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INDEPENDENCE OF THE AXIOMATIC SYSTEM FOR MV-ALGEBRAS

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ABSTRACT. We prove that the well-known axiomatic system of MV-algebras is not independent. The axiom of commutativity can be deleted and the remaining axioms are shown to be independent.

We describe an independent axiomatic system for MV-algebras. The concept of an MV-algebra was introduced by C. C. Chang [2] as an axiomatization of the Łukasiewicz many-valued logic. The definition used in nowadays is taken from the monograph [3] (with a different order of axioms):

Definition 1. By an *MV-algebra* is meant an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following axioms

(MV1) $x \oplus 0 = x$

(MV2) $\neg\neg x = x$

(MV3) $x \oplus y = y \oplus x$

(MV4) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$

(MV5) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

(MV6) $x \oplus \neg 0 = \neg 0$.

We show that the axiomatic system (MV1)–(MV6) is redundant.

Theorem 1. *An algebra* $\mathcal{A} = (A; \oplus, \neg, 0)$ *of type* $(2, 1, 0)$ *is an MV-algebra if and only if it satisfies the axioms* (MV1), (MV2), (MV4), (MV5), (MV6).

Proof. We need to show that (MV3) follows from the axioms (MV1), (MV2), (MV4), (MV5) and (MV6). For this, take $y = 0$ and substitute z by y in (MV4) to obtain

$$(x \oplus 0) \oplus y = x \oplus (0 \oplus y).$$

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Applying (MV1), we get

$$x \oplus y = x \oplus (0 \oplus y). \tag{1}$$

Further, take $x = 0$ and substitute y by x in (MV5) to compute

$$\neg(\neg 0 \oplus x) \oplus x = \neg(\neg x \oplus 0) \oplus 0$$

thus, applying (MV1) and (MV2), we obtain

$$\neg(\neg 0 \oplus x) \oplus x = x. \tag{2}$$

For the next step, we start with (2) where instead of all x is $0 \oplus x$. Thus we have

$$0 \oplus x = \neg\big(\neg 0 \oplus (0 \oplus x)\big) \oplus (0 \oplus x).$$

The right hand side of the last identity can be reduced using (1) twice and (2) as follows

$$\neg\big(\neg 0 \oplus (0 \oplus x)\big) \oplus (0 \oplus x) = \neg\big(\neg 0 \oplus (0 \oplus x)\big) \oplus x = \neg(\neg 0 \oplus x) \oplus x = x.$$

Therefore, we have

$$0 \oplus x = x. \tag{3}$$

Now, put $\neg(\neg x \oplus y)$ instead of x in (MV4) to obtain

$$\neg(\neg x \oplus y) \oplus (y \oplus z) = \big(\neg(\neg x \oplus y) \oplus y\big) \oplus z.$$

Since $\big(\neg(\neg x \oplus y) \oplus y\big) \oplus z = \big(\neg(\neg y \oplus x) \oplus x\big) \oplus z = \neg(\neg y \oplus x) \oplus (x \oplus z)$ by (MV5) and (MV4), we get

$$\neg(\neg x \oplus y) \oplus (y \oplus z) = \neg(\neg y \oplus x) \oplus (x \oplus z). \tag{4}$$

For $y = \neg 0$ in (MV5) we compute

$$\neg(\neg x \oplus \neg 0) \oplus \neg 0 = \neg(\neg\neg 0 \oplus x) \oplus x.$$

From this, applying (MV6), (MV2) and (3), we have

$$\neg 0 = \neg(0 \oplus x) \oplus x,$$

which, using (3) again, give us

$$\neg 0 = \neg x \oplus x. \tag{5}$$

Now, put $\neg y$ instead of y and y instead of z in (4) to obtain

$$\neg(\neg x \oplus \neg y) \oplus (\neg y \oplus y) = \neg(\neg\neg y \oplus x) \oplus (x \oplus y).$$

By (5) and (MV2) we reduce this to

$$\neg(\neg x \oplus \neg y) \oplus \neg 0 = \neg(y \oplus x) \oplus (x \oplus y).$$

Using (MV6), we have

$$\neg 0 = \neg(y \oplus x) \oplus (x \oplus y). \tag{6}$$

Finally, we are going to prove that $x \oplus y = y \oplus x$. Using (3) and (MV2) we compute

$$x \oplus y = 0 \oplus (x \oplus y) = \neg\neg 0 \oplus (x \oplus y).$$

Further, with (6) and (MV5),

$$\neg\neg 0 \oplus (x \oplus y) = \neg\big(\neg(y \oplus x) \oplus (x \oplus y)\big) \oplus (x \oplus y) = \neg\big(\neg(x \oplus y) \oplus (y \oplus x)\big) \oplus (y \oplus x)$$

which is equal to $\neg\neg 0 \oplus (y \oplus x)$ by (6), where x is substituted by y and vice versa. Since $\neg\neg 0 \oplus (y \oplus x) = 0 \oplus (y \oplus x) = y \oplus x$, by (MV2) and (3), we are done. □

Next, we show that the remaining axioms (MV1), (MV2), (MV4), (MV5) and (MV6) are independent. Hence, none of them can be removed.

