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SPECTRAL AUTOMORPHISMS IN CB-EFFECT ALGEBRAS

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Dedicated to Prof. David J. Foulis on the occasion of his 80th birthday

(*Communicated by Sylvia Pulmannová*)

ABSTRACT. Spectral automorphisms have been introduced in [IVANOV, A.—CARAGHEORGHEOPOL, D.: *Spectral automorphisms in quantum logics*, Internat. J. Theoret. Phys. **49** (2010), 3146–3152] in an attempt to construct, in the abstract framework of orthomodular lattices, an analogue of the spectral theory in Hilbert spaces. We generalize spectral automorphisms to the framework of effect algebras with compression bases and study their properties. Characterizations of spectral automorphisms as well as necessary conditions for an automorphism to be spectral are given. An example of a spectral automorphism on the standard effect algebra of a finite-dimensional Hilbert space is discussed and the consequences of spectrality of an automorphism for the unitary Hilbert space operator that generates it are shown.

The last section is devoted to spectral families of automorphisms and their properties, culminating with the formulation and proof of a Stone type theorem (in the sense of Stone's theorem on strongly continuous one-parameter unitary groups — see, e.g. [REED, M.—SIMON, B.: *Methods of Modern Mathematical Physics, Vol. I*, Acad. Press, New York, 1975]) for a group of spectral automorphisms.

Orthomodular lattices and orthomodular posets have been considered for a long time to be the appropriate mathematical structures for studying the logic of quantum mechanics [13, 14]. To allow the study of unsharp measurements or observations, represented by quantum effects, the algebraic structure of effect algebra is currently used (see, e.g., [2, 4]). The so-called compressions were introduced by Gudder [7] in the framework of effect algebras as an abstraction of

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the essential features of the mappings $J_P\colon \mathcal{E}(H) \to \mathcal{E}(H)$, $J_P(A) = PAP$ where P is a projection operator on the Hilbert space H, and $\mathcal{E}(H)$ denotes the set of effect operators on H. Let us remark that J_P is just the restriction to $\mathcal{E}(H)$ of the Naimark compression corresponding to P. Also introduced by Gudder [8], effect algebras with compression bases (CB-effect algebras) are an abstraction of the essential features of $\mathcal{E}(H)$, endowed with the family of compressions $(J_P)_{P\in\mathcal{P}(H)}$, where $\mathcal{P}(H)$ denotes the set of projection operators on the Hilbert space H.

Spectral automorphisms were initially introduced in an attempt to create, in the framework of orthomodular lattices, something similar to the spectral theory in Hilbert spaces [10]. Obtaining new characterizations of spectral automorphisms in orthomodular lattices and an attempt to clarify their connection with the classical discussion of unitary time evolution of a system as one parameter family of automorphisms on the associated logic of the system were the focus of [1].

Since CB-effect algebras are currently considered as the appropriate mathematical structures for representing physical systems, including observables, states and symmetries [3], it is only natural that we pursue the goal of generalizing spectral automorphisms, along with most of their properties and characterizations, to the framework of CB-effect algebras. An effect algebra version of the Stone-type theorem in [1] is also obtained.

1. Basic notions and facts

Definition 1.1. An algebraic structure $(E, \oplus, \mathbf{0}, \mathbf{1})$ such that E is a nonempty set, $\mathbf{0}$ and $\mathbf{1}$ are distinct elements of E and $\oplus$ is a partial binary operation on E is an *effect algebra* if for every $a, b, c \in E$ the following conditions hold:

(1) $a \oplus b = b \oplus a$ if $a \oplus b$ exists,
(2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if $(a \oplus b) \oplus c$ exists,
(3) for every $a \in E$, there exists a unique element $a' \in E$ such that $a \oplus a' = \mathbf{1}$ (*orthosupplement*),
(4) if $a \oplus \mathbf{1}$ is defined, then $a = \mathbf{0}$ (*zero-unit law*).

In what follows, we'll often write E instead of $(E, \oplus, \mathbf{0}, \mathbf{1})$, for simplicity. An effect algebra E is endowed with a partial ordering defined by $a \le b$ if there exists $c \in E$ such that $a \oplus c = b$. If the element c exists, it is uniquely determined and will be denoted by $b \ominus a$. We call elements $a, b \in E$ *orthogonal* and write $a \perp b$ if $a \oplus b$ exists (i.e., $a \le b'$). It can be easily verified that $\mathbf{0}$ and $\mathbf{1}$ are the least and the greatest elements of E, respectively, and that orthosupplementation is order-reversing and involutive. Also, $a \oplus \mathbf{0} = a$ for every $a \in E$ and a *cancellation law* holds: $a \oplus b \leq a \oplus c$ implies $b \leq c$ for every $a, b, c \in E$ (see, e.g., Dvurečenskij and Pulmannová [2], Foulis and Bennett [4]). If E and E' are effect algebras, the map $J \colon E \to E'$ is *additive* if $a \perp b$ implies $J(a) \perp J(b)$ and $J(a \oplus b) = J(a) \oplus J(b)$. A subset F of an effect algebra E which is closed under operations $\oplus$ and $'$ and such that $\mathbf{0}, \mathbf{1} \in F$ is a *sub-effect algebra* of E. A *Boolean subalgebra* of an effect algebra E is a sub-effect algebra of E which is a Boolean algebra with $'$ and with the operations $\vee$, $\wedge$ induced by the order in E.

Definition 1.2. An element a of an effect algebra E is *principal* if $b \oplus c \leq a$ for every $b, c \in E$ such that $b, c \leq a$ and $b \perp c$. It is *central* if a and a' are principal and for every $b \in E$, there exist $b_1, b_2 \in E$, $b_1 \leq a$, $b_2 \leq a'$ such that $b = b_1 \oplus b_2$.

For the remainder of this paper, $\widetilde{C}(E)$ will denote the center (i.e., the set of central elements) of an effect algebra E.

Definition 1.3. The elements a, b of an effect algebra E *coexist* ($a \leftrightarrow b$) if there exist $a_1, b_1, c \in E$ such that $a = a_1 \oplus c$, $b = b_1 \oplus c$ and $a_1 \oplus b_1 \oplus c$ exists. Elements $a_1, b_1, c \in E$ fulfilling this conditions are called a *Mackey decomposition* of a, b. For a subset M of E, we write $a \leftrightarrow M$ if $a \leftrightarrow b$ for all $b \in M$. The *commutant* of M in E is the set $K(M) = \{a \in E : \ a \leftrightarrow M\}$.

Remark 1.4. If F is a sub-effect algebra of an effect algebra E, it is easy to verify the following statements:

(1) if $a \in F$ and a is a principal element of E, then it is a principal element of F;

(2) if $a, b \in F$ coexist in F, then they coexist in E.

We shall need the following well known characterization of coexistence, the proof of which we omit.

Proposition 1.5. *Let E be an effect algebra and $a, b \in E$. Then, $a \leftrightarrow b$ if and only if there exist $a_1, a_2 \in E$ such that $a_1 \leq b'$, $a_2 \leq b$ and $a = a_1 \oplus a_2$.*

Definition 1.6. An effect algebra in which every element is principal is an *orthomodular poset*. An orthomodular poset which is a lattice is an *orthomodular lattice*.

We should mention that orthomodular posets are usually defined equivalently as partially ordered sets which are bounded, endowed with an orthocomplementation such that the orthomodular law is valid [14]. They can also be characterized as effect algebras in which, for every orthogonal pair of elements a, b, $a \oplus b = a \vee b$ ([4,5]). It is therefore obvious that, in orthomodular posets, coexistence is the same thing as the usual compatibility/commutativity. Throughout the paper, a pair of coexistent elements of an effect algebra will be sometimes called *compatible* or *commuting*, if the pair belongs to an orthomodular poset.

