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Mathematica Slovaca

DOI: 10.2478/s12175-012-0063-4
Math. Slovaca **62** (2012), No. 6, 1045–1062

A GENERALIZATION OF EFFECT ALGEBRAS AND ORTHOLATTICES

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Dedicated to Prof. David J. Foulis on the occasion of his 80th birthday

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. A common generalization of effect algebras and ortholattices that allows to represent ortholattices in a similar way in which orthomodular lattices are represented in the setting of effect algebras is introduced.

1. Introduction

In the middle of the nineties, two new classes of quantum structures have appeared: effect algebras and D-posets. Foulis and Bennett [3] defined *effect algebras* as algebraic systems $\langle A; +, 0, 1\rangle$ where A is a non-empty set, 0 and 1 two special elements of A (called the zero and the unit), and $+$ is a partial binary operation on A satisfying the following conditions for all $a, b, c \in A$:

(E1) If $a + b$ is defined, then $b + a$ is defined and $a + b = b + a$;

(E2) If $a + b$ and $(a + b) + c$ are defined, then $b + c$ and $a + (b + c)$ are defined, and $(a + b) + c = a + (b + c)$;

(E3) For every $a \in A$ there exists a unique $a' \in A$ such that $a' + a$ is defined and $a' + a = 1$ (a' is referred to as the *orthosupplement* of a);

(E4) If $a + 1$ is defined, then $a = 0$.

2010 Mathematics Subject Classification: Primary 03G12, 06C15, 08A55, 81P10.
Keywords: effect algebra, ortholattice, pre-effect algebra, generalized pre-effect algebra.
Supported by the Czech Government Research Project MSM6198959214 "Mathematical Models and Structures" and by the ESF Project CZ.1.07/2.3.00/20.0051 "Algebraic Methods in Quantum Logic".

D-posets (*difference posets*) were independently introduced by Kôpka and Chovanec [9] (also see [8]) as algebraic structures $\langle A; \leq, -, 1\rangle$ where $\langle A; \leq\rangle$ is a poset with greatest element 1 and $-$ is a partial binary operation on A (called a difference) such that $a - b$ is defined iff $a \geq b$, and the following conditions are satisfied, for all $a, b, c \in A$:

(D1) $a - b \leq a$ and $a - (a - b) = b$;

(D2) if $a \leq b \leq c$, then $c - b \leq c - a$ and $(c - a) - (c - b) = b - a$.

It turned out that the two concepts are equivalent, i.e., every effect algebra can be made a D-poset and vice versa, and both effect algebras and D-posets have received much attention. For the background of effect algebras and D-posets we refer to the comprehensive monograph [2].

It is well-known that orthomodular lattices can be identified with a certain subclass of effect algebras. However, if we consider ortholattices instead of orthomodular lattices, we see that it is impossible to characterize them in the setting of effect algebras because, roughly speaking, the induced lattice of an effect algebra is automatically orthomodular once it is an ortholattice. Therefore, our primary aim is to find a suitable common extension of effect algebras and ortholattices. In an attempt to cope with this problem we introduce *pre-effect algebras* that essentially differ from effect algebras in one respect: the orthosupplements are not necessarily uniquely determined, i.e., for every a there exists a' such that $a + a' = 1$, but we admit the existence of other elements b such that $a + b = 1$.

Various generalizations of effect algebras and D-posets can be found in the literature, so the question *what happens if we weaken the axiom* (E3) *by omitting uniqueness* itself can be seen as a motivation for the introduction of pre-effect algebras.

Let us recall some basic notions, see e.g. [5], [7], or the aforementioned book [2].

A *partial abelian monoid* is a structure $\langle A; +, 0\rangle$, where A is a non-empty set, $+$ is a partially defined binary operation on A and 0 is a distinguished element of A, satisfying the conditions (E1) and (E2) together with the condition that $a + 0$ is always defined and equals a. Thus the partial addition $+$ is both commutative and associative and 0 acts as an identity element. As usual, when we write $a + b = c$, we mean "$a + b$ is defined and equals c". A partial abelian monoid $\langle A; +, 0\rangle$ is

- *positive* if, for all $a, b \in A$, $a + b = 0$ implies $a = b = 0$;
- *cancellative* if, for all $a, b, c \in A$, $a + c = b + c$ implies $a = b$.

A *unital* partial abelian monoid is a structure $\langle A; +, 0, 1\rangle$ where $\langle A; +, 0\rangle$ is a partial abelian monoid and 1 is its *unit*, i.e., for every $a \in A$ there exists $b \in A$ such that $a + b = 1$. Using this terminology, effect algebras are exactly positive cancellative unital partial abelian monoids. In a sense, pre-effect algebras that we are going to define in Section 2 are a particular kind of positive unital partial abelian monoids.

Every positive cancellative[1] partial abelian monoid can be naturally ordered by putting

$$a \leq b \quad \text{iff} \quad b = a + c \quad \text{for some } c.$$

Clearly, 0 is the least element of the poset thus obtained. In an effect algebra, the unit 1 is the greatest element, and moreover, we have $b = a + c$ iff $a + b' = c'$, which means that $\leq$ can alternatively be specified by

$$a \leq b \quad \text{iff} \quad a + b' \quad \text{is defined.}$$

This partial order is the link between effect algebras and D-posets. Indeed, given an effect algebra $\langle A; +, 0, 1\rangle$, if we let $b - a$ be the only c such that $b = a + c$ provided $a \leq b$, then $\langle A; \leq, -, 1\rangle$ is a D-poset (observe that $b - a$ can be defined more explicitly by $b - a = (a + b')'$ if $a \leq b$). Conversely, in a D-poset $\langle A; \leq, -, 1\rangle$, if we define $a + b$ to be the only c such that $c \geq b$ and $c - b = a$, then $\langle A; +, 0, 1\rangle$ with $0 = 1 - 1$ becomes an effect algebra in which $a' = 1 - a$ (observe that $a + b = (b' - a)'$ if $a \leq b'$). This correspondence is one-to-one.