Theorem 2. *The axioms* (MV1), (MV2), (MV4), (MV5) *and* (MV6) *are independent.*

Proof. Denote by B the two-element set $\{0, 1\}$.
(I) Consider an algebra $(B; \oplus, \neg, 0)$ where $\oplus$ is a constant operation: $x \oplus y = 0$ for all $x, y \in B$ and $\neg 0 = 0$, $\neg 1 = 1$. One can easily check that this algebra satisfies (MV2), (MV4), (MV5), (MV6) but not (MV1) since $1 \oplus 0 = 0 \neq 1$.
(II) Now, let $(B; \oplus)$ be a join-semilattice and $\neg x = 1$ for all $x \in B$. Then $(B; \oplus, \neg, 0)$ satisfies (MV1), (MV4), (MV5) and (MV6) but, trivially, not (MV2).
(III) Let $\mathcal{C} = (\{0, 1, 2\}; \oplus, \neg, 0)$ be an algebra of type $(2, 1, 0)$ where the operations $\oplus$ and $\neg$ are defined by the following tables:

$\oplus$	0	1	2
0	0	1	1
1	1	1	2
2	2	1	2

x	0	1	2
$\neg x$	1	0	2

Evidently $\mathcal{C}$ satisfies (MV1), (MV2), (MV6). We can show that (MV4) is not satisfied: take $x = 0$, $y = 1$, $z = 2$. Then

$$(0 \oplus 1) \oplus 2 = 1 \oplus 2 = 2 \neq 1 = 0 \oplus 2 = 0 \oplus (1 \oplus 2).$$

It remains to prove that $\mathcal{C}$ satisfies (MV5).

(a) If $y = 0$ then (MV5) is reduced to $x = \neg(1 \oplus x) \oplus x$ which is plain to check.

(b) If $y = 1$ then (MV5) is $1 = \neg(0 \oplus x) \oplus x$ which one can easily check.

(c) For $y = 2$ is (MV5) as follows: $\neg(\neg x \oplus 2) \oplus 2 = \neg(\neg 2 \oplus x) \oplus x$ which also holds for each $x \in \{0, 1, 2\}$.

(IV) Let $\mathcal{D} = (\{0, 1, 2\}; \oplus, \neg, 0)$ be an algebra of type $(2, 1, 0)$ where the operations $\oplus$ and $\neg$ are defined by the following tables:

$\oplus$	0	1	2
0	0	1	0
1	1	1	1
2	2	1	2

x	0	1	2
$\neg x$	1	0	2

Evidently $\mathcal{D}$ satisfies (MV1), (MV2) and (MV6). We can show that (MV5) is not satisfied: take $x = 0$, $y = 2$. Then

$$\neg(\neg 0 \oplus 2) \oplus 2 = \neg(1 \oplus 2) \oplus 2 = \neg 1 \oplus 2 = 0 \neq 2 = \neg\neg 2 = \neg(\neg 2 \oplus 0) \oplus 0.$$

It remains to prove that $\mathcal{D}$ satisfies (MV4).

(a) If $z = 0$ or $z = 2$ then (MV4) is reduced to $x \oplus y = x \oplus y$ which is always true.

(b) If $z = 1$ then (MV4) is $(x \oplus y) \oplus 1 = x \oplus (y \oplus 1)$ which is evidently true.

(V) Finally, let $(B; \oplus)$ be a join-semilattice and $\neg$ be the identity mapping on B. Then clearly (MV1), (MV2) and (MV4) are satisfied. To prove (MV5) we mention that for $x = y$ it is trivial as well as for the general case $\{x, y\} = \{0, 1\}$ since

$$\neg(\neg 1 \oplus 0) \oplus 0 = 1 \oplus 0 = 1 = 1 \oplus 1 = \neg(\neg 0 \oplus 1) \oplus 1.$$

It remains to show that (MV6) is violated. For this, take $x = 1$. Then clearly the left-hand side of (MV6) equals to 1 but the right-hand side is 0. □

It is an open problem whether another axiom, different from (MV3), can be removed from (MV1)–(MV6) to obtain an axiomatic system of MV-algebras. After a preliminary inspection, the author conjectures that this is not possible.

Remark 1. In [1: Theorem 2.2, Theorem 3.2.] it was shown another independent axiomatic system for MV-algebras $(A; \oplus, \neg, 0)$:

(P1) $(x \oplus y) \oplus z = y \oplus (z \oplus x)$

(P2) $x \oplus 0 = x$

(P3) $x \oplus \neg 0 = \neg 0$

(P4) $\neg\neg 0 = 0$

(P5) $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$.

For more details see [1].

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