Remark 1.7. It is a well known fact that an orthomodular poset whose elements are pairwise compatible is a Boolean algebra (see, e.g., [11, 14]). In view of Definition 1.6, it follows that an effect algebra with every element principal and every pair of elements coexistent is a Boolean algebra. This entails, according to Proposition 1.5 and Definition 1.2, that an effect algebra E is a Boolean algebra if and only if $E = \widetilde{C}(E)$.

Definition 1.8. An orthomodular poset P is called *regular* if, for every set of pairwise compatible elements $\{p, q, r\} \subseteq P$, $p \leftrightarrow r \vee q$ and $p \leftrightarrow r \wedge q$.

Compressions and compression bases in effect algebras are discussed in details in [7, 8]. Let us briefly present a few basic facts about them that we will use.

Definition 1.9. Let E be an effect algebra and $J\colon E \to E$ be an additive map. If $a \leq J(\mathbf{1})$ implies $J(a) = a$, then J is a *retraction* and $J(\mathbf{1})$ is its *focus*. If, moreover, $J(a) = \mathbf{0}$ implies $a \leq J(\mathbf{1})'$, J is called a *compression*. The element $p \in E$ is a *projection* if it is the focus of some retraction on E.

Remark 1.10. Every element of an effect algebra E which is a projection is a principal element of E ([7: Lemma 3.1]).

Definition 1.11. A sub-effect algebra F of an effect algebra E is *normal* if $b \in F$ for every $a, b, c \in E$ such that $a \oplus b \oplus c$ exists in E, $a \oplus b \in F$ and $b \oplus c \in F$.

It is not difficult to see that if two elements of a normal sub-effect algebra F of an effect algebra E coexist in E, they also coexist in F ([8: Lemma 3.1]).

Definition 1.12. A family $(J_p)_{p\in P}$ of compressions on the effect algebra E, indexed by a normal sub-effect algebra P of E is a *compression base* for E if the following conditions are fulfilled:

(1) The focus of compression J_p is p, for every $p \in P$.
(2) $J_{p\oplus r} \circ J_{r\oplus q} = J_r$ for $p, q, r \in P$ such that $p \oplus q \oplus r$ is defined in E.

Let us that remark that, for every effect algebra E, $(J_p)_{p\in\widetilde{C}(E)}$ with $J_p(a) = p \wedge a$ is a compression base ([15]).

Definition 1.13. A minimal non-zero element of an effect algebra E is called an *atom* of E. E is called *atomic* if every element of E dominates an atom.

Let us now present briefly the so-called standard Hilbert space effect algebra — the prototypical example of an effect algebra. Let H be a Hilbert space and denote by $\mathcal{E}(H)$ the set of selfadjoint operators between the null and the identity operator. With the partial binary operation $\oplus\colon \mathcal{E}(H) \to \mathcal{E}(H)$, defined by $A \oplus B = A + B$ if $A + B \in \mathcal{E}(H)$, $\mathcal{E}(H)$ becomes an effect algebra. The set $\mathcal{P}(H)$ of projection operators on H is the set of principal elements of $\mathcal{E}(H)$ and an orthomodular lattice. If we define $J_P\colon \mathcal{E}(H) \to \mathcal{E}(H)$ by $J_P(A) = PAP$ for every $P \in \mathcal{P}(H)$, then $(J_P)_{P\in\mathcal{P}(H)}$ is a compression base for $\mathcal{E}(H)$. Later on, we shall refer to this compression base for the standard Hilbert space effect algebra as its *canonical* compression base.

Definition 1.14. Let E be an effect algebra. A map $\varphi\colon E \to E$ is an *automorphism* of E if it is bijective and, for every $a, b \in E$, $a \perp b$ if and only if $\varphi(a) \perp \varphi(b)$, in which case $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$.

Remark 1.15. Let us notice that an automorphism φ of an effect algebra E is clearly order-preserving (i.e., $a \leq b$ if and only if $\varphi(a) \leq \varphi(b)$ for every $a, b \in E$). It is therefore not difficult to see that such an automorphism also preserves the infimum (hence the supremum too, by duality). More precisely, if $a, b \in E$ and $a \wedge b$ exists in E, then $\varphi(a) \wedge \varphi(b)$ also exists in E and $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$ (see, e.g., [2: Section 1.3]).

2. Spectral automorphisms: the idea and definitions

In the classical formulation of Hilbert space quantum mechanics, an observable is a self-adjoint operator which can be represented by means of its associated projection-valued spectral measure, according to spectral theory (see, e.g., [9,16]). In current formulations of quantum theory, observables which may be unsharp are represented as positive operator-valued measures, taking values in the set $\mathcal{E}(H)$ of quantum effects ([3,4]). In abstract effect algebras, on the other hand, observables are mathematical objects similar to such measures, except their values are elements of the effect algebra ([3]). Therefore, if we intend to replace the Hilbert space-based structure of the quantum theory with one based on the more abstract and general structure of CB-effect algebra, the study of effect algebra automorphisms might be a useful tool.

Before enouncing the definition of the spectral automorphism notion in CB-effect algebras, let us see what are the facts, in the framework of standard Hilbert space effect algebra, which suggest this notion. Let H be a Hilbert space and $\mathcal{E}(H)$ the corresponding standard effect algebra. Automorphisms of $\mathcal{E}(H)$ are of the form $\varphi_U\colon \mathcal{E}(H) \to \mathcal{E}(H)$, $\varphi_U(A) = UAU^{-1}$, where U is a unitary or antiunitary Hilbert space operator ([3]). An element $A \in \mathcal{E}(H)$ is φ_U-invariant if and only if $\varphi_U(A) = UAU^{-1} = A$, i.e., operators U and A commute. Let B_U be the Boolean algebra of projection operators that is the image of the projection-valued spectral measure associated to U. Then, operators A and U commutes if and only if A commutes with B_U (i.e., with every projection operator in B_U) [9]. We are therefore led to the following definition of spectral automorphisms in CB-effect algebras:

Definition 2.1. Let E be an effect algebra and $(J_p)_{p\in P}$ be a compression base for E. An automorphism $\varphi\colon E \to E$ is *spectral* if there exists a Boolean subalgebra B of P with the property:

$$\varphi(a) = a \iff a \leftrightarrow B. \tag{P1}$$

Before we can formulate, as it would be expected, the definition of the spectrum of a spectral automorphism as the greatest Boolean subalgebra of P fulfilling (P1), some more work is required. Indeed, it is not clear at all if such a Boolean algebra exists. In order to prove that it does, we will make use of a number of well known properties of commutativity/compatibility in orthomodular posets. For complete details, we refer to [14: Section 1.3]. However, for readers convenience, we'll reproduce some of the most important results we use.

Recall that in orthomodular lattices, every subset of pairwise compatible elements is a subset of a Boolean subalgebra of the lattice. However, this is not the case in orthomodular posets, unless they satisfy the regularity property.

Proposition 2.2. ([14: Proposition 1.3.29]) *An orthomodular poset P is regular if and only if every pairwise compatible subset of it admits an enlargement to a Boolean subalgebra of P.*

It is therefore fortunate that we can take advantage of the following result:

Proposition 2.3. ([15: Corollary 4.2(ii)]) *Let E be an effect algebra and $(J_p)_{p\in P}$ be a compression base for E. If $p, q, r \in P$ are pairwise compatible, then $p \leftrightarrow r\wedge q$ and $p \leftrightarrow r \vee q$, hence P is a regular orthomodular poset.*

Although it might be considered as a known fact, the content of the next lemma is tailored to suit our needs, as it will be used several times throughout the paper. The construction that is used in its proof is inspired from [14: Proposition 1.3.23].

Lemma 2.4. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and $M \subseteq P$ be a set of pairwise compatible elements. Then there exists the smallest Boolean subalgebra B of P such that $M \subseteq B$. Moreover, for this Boolean subalgebra B, $K(M) = K(B)$ holds.*

Proof. According to Propositions 2.2 and 2.3, there exists a Boolean subalgebra of P which includes M. The smallest Boolean subalgebra of P which includes M is then just the set-theoretical intersection of all such Boolean algebras.