An effect algebra which is a lattice with respect to its natural order $\leq$ is said to be *lattice-ordered*; it is also called a *lattice effect algebra* (and the associated D-poset is a *D-lattice*).

In order to describe the connections between effect algebras and orthomodular lattices, we need one more notion: An *orthoalgebra* is an effect algebra satisfying the additional condition that $a + a$ is defined only if $a = 0$, which is equivalent to saying that for every a, the orthosupplement a' is a complement of a in the underlying poset. For completeness we recall that an *ortholattice* is an algebra $\langle L; \vee, \wedge, ', 0, 1\rangle$ such that $\langle L; \vee, \wedge, 0, 1\rangle$ is a bounded lattice and $'$ is an orthocomplementation on it (i.e., $x \mapsto x'$ is an antitone involution such that x' is a complement of x), and an *orthomodular lattice* is an ortholattice satisfying the orthomodular law $x \leq y \implies x \vee (x' \wedge y) = y$. Now, orthomodular lattices and lattice-ordered orthoalgebras are equivalent:

[1] In fact, a condition weaker than cancellativity is sufficient for $\leq$ to be a partial order; namely, it suffices to assume that $a + b = b$ implies $a = 0$ (cf. Proposition 2.3).

- Given $\langle L; \vee, \wedge, ', 0, 1\rangle$ an orthomodular lattice, let $+$ be the restriction of $\vee$ to the pairs $\langle a, b\rangle$ with $a \leq b'$. Then $\langle L; +, 0, 1\rangle$ is a lattice-ordered orthoalgebra. Thus we obtain an orthoalgebra if we define $a + b = a \vee b$ for $a \leq b'$.
- If $\langle A; +, 0, 1\rangle$ is a lattice-ordered orthoalgebra with the induced lattice operations $\vee$ and $\wedge$, then $\langle A; \vee, \wedge, ', 0, 1\rangle$ is an orthomodular lattice.

2. Pre-effect algebras

As we have already pointed out, we cannot describe ortholattices in the setting of lattice effect algebras because when $\langle A; +, 0, 1\rangle$ is a lattice effect algebra such that $\langle A; \vee, \wedge, ', 0, 1\rangle$ is an ortholattice, then $\langle A; +, 0, 1\rangle$ is an orthoalgebra and it follows that $\langle A; \vee, \wedge, ', 0, 1\rangle$ is an orthomodular lattice. Therefore, we modify the definition of effect algebras so that we could characterize ortholattices in a similar way in which orthomodular lattices are characterized within lattice effect algebras.

Definition 2.1. A *pre-effect algebra* is a structure $\langle A; +, ', 0, 1\rangle$ where $\langle A; +, 0\rangle$ is a partial abelian monoid, 1 is an element of A and $'$ is a unary operation such that $a' + a = 1$ for all $a \in A$, and the relation $\leq$ given by the rule

$$a \leq b \quad \text{iff} \quad a + b' \quad \text{is defined} \tag{1}$$

is a partial order. A pre-effect algebra satisfying the condition that $a = 0$ whenever $a + a$ is defined (i.e. $a = 0$ if $a \leq a'$) is called a *pre-orthoalgebra*.

Comparing unital partial abelian monoids and pre-effect algebras, it is evident that every pre-effect algebra is a unital partial abelian monoid (it need not be cancellative, but it is positive by Lemma 2.2(vii)). The difference is that in a pre-effect algebra, for each a we fix one of the elements x with the property $a + x = 1$, and this is done in such a way that (1) defines a partial order.

Pre-effect algebras generalize effect algebras. The axiom (E3) says that to each $a \in A$ there corresponds a *unique* $a' \in A$ such that $a + a' = 1$, hence we may extend the signature of effect algebras with a unary symbol $'$ to denote the operation of taking orthosupplements. In other words, every effect algebra $\langle A; +, 0, 1\rangle$ can be regarded as the structure $\langle A; +, ', 0, 1\rangle$ which apparently is a pre-effect algebra.

Lemma 2.2. *For any pre-effect algebra $\langle A; +, ', 0, 1\rangle$, the poset $\langle A; \leq\rangle$ is bounded, 0 and 1 being the least and the greatest element, and for all $a, b \in A$ we have:*

(i) $0' = 1$,

(ii) $a'' = a$,

(iii) $1' = 0$,

(iv) $a \leq b$ *iff* $b' \leq a'$,

(v) $a + 1$ *is defined iff* $a = 0$,

(vi) *if* $a + b = b$, *then* $a = 0$,

(vii) *if* $a + b = 0$, *then* $a = b = 0$.

P r o o f. First, 0 is the least element of $\langle A; \leq \rangle$ because $0 + a'$ is defined for every $a \in A$.

(i) Trivially, $0' = 0' + 0 = 1$.

(ii) Since $a'' + a' = 1$, we have $a'' \leq a$ for all $a \in A$. Replacing a respectively with a' and a'', we get $a''' \leq a'$ and $a'''' \leq a'' \leq a$, which means that $a'''' + a'$ is defined, and so $a' \leq a'''$. Thus $a' = a'''$ whence $1 = a' + a = a''' + a$, so $a \leq a''$.

(iii) Clearly, $1' = 0'' = 0$. Notice that this shows that 1 is the greatest element in $\langle A; \leq \rangle$ since $a \leq 1$ iff $a + 1' = a + 0$ is defined.

(iv) By definition, $b' \leq a'$ iff $b' + a''$ is defined iff $b' + a$ is defined iff $a \leq b$.

(v) The existence of $a + 1$ implies $a \leq 1' = 0$, i.e. $a = 0$.

