X_i\cap X_1}\cdot g_{X_i\cap X_2}\cdot \cdots \cdot g_{X_i\cap X_{\binom{m}{k}\binom{n}{l}}}\Big)^{\frac{1}{\binom{m}{k}\binom{n}{l}}}. \tag{2.5}$$

2.1. Proof of Theorem

For an $m\times n$ matrix $B = (b_{ij})_{m\times n}$ with nonnegative entries $b_{ij} \geq 0$, let $B(k, l)$ denote the set of all $k \times l$ submatrices of B. For each element of $B(k, l)$, which is composed by the rows $(i_1, i_2, \dots, i_k)$ of B and by the columns $(j_1, j_2, \dots, j_l)$

of B, by the partial order of $(i_1, i_2, \ldots, i_k)$ and $(j_1, j_2, \ldots, j_l)$,[1] we could get a sequence of submatrices $B_1, B_2, \ldots, B_{\binom{m}{k}\binom{n}{l}}$.

In Lemma 2.1, let X_i be B_i and X_j, $j = 1, 2, \ldots, \binom{m}{k}\binom{n}{l}$, be corresponding to $B_1, B_2, \ldots, B_{\binom{m}{k}\binom{n}{l}}$, then (2.4) is

$$a_{B_i} = \frac{1}{\binom{m}{k}\binom{n}{l}}\Big(a_{B_i\cap B_1} + a_{B_i\cap B_2} + \cdots + a_{B_i\cap B_{\binom{m}{k}\binom{n}{l}}}\Big), \tag{2.6}$$

and (2.5) is

$$g_{B_i} = \Big(g_{B_i\cap B_1} \cdot g_{B_i\cap B_2} \cdot \cdots \cdot g_{B_i\cap B_{\binom{m}{k}\binom{n}{l}}}\Big)^{\frac{1}{\binom{m}{k}\binom{n}{l}}}. \tag{2.7}$$

By the arithmetic-geometric inequality, it follows that

$$a_{B_i\cap B_j} \geq g_{B_i\cap B_j}, \qquad j = 1, 2, \ldots, \binom{m}{k}\binom{n}{l}. \tag{2.8}$$

From (2.6) and (2.8), we infer that

$$a_{B_i} \geq \frac{1}{\binom{m}{k}\binom{n}{l}}\Big(g_{B_i\cap B_1} + g_{B_i\cap B_2} + \cdots + g_{B_i\cap B_{\binom{m}{k}\binom{n}{l}}}\Big).$$

Therefore

$$\begin{aligned}\Big(\prod_{A\in B(k,l)} a_A\Big)^{\frac{1}{\binom{m}{k}\binom{n}{l}}} &= \Big(\prod_{i=1}^{\binom{m}{k}\binom{n}{l}} a_{B_i}\Big)^{\frac{1}{\binom{m}{k}\binom{n}{l}}} \\ &\geq \frac{1}{\binom{m}{k}\binom{n}{l}}\Bigg(\prod_{i=1}^{\binom{m}{k}\binom{n}{l}}\Big(\sum_{j=1}^{\binom{m}{k}\binom{n}{l}} g_{B_i\cap B_j}\Big)\Bigg)^{\frac{1}{\binom{m}{k}\binom{n}{l}}}.\end{aligned} \tag{2.9}$$

On the other hand, using the discrete case of Hölder's inequality in the form

$$\sum_{k=1}^{n}\Big(\prod_{j=1}^{m} x_{jk}\Big)^{\frac{1}{m}} \leq \Big(\prod_{j=1}^{n}\Big(\sum_{k=1}^{m} x_{jk}\Big)\Big)^{\frac{1}{m}},$$

where n, m are positive integers and $x_{jk} > 0$ $(j, k = 1, 2, \ldots, m)$, we obtain

$$\Bigg(\prod_{i=1}^{\binom{m}{k}\binom{n}{l}}\Big(\sum_{j=1}^{\binom{m}{k}\binom{n}{l}} g_{B_i\cap B_j}\Big)\Bigg)^{\frac{1}{\binom{m}{k}\binom{n}{l}}} \geq \sum_{i=1}^{\binom{m}{k}\binom{n}{l}}\Big(\prod_{j=1}^{\binom{m}{k}\binom{n}{l}} g_{B_i\cap B_j}\Big)^{\frac{1}{\binom{m}{k}\binom{n}{l}}}. \tag{2.10}$$

[1]If we consider two elements of $B(k,l)$, B_m composed by the rows $(i_1, i_2, \ldots, i_k)$ of B and by the columns $(j_1, j_2, \ldots, j_l)$ of B and B_n composed by the rows $(i_1', i_2', \ldots, i_k')$ of B and by the columns $(j_1', j_2', \ldots, j_l')$ of B, then here the partial order $B_m \prec B_n$ means that $i_s < i_s'$ for some $1 \leq s \leq k$ or $i_s = i_s'$ for all $1 \leq s \leq k$ and $j_t < j_t'$ for some $1 \leq t \leq l$.

Combining (2.7), (2.9) and (2.10), it follows that

$$\Big(\prod_{A\in B(k,l)} a_A\Big)^{\frac{1}{\binom{m}{k}\binom{n}{l}}} \geq \frac{1}{\binom{m}{k}\binom{n}{l}} \sum_{i=1}^{\binom{m}{k}\binom{n}{l}} \Big(\prod_{j=1}^{\binom{m}{k}\binom{n}{l}} g_{B_i\cap B_j}\Big)^{\frac{1}{\binom{m}{k}\binom{n}{l}}}$$

$$= \frac{1}{\binom{m}{k}\binom{n}{l}} \sum_{i=1}^{\binom{m}{k}\binom{n}{l}} g_{B_i} = \frac{1}{\binom{m}{k}\binom{n}{l}}\Big(\sum_{A\in B(k,l)} g_A\Big),$$

which is just the inequality (1.2).

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