Let B be the smallest Boolean subalgebra of P which includes M and $a \in E$. Clearly, if $a \leftrightarrow B$, then $a \leftrightarrow M$, since $M \subseteq B$. Let us prove the converse. Let $Q \subseteq M$ be a finite set, $Q = \{q_1, q_2, \dots, q_r\}$ and let us denote $F_Q = \{q_1^{a_1} \wedge q_2^{a_2} \wedge \dots \wedge q_r^{a_r} : (a_1, a_2, \dots, a_r) \in A^r\}$, where $A = \{-1, 1\}$ and $q_i^1 = q_i$, $q_i^{-1} = q_i'$ for every $i \in \{1, 2, \dots, r\}$. Let $B(Q)$ be the set of suprema of all subsets of

F_Q and let $\widetilde{B} = \bigcup\{B(Q) : \ Q \subseteq M, \ Q \text{ finite}\}$. We omit here the routine verification of the fact that $\widetilde{B}$ is a Boolean algebra, which can be found in [14: Proposition 1.3.23]. Since $\widetilde{B}$ is a Boolean algebra that includes M, and every Boolean algebra that includes M must include $\widetilde{B}$ (obviously), it follows that $B = \widetilde{B}$. In view of Proposition 2.3, $a \leftrightarrow M$ implies $a \leftrightarrow B(Q)$ for every $Q \subseteq M$, Q finite. It follows that $a \leftrightarrow B$, which concludes our proof. □

Let us now state the result that will allow us to define the spectrum of a spectral automorphism.

Proposition 2.5. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and $\varphi\colon E \to E$ be a spectral automorphism. There exists the greatest Boolean subalgebra $B \subseteq P$ satisfying* (P1).

Proof. Let $\{B_i : \ i \in I\}$ be the set of all Boolean subalgebras of P satisfying (P1). We'll prove that $\bigcup\limits_{i\in I} B_i$ is a set of pairwise compatible elements of P. Indeed, for every $i, j \in I$, from $a \in B_i$ it follows that $a \leftrightarrow B_i$ and, due to (P1), $\varphi(a) = a$. Applying (P1) again for B_j, we conclude that $a \leftrightarrow B_j$. Hence $B_i \leftrightarrow B_j$ for every $i, j \in I$ and every pair of elements of $\bigcup\limits_{i\in I} B_i$ is compatible. According to Lemma 2.4, there exists the smallest Boolean subalgebra B of P containing $\bigcup\limits_{i\in I} B_i$.

We will now prove that B satisfies (P1), hence being the greatest Boolean subalgebra of P with this property. Clearly, if $a \leftrightarrow B$, then $a \leftrightarrow B_i$ for every $i \in I$, hence $\varphi(a) = a$. Conversely, let us assume $\varphi(a) = a$. It follows that $a \leftrightarrow B_i$ for every $i \in I$, hence $a \leftrightarrow \bigcup\limits_{i\in I} B_i$. In view of the second assertion of Lemma 2.4, we conclude that $a \leftrightarrow B$. □

Definition 2.6. Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and $\varphi\colon E \to E$ be a spectral automorphism. The greatest Boolean subalgebra of P fulfilling (P1) is the *spectrum* of the automorphism φ, denoted by σ_φ.

Proposition 2.7. *Let E be an effect algebra and $(J_p)_{p\in P}$ be a compression base for E. If $P \subseteq \widetilde{C}(E)$, then the identity is the only spectral automorphism of E.*

Proof. Let $\varphi\colon E \to E$ be a spectral automorphism. Then $\sigma_\varphi \subseteq P \subseteq \widetilde{C}(E)$. It follows that $a \leftrightarrow \sigma_\varphi$ for every $a \in E$ and, due to (P1), every element of E is φ-invariant. □

Remark 2.8. As a particular case, if E is a Boolean algebra, then its identity is its only spectral automorphism. Therefore, the presence of nontrivial spectral automorphisms allows us to distinguish between classical (Boolean) and nonclassical theories.

Before ending this section, let us discuss an example of a spectral automorphism.

Example 2.9. Consider H a 3-dimensional complex Hilbert space, $\mathcal{E}(H)$ the corresponding standard effect algebra and $\mathcal{P}(H)$ the set of its projection operators or, equivalently, the set of its subspaces. Let $\mathcal{E}(H)$ be endowed with its canonical compression base $(J_P)_{P\in\mathcal{P}(H)}$. Let $P \in \mathcal{P}(H)$ be a 1-dimensional projector and denote S_P the corresponding subspace (i.e., its range). Then P' is its orthogonal complement and define $U\colon H \to H$ as the symmetry of H with respect to the plane $S_{P'}$ corresponding to P'. Clearly U is a unitary operator, hence $\varphi\colon \mathcal{E}(H) \to \mathcal{E}(H)$ defined by $\varphi(A) = UAU^{-1}$ is an automorphism of $\mathcal{E}(H)$. We assert that $\mathcal{B} = \{\mathbf{0}, P, P', \mathbf{1}\}$ is a Boolean subalgebra of $\mathcal{P}(H)$ satisfying (P1). To prove this assertion, let us notice that $A \in \mathcal{E}(H)$ is φ-invariant if and only if A commutes with U (as operators). On the other hand, $A \leftrightarrow \mathcal{B}$ if and only if $A \leftrightarrow P$ and it is well known, since P is a projection operator, that this is equivalent to the fact that A and P commute as operators (see, e.g., [12]). However, by a classic Hilbert space theory result [9: §27, Theorem 2], A and P commutes if and only if the range of P reduces A (i.e., S_P and $S_{P'}$ are invariant under A). To complete our proof, we only need to show that A and U commute if and only if S_P and $S_{P'}$ are invariant under A. In order to do that, we need to remark that, considering the definition of U, $x \in S_P$ if and only if $Ux = -x$ and $x' \in S_{P'}$ if and only if $Ux' = x'$. Now, let us assume that $AU = UA$. If $x \in S_P$ then $UAx = AUx = -Ax$ and therefore $Ax \in S_P$. Also, if $x' \in S_{P'}$ then $UAx' = AUx' = Ax'$ and therefore $Ax' \in S_{P'}$. Conversely, let us assume that $A(S_P) \subseteq S_P$ and $A(S_{P'}) \subseteq S_{P'}$. For an arbitrary $y \in H$, there exist $x \in S_P$ and $x' \in S_{P'}$ such that $y = x + x'$. Then $AUy = A(Ux+Ux') = A(-x+x') = -Ax+Ax' = UAx+UAx' = UAy$, which proves the commutativity. We conclude that φ is a spectral automorphism.

3. Characterizations and properties of spectral automorphisms

For an automorphism φ of an effect algebra E, we will denote by E_φ the set of φ-invariant elements of E. Due to the definition properties of automorphisms, it is clear that E_φ is a sub-effect algebra of E.

The fact that the spectrum σ_φ of a spectral automorphism $\varphi\colon E \to E$ fulfills (P1) can be written equivalently in the useful form of the equality $E_\varphi = K(\sigma_\varphi)$.

Let E be an effect algebra and $M, N \subseteq E$. The following properties of commutants can be easily verified:

(1) $M \subseteq K(K(M))$ and

(2) if $M \subseteq N$ then $K(N) \subseteq K(M)$.