(vi) If $a + b = b$, then $1 = b + b' = a + b + b' = a + 1$, whence $a = 0$.

(vii) If $a + b = 0$, then $a' = a' + a + b = 1 + b$, and so $b = 0$ and $a = 0$. $\square$

In addition to the partial order $\leq$, which is specified by $+$ and $'$, pre-effect algebras have another partial order that is induced solely by $+$ as follows:

Proposition 2.3. *Let $\langle A; +,', 0, 1 \rangle$ be a pre-effect algebra. The relation $\sqsubseteq$ defined by*

$$a \sqsubseteq b \quad \text{iff} \quad b = a + c \quad \text{for some } c \in A \tag{2}$$

is a partial order such that, for all $a, b \in A$, $a \sqsubseteq b$ implies $a \leq b$. The two orders coincide if and only if $\langle A; +,', 0, 1 \rangle$ is an effect algebra.

P r o o f. It is easy to see that $\sqsubseteq$ is a partial order: reflexivity and transitivity are trivial since $\langle A; +, 0 \rangle$ is a partial abelian monoid, and $\sqsubseteq$ is antisymmetric owing to Lemma 2.2(vi) and (vii), because if $b = a + x$ and $a = b + y$ for some $x, y \in A$, then $a = a + x + y$ yields $x + y = 0$, whence $x = y = 0$ and so $a = b$.

Moreover, if $a \sqsubseteq b$, then $b = a + c$ for some $c \in A$, whence $1 = b' + b = b' + a + c$. Thus $b' + a = a + b'$ is defined, showing $a \leq b$.

Let $\sqsubseteq$ and $\leq$ coincide. We have to show that a' is the only element with $a' + a = 1$. For, assume $x + a = 1$. Then $x \leq a'$, which implies $x \sqsubseteq a'$, so there exists $y \in A$ such that $a' = x + y$. Then $1 = a + x + y = 1 + y$ whence $y = 0$ proving $a' = x$. Therefore $\langle A; +,', 0, 1 \rangle$ is an effect algebra.

On the other hand, if $\langle A; +,', 0, 1\rangle$ is an effect algebra, then it follows from basic properties of effect algebras that we have already mentioned above that $a + b'$ is defined iff $b = a + c$ for some c (in fact, $c = (a + b')'$, see e.g. [2: Lemma 1.2.5(i)]), thus $\sqsubseteq$ and $\leq$ coincide. □

Lemma 2.4. *Let $\langle A; +,', 0, 1\rangle$ be a pre-effect algebra and $a, b, c, d \in A$. If $a \leq b$, $c \leq d$ and $b+d$ is defined, then $a+c$ is defined too, and $a+c \leq b+d$. The same holds true for $\sqsubseteq$.*

Proof. Since $1 = b + d + (b + d)'$, we have $a \leq b \leq (d + (b + d)')'$, thus $a + d + (b + d)'$ is defined and $a + d \leq b + d$. Similarly, $a + c \leq a + d$, and hence $a + c \leq b + d$.

Let now $a \sqsubseteq b$ and $c \sqsubseteq d$, i.e., $b = a + x$ and $d = c + y$ for some $x, y \in A$. Then $b + d = a + x + c + y = a + c + x + y$, so $a + c \sqsubseteq b + d$. □

We now turn our attention to ortholattices.

Lemma 2.5. *Let $\langle A; +,', 0, 1\rangle$ be a pre-effect algebra. Then $\langle A; +,', 0, 1\rangle$ is a pre-orthoalgebra if and only if for every $a \in A$, a' is a complement of a in $\langle A; \leq\rangle$. In this case, a' is a complement of a in $\langle A; \sqsubseteq\rangle$.*

Proof. Let $\langle A; +,', 0, 1\rangle$ be a pre-orthoalgebra. If $x \leq a$ and $x \leq a'$, then by Lemma 2.4, $x+x$ is defined, which is possible only if $x = 0$. Thus $\inf\{a, a'\} = 0$. Consequently, if $x \geq a$ and $x \geq a'$, then $x' \leq a'$ and $x' \leq a'' = a$ by Lemma 2.2(iv), so $x' = 0$ and $x = x'' = 1$, proving $\sup\{a, a'\} = 1$. Since the order $\leq$ exceeds $\sqsubseteq$ (i.e. $x \sqsubseteq y$ implies $x \leq y$), it follows that $\inf\{a, a'\} = 0$ and $\sup\{a, a'\} = 1$ in $\langle A; \sqsubseteq\rangle$.

Conversely, assume that for every $a \in A$, a' is a complement of a in $\langle A; \leq\rangle$. If $a + a$ is defined, then $a \leq a'$ and hence $a = \inf\{a, a'\} = 0$. Thus $\langle A; +,', 0, 1\rangle$ is a pre-orthoalgebra. □

Before the next definition, it is worth emphasizing that in view of Proposition 2.3, if $a + b$ is defined, then it is a common upper bound of a, b in $\langle A; \sqsubseteq\rangle$ as well as in $\langle A; \leq\rangle$. In orthoalgebras, where $\sqsubseteq$ and $\leq$ coincide, $a + b$ is even a minimal one, but this need not be the case for pre-orthoalgebras.

Definition 2.6. We say that a pre-effect algebra $\langle A; +,', 0, 1\rangle$ is a *strong pre-orthoalgebra* if for all $a, b \in A$ for which $a + b$ is defined, $a + b$ is a minimal common upper bound of a, b in the poset $\langle A; \leq\rangle$.

The definition is correct because when $a + a$ is defined, then the condition guarantees $a = a + a$, whence $a = 0$ by Lemma 2.2(vi), i.e., every strong pre-orthoalgebra is a pre-orthoalgebra.