Proposition 3.1. *Let E be an effect algebra and $(J_p)_{p\in P}$ be a compression base for E. If $M \subseteq P$ then the commutant $K(M)$ of M is a sub-effect algebra of E.*

Proof. Clearly, $\mathbf{0}, \mathbf{1} \in K(M)$. According to [2: Proposition 1.10.3], $a \leftrightarrow b$ if and only if $a' \leftrightarrow b$ for every $a, b \in E$. Hence $a \in K(M)$ if and only if $a' \in K(M)$. It remains to prove that, for every orthogonal pair of elements $a, b \in E$, if $a, b \leftrightarrow M$, then $a \oplus b \leftrightarrow M$. Towards this end, we will use the characterization of coexistence given in Proposition 1.5. Let $c \in M$. Since $a, b \leftrightarrow c$, there exist $a_1, b_1 \leq c$ and $a_2, b_2 \leq c'$ such that $a = a_1 \oplus a_2$ and $b = b_1 \oplus b_2$. On the other hand, since $a \perp b$, it follows that $a_1, a_2 \leq a \leq b' \leq b_1', b_2'$ and therefore $a_1 \perp b_1$, $a_2 \perp b_2$. It follows that $a_1 \oplus b_1 \leq c$ and $a_2 \oplus b_2 \leq c'$, since c, c' are in P and therefore, according to Remark 1.10, they are principal elements of E. Taking into account that $a \oplus b = (a_1 \oplus b_1) \oplus (a_2 \oplus b_2)$, the desired conclusion that $a \oplus b \leftrightarrow c$ follows. □

Lemma 3.2. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and $\varphi\colon E \to E$ be an automorphism. Then $\widetilde{C}(E_\varphi) \cap P$ is a Boolean subalgebra of P.*

Proof. First, let us remark that since E_φ is a sub-effect algebra of E, it makes sense to speak about its center $\widetilde{C}(E_\varphi)$, which is even a Boolean algebra ([6]). It is then clear that $\widetilde{C}(E_\varphi) \cap P$ is a sub-effect algebra of E (and of P as well). All its elements are principal in E, according to Remark 1.10, and therefore in all sub-effect algebras of E, in view of Remark 1.4(1). It follows that $\widetilde{C}(E_\varphi) \cap P$ is an orthomodular poset. To prove it's a Boolean algebra, we need to show that its elements are pairwise coexistent. Let $a, b \in \widetilde{C}(E_\varphi) \cap P$. Since $\widetilde{C}(E_\varphi)$ is a Boolean algebra, a and b coexist in $\widetilde{C}(E_\varphi)$. Let $a = a_1 \oplus c$, $b = b_1 \oplus c$ be a Mackey decomposition of a, b in $\widetilde{C}(E_\varphi)$. Then $a_1 \oplus c \oplus b_1$ exists in $\widetilde{C}(E_\varphi)$, hence in E, $a = a_1 \oplus c \in P$ and $b = b_1 \oplus c \in P$. Since P is a normal sub-effect algebra of E, it follows that $c \in P$ and therefore $a_1, b_1 \in P$. This proves that a_1, b_1, c is a Mackey decomposition of a, b in $\widetilde{C}(E_\varphi) \cap P$, hence a, b coexist in $\widetilde{C}(E_\varphi) \cap P$, which concludes our proof. □

The following lemma and corollary, that will be useful in the sequel, are related to [6: Theorem 4.2, Lemma 5.2]. However, the statements we prove are slightly more general and could be of some interest by themselves.

Lemma 3.3. *Let E be an effect algebra, $\{e_1, e_2, \dots, e_n\}$ be an orthogonal set of its elements (i.e., the sum $\bigoplus_{i=1}^{n} e_i$ exists) and consider $p \in E$ such that $p = \bigoplus_{i=1}^{n} p_i$ with $p_i \leq e_i$. If e_j is principal for some $j \in \{1, 2, \dots, n\}$, then $p \wedge e_j$ exists in E and $p_j = p \wedge e_j$.*

P r o o f. Let us assume e_j is principal. Clearly $p_j \leq e_j, p$, and for an arbitrary $x \in E$, $x \leq e_j, p$, we have to prove that $x \leq p_j$. Let us denote by $q_i = e_i \ominus p_i$ for all $i \in \{1, 2, \ldots, n\}$. Then $\bigoplus_{i=1}^{n} e_i = \bigoplus_{i=1}^{n} (p_i \oplus q_i) = \big(\bigoplus_{i=1}^{n} p_i\big) \oplus \big(\bigoplus_{i=1}^{n} q_i\big) = p \oplus q$, where $q = \bigoplus_{i=1}^{n} q_i$. It follows that $q_j \leq q \leq p' \leq x'$, hence $x \perp q_j$. Since $x, q_j \leq e_j$ and e_j is principal, it results that $x \oplus q_j \leq e_j = p_j \oplus q_j$ and therefore, by the cancellation law, $x \leq p_j$. □

Corollary 3.4. *If a, a' are principal elements of the effect algebra E, $b \in E$ and $a \leftrightarrow b$, then $a \wedge b$ and $a' \wedge b$ exist in E and $b = (a \wedge b) \oplus (a' \wedge b)$.*

Lemma 3.5. *Let E be an effect algebra, $(J_p)_{p \in P}$ be a compression base for E and $\varphi \colon E \to E$ be an automorphism. If a, a' are principal in E, $a, b \in E_\varphi$ and a, b coexist in E, then they coexist in E_φ as well.*

P r o o f. According to Corollary 3.4, $a \wedge b$ and $a' \wedge b$ exist in E and $b = (a \wedge b) \oplus (a' \wedge b)$. Since $a, a', b \in E_\varphi$, according to Remark 1.15, $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b) = a \wedge b$ and $\varphi(a' \wedge b) = \varphi(a') \wedge \varphi(b) = a' \wedge b$, and therefore $a \wedge b, a' \wedge b \in E_\varphi$. Considering that $a \wedge b \leq a$ and $a' \wedge b \leq a'$, it follows that a, b coexist in E_φ, according to Proposition 1.5. □

The following theorem allows us to find the spectrum of a spectral automorphism.

Theorem 3.6. *Let E be an effect algebra and $(J_p)_{p \in P}$ be a compression base for E. If $\varphi \colon E \to E$ is a spectral automorphism, then $\sigma_\varphi = \widetilde{C}(E_\varphi) \cap P$.*

P r o o f. Let $s \in \sigma_\varphi$. Since σ_φ is a Boolean algebra, s commutes with it and, according to (P1), $s \in E_\varphi$. Since $s \in \sigma_\varphi \subseteq P$, s and s' are principal elements of E, according to Remark 1.10, and of E_φ as well, according to Remark 1.4(1). Clearly, $s \leftrightarrow K(\sigma_\varphi)$ and using (P1) again, $K(\sigma_\varphi) = E_\varphi$, hence $s \leftrightarrow E_\varphi$, i.e., s coexists with every element of E_φ in E. To show that $s \in \widetilde{C}(E_\varphi)$, we need this coexistence to take place in E_φ as well. This happens, according to Lemma 3.5. It follows that $s \in \widetilde{C}(E_\varphi) \cap P$, which proves that $\sigma_\varphi \subseteq \widetilde{C}(E_\varphi) \cap P$.

To prove the converse inclusion, it suffices to show that $\widetilde{C}(E_\varphi) \cap P$ satisfies (P1), since it's a Boolean subalgebra of P, according to Lemma 3.2, and σ_φ is the greatest Boolean subalgebra of P with this property. Let $a \in E$ be φ-invariant, i.e., $a \in E_\varphi$. Then $a \leftrightarrow \widetilde{C}(E_\varphi)$, hence $a \leftrightarrow \widetilde{C}(E_\varphi) \cap P$. Conversely, since $\widetilde{C}(E_\varphi) \cap P \supseteq \sigma_\varphi$, $a \leftrightarrow \widetilde{C}(E_\varphi) \cap P$ implies $a \leftrightarrow \sigma_\varphi$ and, due to the spectrality of φ, this entails that $\varphi(a) = a$. □

Corollary 3.7. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and $\varphi\colon E \to E$ be an automorphism. Then φ is spectral if and only if $K(\widetilde{C}(E_\varphi) \cap P) \subseteq E_\varphi$ (the converse inclusion is always true).*

P r o o f. According to Theorem 3.6, if φ is spectral, then $\sigma_\varphi = \widetilde{C}(E_\varphi) \cap P$, and due to (P1), $K(\sigma_\varphi) = E_\varphi$.