The following theorem gives the promised characterization of ortholattices within pre-effect algebras: there is a one-to-one correspondence between ortholattices and those strong pre-orthoalgebras which are lattice-ordered under $\leq$. Recalling Lemma 2.5, the proof of the theorem is straightforward, hence we omit it.

Theorem 2.7. *Let $\langle A; +, ', 0, 1\rangle$ be a pre-orthoalgebra such that $\langle A; \leq\rangle$ is a lattice with the associated lattice operations $\vee$ and $\wedge$. Then $\langle A; \vee, \wedge, ', 0, 1\rangle$ is an ortholattice. Conversely, let $\langle L; \vee, \wedge, ', 0, 1\rangle$ be an ortholattice. If we define $+$ by stipulating that $a + b$ is defined iff $a \leq b'$ in which case $a + b = a \vee b$, then $\langle L; +, ', 0, 1\rangle$ is a strong pre-orthoalgebra.*

We cannot skip the adjective "strong" since two (or more) distinct pre-orthoalgebras can determine the same ortholattice.

Proposition 2.8. *Let $\langle L; \vee, \wedge, ', 0, 1\rangle$ be an ortholattice. Let L be equipped with $+$ as follows: $a + 0 = 0 + a = a$ for every $a \in A$, and if $a, b \in A \setminus \{0\}$, then $a + b = b + a$ is defined iff $a \leq b'$, in which case $a + b = b + a = 1$. Then $\langle L; +, ', 0, 1\rangle$ is a pre-orthoalgebra which induces the ortholattice $\langle L; \vee, \wedge, ', 0, 1\rangle$.*

Proof. It is obvious that the only thing we have to check is associativity of $+$. Trivially, if $0 \in \{a, b, c\}$, then $(a + b) + c$ is defined iff so is $a + (b + c)$, and $(a+b)+c = a+(b+c)$. Let $0 \notin \{a, b, c\}$. In this case, if $(a+b)+c$ were defined, then $1 = a + b \leq c'$ and we would get $c = 0$. Also, if $a + (b + c)$ were defined, then $a \leq (b + c)' = 1' = 0$, so $a = 0$. Thus neither $(a + b) + c$ nor $a + (b + c)$ is defined when $0 \notin \{a, b, c\}$. $\square$

Theorem 2.9. *Let $\langle A; +, ', 0, 1\rangle$ be a pre-orthoalgebra such that $\langle A; \sqsubseteq\rangle$ is a lattice with the associated lattice operations $\sqcup$ and $\sqcap$. If $\langle A; \sqcup, \sqcap, ', 0, 1\rangle$ is an ortholattice,[2] then it is an orthomodular lattice.*

Proof. Suppose by way of contradiction that the ortholattice $\langle A; \sqcup, \sqcap, ', 0, 1\rangle$ is not orthomodular, so it contains a subalgebra $\{0, a, b, a', b', 1\}$ where $a \sqsubseteq b$, $b' \sqsubseteq a'$, $x \sqcup y = 1$ and $x \sqcap y = 0$ for $x \in \{a, b\}$, $y \in \{a', b'\}$. Since $a \sqsubseteq b$, there exists $c \in A$ such that $b = a + c$, and hence $1 = b' + b = b' + a + c$. Thus $b' + a$ is defined and is equal to 1 because $b' + a$ is a common upper bound of a, b' and $a \sqcup b' = 1$. Hence $1 = 1 + c$, which is possible only if $c = 0$. Then $b = a + c = a$, a contradiction. $\square$

Remark. After submitting the paper, thanks to the referee's comments, we found out that in the literature, there already exist structures that generalize

[2]By Lemma 2.5, $\langle A; \sqcup, \sqcap, ', 0, 1\rangle$ is a lattice with complementation, but we don't know if the de Morgan laws hold, hence we assume that it is an ortholattice.

effect algebras and have most of the features that we wanted our pre-effect algebras to have. Namely, in [1], *quasi effect algebras* are defined as structures $\langle A; \leq, +, ', 0, 1\rangle$ where $\langle A; \leq, ', 0, 1\rangle$ is a bounded poset with an antitone involution (i.e., the map $x \mapsto x'$ is an antitone involution) and $+$ is a partial binary operation on A such that:

(i) $a + b = b + a$ when one side is defined,

(ii) $a + 0 = a$,

(iii) $a' + a = 1$,

(iv) if $a + 1$ is defined, then $a = 0$,

(v) if $a \leq b$ and $a + c, b + c$ are defined, then $a + c \leq b + c$.

It is obvious by Lemmata 2.2 and 2.4 that if $\langle A; +, ', 0, 1\rangle$ is a pre-effect algebra, then $\langle A; \leq, +, ', 0, 1\rangle$ where $\leq$ is given by (1) is a quasi effect algebra. On the other hand, if $\langle A; \leq, +, ', 0, 1\rangle$ is a quasi effect algebra, then the reduct $\langle A; +, ', 0, 1\rangle$ is a pre-effect algebra if and only if the conditions (E2) and (1) are satisfied.

Besides associativity/non-associativity, the difference between pre-effect algebras and quasi effect algebras is that in the latter case the partial order $\leq$ cannot be eliminated from the signature because it need not be specified by $+$ (or by $+$ and $'$) as it is in pre-effect algebras. This is demonstrated by the following simple example.

Let $\langle A; \vee, \wedge, ', 0, 1\rangle$ be the 4-element Boolean lattice with $A = \{0, a, b, 1\}$. If we make it a lattice effect algebra, we get

$+$	0	a	b	1
0	0	a	b	1
a	a	.	1	.
b	b	1	.	.
1	1	.	.	.

because $a' = b$ and $b' = a$, and the underlying lattice order $\leq$ obeys both (1) and (2). Let us equip A with the linear order $\preceq$ such that $0 \preceq a \preceq b \preceq 1$. Then $\langle A; \preceq, +, ', 0, 1\rangle$ is a quasi effect algebra in which $\preceq$ obeys neither (1) nor (2). In other words, $\langle A; \leq, +, ', 0, 1\rangle$ and $\langle A; \preceq, +, ', 0, 1\rangle$ are non-isomorphic quasi effect algebras, though the reduct $\langle A; +, ', 0, 1\rangle$ is the same.