Conversely, if $K(\widetilde{C}(E_\varphi) \cap P) \subseteq E_\varphi$, then $K(\widetilde{C}(E_\varphi) \cap P) = E_\varphi$ and therefore $\widetilde{C}(E_\varphi) \cap P$ is a Boolean subalgebra of P (according to Lemma 3.2) which satisfies (P1), and it follows that φ is spectral. □

The following result characterizes spectral automorphisms in CB-effect algebras.

Theorem 3.8. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and $\varphi\colon E \to E$ be an automorphism. Then φ is spectral if and only if $a \wedge b \in E_\varphi$ for every $a \in \widetilde{C}(E_\varphi) \cap P$, $b \in K(\widetilde{C}(E_\varphi) \cap P)$.*

P r o o f.

" $\Longleftarrow$ " Let us denote $\widetilde{C}(E_\varphi) \cap P = B$. According to Lemma 3.2, B is a Boolean subalgebra of P. Let b be an element of E such that $b \leftrightarrow B$. For every $a \in B$ we have $a' \in B$, a, a' are principal in E, according to Remark 1.10, and $a \leftrightarrow b$, therefore, according to Corollary 3.4, $b = (a \wedge b) \oplus (a' \wedge b)$. It then follows from our hypothesis that $a \wedge b, a' \wedge b \in E_\varphi$, hence $b \in E_\varphi$. Conversely, if $b \in E_\varphi$, clearly $b \leftrightarrow \widetilde{C}(E_\varphi) \cap P = B$. It follows that B is a Boolean subalgebra of P satisfying (P1), hence φ is a spectral automorphism.

" $\Longrightarrow$ " Let us assume φ is spectral and $b \in K(\widetilde{C}(E_\varphi) \cap P)$, $a \in \widetilde{C}(E_\varphi) \cap P$. Then a, a' are principal elements of E, according to Remark 1.10, $a \leftrightarrow b$ and, according to Corollary 3.4, the infimum $a \wedge b$ exists in E. Then, according to Remark 1.15, $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b) = a \wedge b$, since $a \in E_\varphi$ and, according to Corollary 3.7, $b \in E_\varphi$ as well. It follows that $a \wedge b \in E_\varphi$. □

The search for the conditions that a Boolean algebra must fulfill in order to be the spectrum of a spectral automorphism leads to the following notion.

Definition 3.9. Let E be an effect algebra and $(J_p)_{p\in P}$ be a compression base for E. A Boolean subalgebra $B \subseteq P$ is *C-maximal* if $\widetilde{C}(K(B)) \cap P \subseteq B$.

Let us remark that, according to Proposition 3.1, $K(B)$ is an effect algebra, therefore its center exists. It is easy to see that, for example, every block (i.e., maximal Boolean subalgebra) of P is C-maximal.

The following results from [15] will be used for the proof of our next theorem.

Lemma 3.10. (see [15: Lemma 4.1, Theorem 4.5, Corollary 4.1]) *Let E be an effect algebra and $(J_p)_{p\in P}$ be a compression base for E.*

(1) *If $p, q \in P$ and $p \leftrightarrow q$, then, for every $a \in E$, $a \leftrightarrow \{p, q\}$ implies $a \leftrightarrow p \wedge q$.*

(2) *If $p_1, p_2 \in P$ are orthogonal and $a \in E$, $a \leftrightarrow \{p_1, p_2\}$, then $(p_1 \oplus p_2) \wedge a = (p_1 \wedge a) \oplus (p_2 \wedge a)$.*

Theorem 3.11. *Let E be an effect algebra and $(J_p)_{p\in P}$ be a compression base for E. A Boolean subalgebra $B \subseteq P$ is C-maximal if and only if $B = K(K(B)) \cap P$.*

Proof.

" $\Longleftarrow$ " Let B be a Boolean subalgebra of P such that $B = K(K(B))\cap P$, and let $a \in \widetilde{C}(K(B))\cap P$. Then $a \in P$ and $a \leftrightarrow K(B)$, hence $a \in K(K(B))\cap P = B$. It follows that $\widetilde{C}(K(B)) \cap P \subseteq B$, hence B is C-maximal.

" $\Longrightarrow$ " Let B be a Boolean subalgebra of P such that $\widetilde{C}(K(B)) \cap P \subseteq B$. The inclusion $B \subseteq K(K(B)) \cap P$ is trivial. Let $a \in K(K(B)) \cap P$. It suffices to prove that $a \in \widetilde{C}(K(B))$. Since $a \leftrightarrow K(B)$ and $B \subseteq K(B)$, also $a \leftrightarrow B$, hence $a \in K(B)$. Since $a, a' \in P$ are principal elements of E, and, according to Remark 1.4(1), also of $K(B)$, we only need to prove that a coexists with every element of $K(B)$ in $K(B)$ too (not just in E). Let $b \in K(B)$. Since $a \in K(K(B))$, $a \leftrightarrow b$ in E and, according to Corollary 3.4, $a \wedge b$ and $a' \wedge b$ exist in E and $b = (a \wedge b) \oplus (a' \wedge b)$. To prove $a \leftrightarrow b$ in $K(B)$, it suffices to show that $a \wedge b, a' \wedge b \in K(B)$. We will only prove that $a \wedge b \in K(B)$, the proof for $a' \wedge b$ being analogous. Let $d \in B$. We have to show that $d \leftrightarrow a \wedge b$ in E. Let us remark that, although a, b, d are pairwise coexistent, b need not be in P and therefore we cannot use the regularity of P. We shall, instead, use Lemma 3.10 (1). Indeed, $a, d \in P$, $a \leftrightarrow d$ (in E and in P, since P is a normal sub-effect algebra of E) and $b \leftrightarrow \{a, d\}$, hence $b \leftrightarrow a \wedge d$. Similarly, $b \leftrightarrow a \wedge d'$. On the other hand, applying Corollary 3.4 in P, we find that $a \wedge d$ and $a \wedge d'$ exist in P and $a = (a \wedge d) \oplus (a \wedge d')$. Applying Lemma 3.10 (2) with $a \wedge d, a \wedge d'$ as p_1, p_2, we find that $\big((a \wedge d) \oplus (a \wedge d')\big) \wedge b = \big((a \wedge d) \wedge b\big) \oplus \big((a \wedge d') \wedge b\big)$. Therefore, $a \wedge b = \big((a \wedge d) \oplus (a \wedge d')\big) \wedge b = \big((a \wedge b) \wedge d\big) \oplus \big((a \wedge b) \wedge d'\big)$, hence $a \wedge b \leftrightarrow d$, according to Proposition 1.5. □

Let us apply the just proved result to the spectrum of a spectral automorphism.

Corollary 3.12. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and $\varphi\colon E \to E$ be a spectral automorphism. Then:*

(1) $\sigma_\varphi = \widetilde{C}(E_\varphi) \cap P$ *is C-maximal;*

(2) $\sigma_\varphi = K(K(\sigma_\varphi)) \cap P$*;*

(3) $\sigma_\varphi = K(E_\varphi) \cap P$.

Proof.

(1) According to Theorem 3.6, $\sigma_\varphi = \widetilde{C}(E_\varphi) \cap P$ and that, according to (P1), $E_\varphi = K(\sigma_\varphi)$. It results that $\sigma_\varphi = \widetilde{C}(K(\sigma_\varphi)) \cap P$. Since σ_φ is a Boolean subalgebra of P, its C-maximality follows.

(2) It follows directly from (1) and Theorem 3.11.