In the rest of this section we present *pre-difference posets*, the "pre-version" of difference posets (D-posets), and prove that they are equivalent to pre-effect algebras. Observe that if $\langle A; \leq, -, 1\rangle$ is a D-poset and if we add the constant $0 = 1 - 1$ to the signature, then $\langle A; \leq, -, 0, 1\rangle$ is a pre-difference poset in the sense of the following definition.

Definition 2.10. By a *pre-difference poset* we mean a structure $\langle A; \leq, -, 0, 1\rangle$, where $\langle A; \leq\rangle$ is a poset with greatest element 1 and $-$ is a partial binary operation such that $a - b$ is defined iff $b \leq a$, satisfying the following conditions (for all $a, b, c \in A$):

(QD1) $a - a = 0$,

(QD2) $1 - (1 - a) = a$,

(QD3) if $a \geq b$ and $a - b \geq c$, then $a \geq c$ and $a - c \geq b$, and $(a-b)-c = (a-c)-b$.

Theorem 2.11. *For every pre-effect algebra $\langle A; +, ', 0, 1\rangle$, the structure $\langle A; \leq, -, 0, 1\rangle$, where $\leq$ is given by* (1) *and $a - b = (a' + b)'$ for $b \leq a$, is a pre-difference poset.*

Proof. We know that $\langle A; \leq\rangle$ is a poset whose bounds are 0 and 1. Clearly, $a - b = (a' + b)'$ is defined iff $a' + b$ is defined iff $b \leq a$. We also have $a - a = (a' + a)' = 1' = 0$ and $1 - a = (1' + a)' = (0 + a)' = a'$, and hence $1 - (1 - a) = a'' = a$ for all $a \in A$. There remains to verify (QD3). By definition, $a \geq b$ and $a - b = (a' + b)' \geq c$ iff $(a' + b) + c$ is defined iff $(a' + c) + b$ is defined iff $a \geq c$ and $a - c = (a' + c)' \geq b$. If this is the case, then $(a - b) - c = ((a' + b) + c)' = ((a' + c) + b)' = (a - c) - b$. ☐

For the reverse passage we need a technical lemma:

Lemma 2.12. *In any pre-difference poset $\langle A; \leq, -, 0, 1\rangle$, for all $a, b \in A$ we have:*

(a) *$0 \leq a$ and $a - 0 = a$;*

(b) *$a \geq b$ iff $1 - a \leq 1 - b$, in which case $a - b = (1 - b) - (1 - a)$.*

Proof.

(a) Using (QD3), since $a \geq a$ and $a - a \geq 0$, we have $a \geq 0$ and $a - 0 \geq a$. Then $a \geq 0$ together with $a - 0 \geq a - 0$ implies $a \geq a - 0$ again by (QD3). Thus $a - 0 = a$.

(b) We have $1 \geq 1 - a$ and $1 - (1 - a) = a \geq b$, hence $1 - b \geq 1 - a$ and $a - b = (1 - (1 - a)) - b = (1 - b) - (1 - a)$ by (QD3). ☐

Theorem 2.13. *Let $\langle A; \leq, -, 0, 1\rangle$ be a pre-difference poset. If we define $a' = 1 - a$ and $a + b = (a' - b)'$ for $a' \geq b$, then $\langle A; +, ', 0, 1\rangle$ is a pre-effect algebra. Moreover, the partial order defined by* (1) *coincides with $\leq$.*

Proof. We prove that $\langle A; +, 0\rangle$ is a partial abelian monoid. Obviously, $a + 0 = (a' - 0)' = a'' = a$ for every $a \in A$. We have $a' \geq b$ and $a' - b = (a' - b)'' = (a + b)' \geq c$ iff $a' \geq c$ and $a' - c = (a' - c)'' = (a + c)' \geq b$. So $(a + b) + c = (a' - b)' + b$ is defined iff $(a + c) + b = (a' - c)' + b$ is defined, and we have

$(a+b)+c = ((a'-b)-c)' = ((a'-c)-b)' = (a+c)+b$, which proves both commutativity and associativity of $+$.

Moreover, it is easily seen that $a'+a = (a''-a)' = 0' = 1$. Finally, the relation $\preceq$ defined by (1), i.e. $a \preceq b$ iff $a+b'$ exists, is a partial order because $\preceq$ is exactly the initial partial order $\leq$. □

3. Generalized pre-effect algebras

Positive cancellative partial abelian monoids are sometimes called *generalized effect algebras* (see e.g. [2]). We have already mentioned in Section 1 that the stipulation $a \leq b$ iff $b = a+c$ for some c defines a partial order, but there is no upper bound in general, and a generalized effect algebra which has greatest element is nothing but an effect algebra. Furthermore, like in effect algebras, a partial subtraction is implicitly determined by $+$; namely, if $a \leq b$, then $b-a$ is the only c such that $b = a+c$. Unfortunately, in pre-effect algebras or pre-difference posets, $+$ and $-$ are related via the unit 1 (see Theorems 2.11 and 2.13), and hence if we want to generalize pre-effect algebras by dropping units, we have to work with both $+$ and $-$.

Definition 3.1. A *generalized pre-effect algebra* is a structure $\langle A; +, -, 0\rangle$ where $+$ and $-$ are partial binary operations on A such that

(GQE1) $+$ is commutative, i.e., $a+b = b+a$ if one side is defined,

(GQE2) $a-a = 0$ for all $a \in A$,

(GQE3) the relation $\leq$ defined by $a \leq b$ iff $b-a$ exists is a partial order,

(GQE4) for all $a, b, c \in A$, $a \geq b$ and $a-b \geq c$ iff $b+c$ is defined and $a \geq b+c$, in which case $(a-b)-c = a-(b+c)$.