(3) It is a direct result of (2) and the fact that $K(\sigma_\varphi) = E_\varphi$, since φ is spectral. □

4. An application of spectral automorphisms to $\mathcal{E}(H)$

The notion of spectral automorphism was introduced with the declared intention to obtain an analogue of the Hilbert space spectral theory in the abstract setting of compression base effect algebras. It is time to see if this attempt was successful, by applying the abstract theory to the particular case of the standard Hilbert space effect algebra. Therefore, we devote this section to the proof of a "spectral theorem" in $\mathcal{E}(H)$, for a finite-dimensional Hilbert space H.

Before we can prove the main result of this section, some preparations are needed. Let us recall some well known results and prove a few other concerning orthomodular lattices in general and the set $\mathcal{P}(H)$ of projection operators of a Hilbert space H in particular.

Proposition 4.1. ([11: §4, Lemma 1]) *In an orthomodular lattice L, every subset of mutually commuting elements is included in a block of L.*

Proposition 4.2. ([10: Proposition 2.4]) *If B is an atomic Boolean subalgebra of an orthomodular lattice L, a is an atom of B and $\omega \in L$ with $\omega \leq a$, then $\omega \leftrightarrow B$.*

Proposition 4.3. ([11: §4, Lemma 2]) *If B is a block of an orthomodular lattice L, then the atoms of B are atoms of L.*

It is, of course, a well known fact that the set $\mathcal{P}(H)$ of projection operators (or equivalently, the set of closed subspaces) of a Hilbert space H is an atomic complete orthomodular lattice (see, e.g., [11: §5]). Its atoms are the 1-dimensional subspaces, or the corresponding projectors. Let us denote in the sequel by $\hat{e}$ the 1-dimensional subspace generated by $e \in H$, $\|e\| = 1$ and $P_{\hat{e}}$ the corresponding projection operator, i.e., $P_{\hat{e}} \colon H \to H$, $P_{\hat{e}}x = \langle x, e\rangle e$ (where $\langle\cdot,\cdot\rangle$ denotes the inner product of H).

Lemma 4.4. *Let H be a Hilbert space. For every automorphism $\varphi \colon \mathcal{E}(H) \to \mathcal{E}(H)$ defined by $\varphi(A) = UAU^{-1}$, where U is a unitary operator on H, and every atom $P_{\hat{e}} \in \mathcal{P}(H)$, we have $\varphi(P_{\hat{e}}) = P_{\widehat{Ue}}$.*

Proof. As previously mentioned, for $e \in H$, $\|e\| = 1$, $P_{\hat{e}}$ is defined by $P_{\hat{e}}\colon H \to H$, $P_{\hat{e}}x = \langle x, e\rangle e$. Since U is unitary, we have $\|Ue\| = 1$ and U^{-1} is also the adjoint of U. Then $\varphi(P_{\hat{e}})x = UP_{\hat{e}}U^{-1}x = U\langle U^{-1}x, e\rangle e = \langle x, Ue\rangle Ue = P_{\widehat{Ue}}x$ for every $x \in H$. □

Theorem 4.5. *Let H be an n-dimensional Hilbert space, $\mathcal{E}(H)$ be its standard effect algebra and $(J_P)_{P\in\mathcal{P}(H)}$ be the canonical compression base for $\mathcal{E}(H)$. Let $U\colon H \to H$ be a unitary operator and $\varphi\colon \mathcal{E}(H) \to \mathcal{E}(H)$ be an automorphism defined by $\varphi(A) = UAU^{-1}$. If φ is spectral, then:*

(1) *There is an orthogonal basis $\{e_1, e_2, \dots, e_n\}$ of H such that for every $i \in \{1, 2, \dots, n\}$, $Ue_i = \lambda_i e_i$ where λ_i is a scalar, $|\lambda_i| = 1$.*

(2) *There exists a partition Π of the set $\{1, 2, \dots, n\}$ such that any φ-invariant atom of $\mathcal{P}(H)$ is a 1-dimensional subspace in exactly one of the subspaces $\bigvee_{j\in J} \hat{e}_j$, $J \in \Pi$.*

(3) *If the subalgebra $\mathcal{E}(H)_\varphi$ of φ-invariant elements of $\mathcal{E}(H)$ is Boolean, then the spectrum $\sigma_\varphi = \mathcal{E}(H)_\varphi \cap \mathcal{P}(H)$ is a block in $\mathcal{P}(H)$. In this case all eigenvalues of U are distinct and $\Pi = \big\{\{1\}, \{2\}, \dots, \{n\}\big\}$.*

(4) *The spectrum σ_φ is a Boolean algebra generated by $\big\{\bigvee_{j\in J} \hat{e}_j : J \in \Pi\big\}$.*

(5) *If the effect $A \in \mathcal{E}(H)$ is φ-invariant and $P \in \mathcal{P}(H)$ is the smallest projection that dominates A (namely the projection on the range of A), then P is φ-invariant too. In view of (2), it follows that if A is a φ-invariant nonzero effect dominated by an atom of $\mathcal{P}(H)$, then the range of A is included in one of the subspaces $\bigvee_{j\in J} \hat{e}_j$, $J \in \Pi$.*

Proof.

(1) φ is a spectral automorphism, therefore $\mathcal{E}(H)_\varphi = K(\sigma_\varphi)$. Since σ_φ is a Boolean subalgebra of $\mathcal{P}(H)$, according to Proposition 4.1 there exists a block B_0 of $\mathcal{P}(H)$ such that $\sigma_\varphi \subseteq B_0$. Obviously $B_0 \subseteq K(\sigma_\varphi)$ and it follows that $B_0 \subseteq \mathcal{E}(H)_\varphi \cap \mathcal{P}(H)$. According to Proposition 4.3, the atoms of B_0 are atoms of $\mathcal{P}(H)$, i.e., 1-dimensional subspaces/projectors of H. Let $\mathcal{B}$ be the set of all atoms of B_0. Since B_0 is Boolean, it follows that its atoms are mutually orthogonal and therefore the corresponding 1-dimensional subspaces and the vectors that generate these subspaces are orthogonal. Since H is n-dimensional, it follows that there are at most n atoms in $\mathcal{B}$. However, since $\bigvee \mathcal{B} = \mathbf{1}$, it follows that there must be exactly n atoms in $\mathcal{B}$. Let $\mathcal{B} = \{\hat{e}_1, \hat{e}_2, \dots, \hat{e}_n\}$. In view of the previous arguments, it is clear that $\{e_1, e_2, \dots, e_n\}$ is an orthogonal basis of H. Recall now that $\mathcal{B} = \{\hat{e}_1, \hat{e}_2, \dots, \hat{e}_n\} \subseteq B_0 \subseteq \mathcal{E}(H)_\varphi$, hence $\hat{e}_1, \hat{e}_2, \dots, \hat{e}_n$ are φ-invariant. Then, according to Lemma 4.4, for every $i \in \{1, 2, \dots, n\}$,

$\varphi(P_{\hat{e}_i}) = P_{\widehat{Ue_i}} = P_{\hat{e}_i}$. It follows that $\widehat{Ue_i} = \hat{e}_i$, hence $Ue_i = \lambda_i e_i$ for some scalar λ_i (which must be of modulus 1 since U is unitary), for every $i \in \{1, 2, \ldots, n\}$.

(2) Let $\hat{e} \in \mathcal{P}(H)$ be a φ-invariant atom such that $\hat{e} \notin \mathcal{B}$. Since $\{e_1, e_2, \ldots, e_n\}$ is a basis of H, there exists $J \subseteq \{1, 2, \ldots, n\}$ such that $e = \sum_{j \in J} a_j e_j$, with $a_j \neq 0$ for all $j \in J$. Due to the φ-invariance of $\hat{e}$, it follows that there exists a scalar λ such that $Ue = \lambda e$ and since $Ue_j = \lambda_j e_j$ for every $j \in J$, we find that $\sum_{j \in J} a_j(\lambda - \lambda_j)e_j = 0$. We conclude that $\lambda_j = \lambda$ for all $j \in J$, and $\bigvee_{j \in J} \hat{e}_j$ is the corresponding eigenspace. It is now clear that each element J of the partition Π that we are looking for corresponds to a distinct eigenspace of U.