We first show that both generalized effect algebras and pre-effect algebras are special cases of generalized pre-effect algebras:

Proposition 3.2. *Let $\langle A; +, 0\rangle$ be a generalized effect algebra. For $a, b \in A$, let $a-b$ be defined iff there exists $c \in A$ such that $a = b+c$, and in this case $a-b = c$. Then $\langle A; +, -, 0\rangle$ is a generalized pre-effect algebra.*

Proof. The definition of $-$ is correct because $+$ is cancellative. Also, $a-b$ is defined iff $b \leq a$, where $\leq$ is the natural order of $\langle A; +, 0\rangle$, hence we only have to check the condition (GQE4). If $a \geq b$ and $a-b \geq c$, then $a = (a-b)+b = ((a-b)-c)+c+b$, so $b+c$ is defined and $a \geq b+c$. Conversely, if $a \geq b+c$, then $a = (a-(b+c))+b+c$ and so $a \geq b$ and $a-b = (a-(b+c))+c \geq c$. In this case we obviously have $a-(b+c) = (a-b)-c$. □

Proposition 3.3. *Let $\langle A; +, ', 0, 1\rangle$ be a pre-effect algebra and let $-$ be defined as in Theorem 2.11, i.e., $a - b = (a' + b)'$ when $b \le a$. Then $\langle A; +, -, 0\rangle$ is a generalized pre-effect algebra.*

Proof. (GQE1), (GQE2) and (GQE3) are obviously satisfied. Further, $a \ge b$ and $(a' + b)' = a - b \ge c$ iff $(a' + b) + c$ is defined iff $a' + (b + c)$ is defined iff $b + c$ is defined and $a \ge b + c$, and then we have $a - (b + c) = (a' + (b + c))' = ((a' + b) + c)' = ((a' + b)'' + c)' = (a - b) - c$. Thus the condition (GQE4) is fulfilled too. $\square$

We now prove some properties of generalized pre-effect algebras. Notice that by (v) and (viii), generalized pre-effect algebras are positive abelian monoids.

Lemma 3.4. *Let $\langle A; +, -, 0\rangle$ be a generalized pre-effect algebra. Then for all $a, b, c \in A$:*

(i) *$a \ge b$ and $a - b \ge c$ iff $a \ge c$ and $a - c \ge b$, and in this case $(a - b) - c = (a - c) - b$;*

(ii) *$0 \le a$ and $a - 0 = a$;*

(iii) *$a + 0 = a$;*

(iv) *if $a + b$ is defined, then $a + b \ge b$ and $(a + b) - b \ge a$;*

(v) *if $a + b = 0$, then $a = b = 0$;*

(vi) *if $a + b = b$, then $a = 0$;*

(vii) *if $a \ge b$, then $a \ge a - b$, $a - (a - b) \ge b$ and $a \ge (a - b) + b$;*

(viii) *$+$ is associative (in the sense of the axiom (E2));*

(ix) *if $a \ge b$ and $a + c$ is defined, then $b + c$ is defined, $a + c \ge b + c$ and $(a + c) - (b + c) \ge a - b$;*

(x) *if $a \ge b \ge c$, then $a - c \ge (a - b) + (b - c)$.*

Proof.

(i) This follows from (GQE4) and commutativity of $+$.

(ii) Using the item (i), $a \ge a$ and $a - a \ge 0$ implies $a \ge 0$ and $a - 0 \ge a$. Analogously, $a \ge 0$ and $a - 0 \ge a - 0$ implies $a \ge a - 0$, hence $a - 0 = a$.

(iii) By (GQE4), from $a \ge a$ and $a - a \ge 0$ we get that $a + 0$ is defined and $a \ge a + 0$. Then $a + 0 \ge a + 0$ implies $a + 0 \ge a$ again by (GQE4), thus $a + 0 = a$.

(iv) If $a + b$ is defined, then $a + b \ge a + b = b + a$ yields $a + b \ge b$ and $(a + b) - b \ge a$ by (GQE4).

(v) Since $a + b$ is a common upper bound of a, b whenever $a + b$ is defined, $a + b = 0$ implies $a = b = 0$.

(vi) If $a + b = b$, then $0 = (a + b) - b \ge a$ by (iv), so $a = 0$.

(vii) If $a \geq b$, then $a - b \geq a - b$ implies $a \geq a - b$, $a - (a - b) \geq b$ and $a \geq (a - b) + b$ by (GQE4) and (i).

(viii) Let $(a + b) + c$ be defined. Then by (iv) we have $(a + b) + c \geq c$ and $((a + b) + c) - c \geq a + b = b + a$, and so $((a + b) + c) - c \geq b$ and $(((a + b) + c) - c) - b \geq a$ by (GQE4). On the other hand, again by (GQE4), $((a+b)+c)-c \geq b$ entails the existence of $c+b$ and $(a+b)+c \geq c+b$, whence $((a+b)+c)-(c+b) = (((a+b)+c)-c)-b \geq a$. But this means that $(c+b)+a$ is defined and $(a+b)+c \geq (c+b)+a = a+(b+c)$. Thus $(a+b)+c \geq a+(b+c)$ whenever the left side is defined. Further, by what we have just shown, and since $+$ is commutative, we have $a + (b + c) = (c + b) + a \geq c + (b + a) = (a + b) + c$.

(ix) If $a + c$ is defined and $a \geq b$, then $(a + c) - c \geq a \geq b$, and so $b + c$ is defined and we have $a + c \geq b + c$. Therefore, since $a \geq (a - b) + b$, we have $a + c \geq (a - b) + b + c$, whence $(a + c) - (b + c) \geq a - b$.