(3) If $\mathcal{E}(H)_\varphi$ is Boolean, according to Theorem 3.6, $\sigma_\varphi = \mathcal{E}(H)_\varphi \cap \mathcal{P}(H)$. Let $A \in \mathcal{E}(H)$ such that $A \leftrightarrow \mathcal{E}(H)_\varphi \cap \mathcal{P}(H)$. Then $A \leftrightarrow \sigma_\varphi$, hence $A \in \mathcal{E}(H)_\varphi$. It follows that $\sigma_\varphi = \mathcal{E}(H)_\varphi \cap \mathcal{P}(H)$ is a block of $\mathcal{P}(H)$, hence $\mathcal{B} = \{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n\}$ are the only φ-invariant atoms in $\mathcal{P}(H)$. This implies that eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ corresponding to vectors $\{e_1, e_2, \ldots, e_n\}$ are distinct. Indeed, if more than one of these vectors correspond to the same eigenvalue, than any subspace of their corresponding eigenspace is φ-invariant, in contradiction to our previous assertion.

(4) Since $\mathcal{B} = \{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n\}$ is the set of all the atoms of B_0 and $\sigma_\varphi \subseteq B_0$, it follows that every element, and in particular every atom of σ_φ is a supremum of a subset of $\mathcal{B}$. Considering the fact that σ_φ is a Boolean algebra and therefore its atoms are mutually orthogonal and their supremum is $\mathbf{1}$, we conclude that there exists a partition Π_1 of $\{1, 2, \ldots, n\}$ such that the atoms of σ_φ are $\big\{ \bigvee_{i \in I} \hat{e}_i : I \in \Pi_1 \big\}$. We have to prove that $\Pi_1 = \Pi$. Let $\omega \in H$, $\widehat{\omega} \leq \bigvee_{i \in I} \hat{e}_i$ for some $I \in \Pi_1$. According to Proposition 4.2, $\widehat{\omega} \leftrightarrow \sigma_\varphi$ and since φ is spectral, $\widehat{\omega}$ is φ-invariant. Therefore, all 1-dimensional subspaces dominated by $\bigvee_{i \in I} \hat{e}_i$ are φ-invariant and it follows that $\bigvee_{i \in I} \hat{e}_i$ is included in some eigenspace of U. Then there exists $J \in \Pi$ such that $I \subseteq J$. Since $\sum_{I \in \Pi_1} \operatorname{card}(I) = \sum_{J \in \Pi} \operatorname{card}(J) = n$, we only need to prove that there are no distinct $I_1, I_2 \in \Pi_1$ such that $I_1, I_2 \subseteq J$ for some $J \in \Pi$. Indeed, if that would be the case, we could choose $\omega_1, \omega_2 \in H$ such that $\widehat{\omega_1} \leq \bigvee_{i \in I_1} \hat{e}_i$ and $\widehat{\omega_2} \leq \bigvee_{i \in I_2} \hat{e}_i$. Then let $\omega = \omega_1 + \omega_2 \in H$ and we have $\widehat{\omega} \nleq \bigvee_{i \in I_1} \hat{e}_i$, $\widehat{\omega} \nleq \bigvee_{i \in I_2} \hat{e}_i$ but $\widehat{\omega} \leq \bigvee_{j \in J} \hat{e}_j$, which in turn implies $\widehat{\omega}$ is φ-invariant, hence it commutes with σ_φ and $\widehat{\omega} \leftrightarrow \bigvee_{i \in I} \hat{e}_i$ for every $I \in \Pi_1$. Since $\widehat{\omega}$ is an atom of $\mathcal{P}(H)$ that is neither included nor orthogonal to $\bigvee_{i \in I_1} \hat{e}_i$, $\bigvee_{i \in I_2} \hat{e}_i$, this is a contradiction.

(5) Let $A \in \mathcal{E}(H)$ be φ-invariant and $P \in \mathcal{P}(H)$ be the projection on the range of A, which is the smallest projection that dominates A. We have to prove that P is also φ-invariant. Since the automorphism φ is order-preserving, $\varphi(P)$ must be the smallest projection that dominates $\varphi(A)$, namely the projection on the range of $\varphi(A)$. Since $\varphi(A) = A$, it follows that $\varphi(P) = P$. □

Remark 4.6. The properties (1)–(5) from Theorem 4.5 were derived only from the fact that φ is spectral, without any other information except for the properties of unitary operators.

5. Spectral families of automorphisms

Let E denote, like in the rest of this paper, an effect algebra endowed with a compression base $(J_p)_{p\in P}$ and let Φ be a family of automorphisms of E.

Definition 5.1. The family Φ of automorphisms of E is called a *spectral family of automorphisms* if there exists a Boolean subalgebra B_Φ of P satisfying:

$$\big(\varphi(a) = a \quad \text{for all } \ \varphi \in \Phi\big) \iff a \leftrightarrow B_\Phi. \tag{P2}$$

In the sequel, we denote $E_\Phi = \{a \in E : \ \varphi(a) = a \ \text{ for all } \ \varphi \in \Phi\}$. Let us remark that $E_\Phi = \bigcap\limits_{\varphi\in\Phi} E_\varphi$ and therefore it's a sub-effect algebra of E.

Proposition 5.2. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and Φ be a spectral family of automorphisms of E. There exists the greatest Boolean subalgebra B_Φ of P satisfying* (P2)*.*

Proof. The proof relies heavily on Lemma 2.4 and it is completely similar to the proof of Proposition 2.5 (except instead of one automorphism we have a family of automorphisms), therefore we omit it. □

Definition 5.3. Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and Φ be a spectral family of automorphisms of E. The *spectrum* (denoted by σ_Φ) of the spectral family Φ of automorphisms is the greatest Boolean subalgebra B of P fulfilling (P2).

Lemma 5.4. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and Φ be a spectral family of automorphisms of E. Then $\widetilde{C}(E_\Phi) \cap P$ is a Boolean subalgebra of P.*

Proof. The proof repeats the proof of Lemma 3.2, with the only modification that φ should be replaced by Φ. □

Lemma 5.5. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and Φ be a spectral family of automorphisms of E. If a, a' are principal in E, $a, b \in E_\Phi$ and a, b coexist in E, then they coexist in E_Φ as well.*

Proof. The proof is analogous to the proof of Lemma 3.5. □

Theorem 5.6. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and Φ be a spectral family of automorphisms of E. Then $\sigma_\Phi = \widetilde{C}(E_\Phi) \cap P$.*

Proof. Property (P2) fulfilled by σ_Φ is equivalent to $E_\Phi = K(\sigma_\Phi)$. Let $a \in \sigma_\Phi$. Then $a \leftrightarrow \sigma_\Phi$ and therefore $a \in K(\sigma_\Phi) = E_\Phi$. Moreover, since $a \in \sigma_\Phi$, we obtain $a \leftrightarrow K(\sigma_\Phi) = E_\Phi$, i.e., a coexists with every element of E_Φ in E. According to Lemma 5.5 and since $a, a' \in P$ are principal elements of E, it follows that a coexists with every element of E_Φ in E_Φ as well, hence $a \in \widetilde{C}(E_\Phi) \cap P$, and we conclude that $\sigma_\Phi \subseteq \widetilde{C}(E_\Phi) \cap P$.