(x) Let $a \geq b \geq c$. Since $a \geq (a - b) + b$ and $b \geq (b - c) + c$, the sum $(a - b) + (b - c) + c$ is defined and $a \geq (a - b) + (b - c) + c$, which entails $a - c \geq (a - b) + (b - c)$. $\square$

Now, like in pre-effect algebras, it can easily be shown that the relation $\sqsubseteq$ defined by (2), i.e., $a \sqsubseteq b$ iff $b = a + c$ for some c, is a partial order such that $\leq$ exceeds $\sqsubseteq$ (i.e., if $a \sqsubseteq b$, then also $a \leq b$).

Proposition 3.5. *Let $\langle A; +, -, 0\rangle$ be a generalized pre-effect algebra. Then $\leq$ and $\sqsubseteq$ coincide if and only if $\langle A; +, 0\rangle$ is a generalized effect algebra.* (*Cf.* Proposition 2.3.)

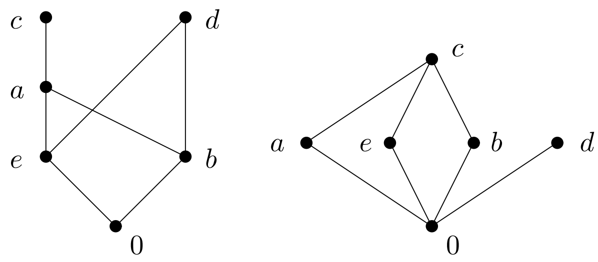
Proof. We have to show that $+$ is cancellative if $\leq$ and $\sqsubseteq$ coincide. To this end, let $d = a + b = a + c$. Then $d - a \geq b$, hence $d - a \sqsupseteq b$ and so $d - a = b + x$ for some $x \in A$. It follows that $d \geq (d - a) + a = b + x + a = d + x \geq d$ by (vii) and (iv) of Lemma 3.4, thus $d = d + x$ which yields $x = 0$ by Lemma 3.4(vi), and so $d - a = b$. Similarly, we get $d - a = c$. $\square$

Example 3.6. Let $A = \{0, a, b, c, d, e\}$ be equipped with the partial operations $+, -$ as follows:

$+$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	c	c	.	.	c
b	b	c	c	.	.	c
c	c	.	.	.	.	.
d	d	.	.	.	.	.
e	e	c	c	.	.	c

$-$	0	a	b	c	d	e
0	0	.	.	.	.	.
a	a	0	0	.	.	0
b	b	.	0	.	.	.
c	c	a	a	0	.	a
d	d	.	0	.	0	0
e	e	.	.	.	.	0

Then $\langle A; +, -, 0\rangle$ is a generalized pre-effect algebra; the Hasse diagrams of $\langle A; \leq\rangle$ and $\langle A; \sqsubseteq\rangle$ are:



It is known that every generalized effect algebra can be embedded into an effect algebra (see [6], [2]). This construction is called *unitization* and we now show that it works for our generalized pre-effect algebras too.

Let $\langle A; +_A, -_A, 0_A\rangle$ be a generalized pre-effect algebra (the underlying order given by (GQE3) is denoted by $\leq_A$). Let $A^* = \{a^* : a \in A\}$ be a disjoint copy of A. We can make $A \cup A^*$ into a pre-effect algebra as follows:

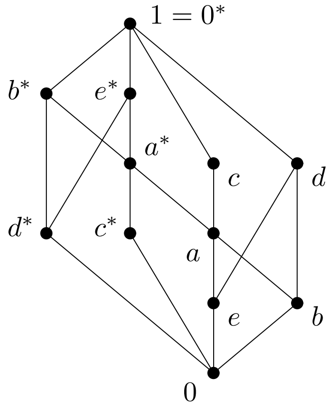
- $a + b$ is defined iff $a +_A b$ is defined, and $a + b = a +_A b$;
- $a + b^*$ is defined iff $b^* + a$ is defined iff $a \leq_A b$, and in this case $a + b^* = b^* + a = (b -_A a)^*$;
- $a^* + b^*$ is not defined;
- $a' = a^*$ and $(a^*)' = a$;
- $0 = 0_A$ and $1 = 0_A^*$.

Thus for the order $\leq$ on $A \cup A^*$ we have:

(i) $a \leq b$ iff $a \leq_A b$ iff $b^* \leq a^*$,
(ii) $a \leq b^*$ iff $b \leq a^*$ iff $a +_A b$ is defined, and
(iii) $a^* \not\leq b$ for all $a, b \in A$.

Theorem 3.7. *For every generalized pre-effect algebra $\langle A; +_A, -_A, 0_A\rangle$, the structure $\langle A \cup A^*; +, ', 0, 1\rangle$ is a pre-effect algebra.*

The proof is straightforward. The unitization of the generalized pre-effect algebra from Example 3.6 is shown in the following figure:



4. Principal and central elements

In this section, we describe two-factor direct product decompositions of pre-effect algebras. It turns out that they are determined by the so-called central elements which are defined just as in effect algebras (see [2], [4]). We also prove that the central elements form a Boolean subalgebra.

Definition 4.1. Let $\langle A; +, ', 0, 1\rangle$ be a pre-effect algebra. We call an element $a \in A$ *principal* if the interval $[0, a]$ is closed under $+$, i.e., for all $x, y \in A$ such that $x, y \leq a$ and $x + y$ is defined we have $x + y \leq a$. Further, we say that $a \in A$ is a *central element* if

(i) both a and a' are principal elements,

(ii) for every $x \in A$ there exist $y, z \in A$ such that $y \leq a$, $z \leq a'$ and $x = y + z$.

In what follows, $\langle A; +, ', 0, 1\rangle$ is a fixed but arbitrary pre-effect algebra.