For the converse inclusion, since, according to Lemma 5.4, $\widetilde{C}(E_\Phi) \cap P$ is a Boolean subalgebra of P, it suffices to prove it fulfills (P2). If $\varphi(a) = a$, for all $\varphi \in \Phi$, it follows $a \in E_\Phi$, hence $a \leftrightarrow \widetilde{C}(E_\Phi) \cap P$. For the converse implication, since $\widetilde{C}(E_\Phi) \cap P \supseteq \sigma_\Phi$, $a \leftrightarrow \widetilde{C}(E_\Phi) \cap P$ implies $a \leftrightarrow \sigma_\Phi$ and therefore, since σ_Φ satisfies (P2), $\varphi(a) = a$ for all $\varphi \in \Phi$. □

Corollary 5.7. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and Φ be a family of automorphisms of E. Then Φ is a spectral family if and only if $K(\widetilde{C}(E_\Phi) \cap P) \subseteq E_\Phi$ (the converse inclusion is trivially satisfied).*

Proposition 5.8. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and Φ be a spectral family of automorphisms of E. Then:*

(1) *$\sigma_\Phi = \widetilde{C}(E_\Phi) \cap P$ is C-maximal;*
(2) *$\sigma_\Phi = K(K(\sigma_\Phi)) \cap P$;*
(3) *$\sigma_\Phi = K(E_\Phi) \cap P$.*

Proof.

(1) The result follows from Theorem 5.6 and the fact that $E_\Phi = K(\sigma_\Phi)$, since Φ is a spectral family of automorphisms.

(2) It is a direct consequence of (1) and Theorem 3.11.

(3) The result follows from (2) and the fact that $K(\sigma_\Phi) = E_\Phi$. □

Theorem 5.9. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and Φ be a family of spectral automorphisms of E. Then Φ is a spectral family of automorphisms if and only if the spectra of the automorphisms in the family are pairwise compatible, i.e., $\sigma_\varphi \leftrightarrow \sigma_\psi$ for every $\varphi, \psi \in \Phi$. In this case, σ_Φ includes all spectra of automorphisms in the family.*

Proof.

“ $\Longrightarrow$ ” Let us assume Φ is a spectral family of automorphisms of E, with the spectrum σ_Φ. We'll prove that $\sigma_\varphi \subseteq \sigma_\Phi$, for all $\varphi \in \Phi$. Since σ_Φ is a Boolean algebra, it will follow that $\sigma_\varphi \leftrightarrow \sigma_\psi$ for every $\varphi, \psi \in \Phi$. Let $\varphi \in \Phi$ and $b \in \sigma_\varphi$. For every $a \in K(\sigma_\Phi)$, (P2) implies $\varphi(a) = a$ for all $\varphi \in \Phi$, hence $a \leftrightarrow \sigma_\varphi$ and in particular $a \leftrightarrow b$. It follows that $b \in K(K(\sigma_\Phi))$, and since $b \in P$, we find that $b \in K(K(\sigma_\Phi)) \cap P$. However, according to Proposition 5.8, $K(K(\sigma_\Phi)) \cap P = \sigma_\Phi$, and therefore $\sigma_\varphi \subseteq \sigma_\Phi$.

“ $\Longleftarrow$ ” Conversely, let us assume $\sigma_\varphi \leftrightarrow \sigma_\psi$ for every $\varphi, \psi \in \Phi$. Then $\bigcup\limits_{\varphi \in \Phi} \sigma_\varphi$ is a set of pairwise compatible elements of P. According to Lemma 2.4, there exists the smallest Boolean subalgebra $B \subseteq P$ which includes $\bigcup\limits_{\varphi \in \Phi} \sigma_\varphi$. Moreover $a \leftrightarrow \bigcup\limits_{\varphi \in \Phi} \sigma_\varphi$ if and only if $a \leftrightarrow B$. It follows that $\varphi(a) = a$ for all $\varphi \in \Phi$ if and only if $a \leftrightarrow \sigma_\varphi$ for all $\varphi \in \Phi$ if and only if $a \leftrightarrow \bigcup\limits_{\varphi \in \Phi} \sigma_\varphi$ if and only if $a \leftrightarrow B$, which means that B is a Boolean subalgebra of P fulfilling (P2). We conclude that Φ is a spectral family of automorphisms. $\square$

Remark 5.10. The following useful facts hardly require verification:

(1) If $\varphi\colon E \to E$ is a spectral automorphism, then φ^{-1} is also a spectral automorphism, and $\sigma_\varphi = \sigma_{\varphi^{-1}}$.

(2) The identity $\mathrm{id}_E\colon E \to E$, $\mathrm{id}_E(a) = a$ for all $a \in E$ is a spectral automorphism and its spectrum is $\sigma_{\mathrm{id}_E} = \widetilde{C}(E) \cap P$.

Theorem 5.11 (A “replica” of Stone's Theorem on strongly continuous uniparametric groups of unitary operators). *Let E be an effect algebra, $(J_p)_{p \in P}$ be a compression base for E and Φ be a family of spectral automorphisms of E. If the following conditions are fulfilled:*

(i) Φ *is an abelian group;*

(ii) $\varphi(E_\psi) = E_{\varphi\psi}$ *for every $\varphi, \psi \in \Phi$ such that $\psi \notin \{\mathrm{id}_E, \varphi^{-1}\}$,*

then:

(1) $E_\varphi = E_\psi$ *for all $\varphi, \psi \in \Phi \setminus \{\mathrm{id}_E\}$;*

(2) $\sigma_\varphi = \sigma_\psi$ *for all $\varphi, \psi \in \Phi \setminus \{\mathrm{id}_E\}$;*

(3) Φ *is a spectral family.*

Proof.

(1) Let $\varphi, \psi \in \Phi$. Then $a \in E_\varphi$ if and only if $\varphi(a) = a$ if and only if $\psi(\varphi(a)) = \psi(a)$. Since Φ is an abelian group, this is equivalent to $\varphi(\psi(a)) = \psi(a)$ and to $\psi(a) \in E_\varphi$. It follows that $\psi(E_\varphi) = E_\varphi$.

Let $\varphi, \psi \in \Phi \setminus \{\mathrm{id}_E\}$ and $\varphi \neq \psi$. Clearly, if $\psi = \varphi^{-1}$, then $E_\varphi = E_\psi$. Let $\psi \neq \varphi^{-1}$ and define $\chi = (\varphi\psi)^{-1} = \psi^{-1}\varphi^{-1}$. It follows that $\chi \in \Phi \setminus \{\mathrm{id}_E, \varphi^{-1}, \psi^{-1}\}$, hence $E_\varphi = E_{\chi^{-1}\varphi\chi} = \chi^{-1}(\varphi(E_\chi)) = \chi^{-1}(\psi(E_\chi)) = E_{\chi^{-1}\psi\chi} = E_\psi$.

(2) The equality of spectra follows easily, since $\sigma_\varphi = \widetilde{C}(E_\varphi) \cap P$ for every $\varphi \in \Phi$.

(3) According to Theorem 5.9, Φ is a spectral family of automorphisms if and only if their spectra are pairwise compatible. Since all spectra except the spectrum of identity coincide, we only have to prove that $\sigma_{\mathrm{id}_E} \leftrightarrow \sigma_\varphi$ for some $\varphi \in \Phi \setminus \{\mathrm{id}_E\}$. We will even prove that $\sigma_{\mathrm{id}_E} \subseteq \sigma_\varphi$. Let $\varphi \in \Phi \setminus \{\mathrm{id}_E\}$. Obviously $K(E) \subseteq K(E_\varphi)$, hence, according to Corollary 3.12, $\sigma_{\mathrm{id}_E} = K(E) \cap P \subseteq K(E_\varphi) \cap P = \sigma_\varphi$. □

Let us remark that Theorem 5.11 generalizes [1: Theorem 6.8] to spectral automorphisms in CB-effect algebras. Its proof is similar to the one for the case of orthomodular lattice spectral automorphisms.

Remark 5.12. An abelian group $\{\varphi_t\}_t$ of automorphisms of the standard effect algebra $\mathcal{E}(H)$ of a Hilbert space H is generated, e.g., by a one-parameter abelian group $\{U_t\}_t$ of unitaries on H by taking $\varphi_t(A) = U_t A U_t^{-1}$.

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