Lemma 4.2. *If $a \in A$ is a principal element, then $a \wedge (x - a) = 0$ for every $x \in A$ with $x \geq a$. In particular, $a \wedge a' = 0$.*

Proof. If $y \leq a$ and $y \leq x - a$, then $y + a$ is defined and $y + a \leq x$. Since a is principal and $y \leq a$, we have $y + a \leq a$, which yields $y + a = a$, and hence $y = 0$. □

Lemma 4.3. *Let $a \in A$ be central. If $x = y + z$ where $y \leq a$ and $z \leq a'$, then $y = x \wedge a$ and $z = x \wedge a'$.*

Proof. Assume that $u \leq x$ and $u \leq a$. Then $u \leq y$ iff $u + y'$ is defined, so we aim at showing that $u + y'$ exists. We can write $y' = p + q$ for some $p \leq a$ and $q \leq a'$. Then $p \leq y'$ and so $p + y$ is defined. Also, since $p, y \leq a$, we have $p+y \leq a \leq z'$ which entails the existence of $p+y+z = p+x$. Since $u \leq x$, $p+u$ is defined too. Moreover, $p+u \leq a \leq q'$ because $p, u \leq a$. Thus $p+u+q = y'+u$ is defined, proving $u \leq y$. Hence $y = x \wedge a$.

In an analogous way we can show that $z = x \wedge a'$. □

Lemma 4.4. *Let $a \in A$ be a central element. If $x \geq a$, then $x = a + (x - a)$ and $x - a = x \wedge a'$. If $x \leq a$, then $a = (a - x) + x$ and $a - (a - x) = x$.*

Proof. Let $x \geq a$. By Lemma 4.3 we know that $x = (x \wedge a) + (x \wedge a') = a + (x \wedge a')$, whence $x - a \geq x \wedge a'$ and so $x \geq a + (x - a) \geq a + (x \wedge a') = x$. Thus $x = a + (x - a)$. Since $x - a = a' - x' \leq a'$, we conclude $x - a = x \wedge a'$ by Lemma 4.3.

Now, let $x \leq a$. Since a' is central and $x' \geq a'$, by the first part of the lemma we have $x' = a' + (x' - a') = a' + (a - x)$. Then $1 = x + x' = x + a' + (a - x)$ where $x + (a - x) \leq a$, and it follows that $a = x + (a - x)$. Furthermore, $x' = a' + (a - x)$ implies $x = 1 - x' = 1 - (a' + (a - x)) = (1 - a') - (a - x) = a - (a - x)$. □

Lemma 4.5. *Let $a \in A$ be central. For all $x, y \leq a$, $x \leq y$ iff $x + (a - y)$ is defined.*

Proof. If $x \leq y$, then $x + y'$ is defined. But $1 \geq a \geq y$ implies $y' = 1 - y \geq a - y$, hence $x + (a - y)$ is defined too. Conversely, assume that $x + (a - y)$ exists. We have $x, a - y \leq a$ and hence $x + (a - y) \leq a$, which yields $y = a - (a - y) \geq x$. □

Proposition 4.6. *Let $a \in A$ be a central element. The structure $\langle [0, a]; +, ^\flat, 0, a \rangle$, where $x^\flat = a - x$, is a pre-effect algebra.*

Proof. $\langle [0, a]; +, 0 \rangle$ is evidently a partial abelian monoid. For all $x \in [0, a]$ we have $x^\flat + x = (a - x) + x = a$. Likewise, the relation $\preceq$ defined by $x \preceq y$ iff $x + y^\flat = x + (a - y)$ is defined in $[0, a]$, is a partial order on $[0, a]$; in fact, $\preceq$ is just the restriction to $[0, a]$ of $\leq$. □

Lemma 4.7. *Let $a \in A$ be central. Assume that $x = y_1 + z_1$ and $x' = y_2 + z_2$ where $y_1, y_2 \leq a$ and $z_1, z_2 \leq a'$. Then $y_1 = a - y_2$ and $y_2 = a - y_1$.*

Proof. We have $1 = x + x' = y_1 + z_1 + y_2 + z_2 = y_1 + y_2 + z_1 + z_2$ where $y_1 + y_2 \leq a$ and $z_1 + z_2 \leq a'$, hence $y_1 + y_2 = a$ and $z_1 + z_2 = a'$ by Lemma 4.3. Then $a - y_2 \geq y_1$ and $a - y_1 \geq y_2$, so it remains to show that $a - y_2 \leq y_1$ and $a - y_1 \leq y_2$.

Since $(a - y_2) + y_2 = a \leq z_2'$ by Lemma 4.4, we have $a + z_2 = (a - y_2) + y_2 + z_2 = (a - y_2) + x'$, and hence $a - y_2 \leq x$. Analogously, $a + z_1 = (a - y_1) + y_1 + z_1 =$

$(a-y_1)+x$, so $x \le (a-y_1)'$. Then $a-y_2 \le x \le (a-y_1)'$ and hence $(a-y_2)$ $+(a-y_1)$ is defined. Moreover, $(a-y_2)+(a-y_1) \le a$ and so $(a-y_2)+(a-y_1)+a'$ is defined. But $y_1' = a' + (y_1' - a') = a' + (a-y_1)$ by Lemma 4.4, thus $(a-y_2)+y_1'$ is defined and we conclude $a - y_2 \le y_1$. Analogously, $y_2' = a' + (y_2' - a') = a' + (a - y_2)$, so $y_2' + (a-y_1)$ is defined, showing $a - y_1 \le y_2$. □

Theorem 4.8. *Let $a \in A$ be a central element. The map $\psi\colon x \mapsto \langle x \wedge a, x \wedge a'\rangle$ is an isomorphism of $\langle A; +,', 0, 1\rangle$ onto the direct product of $\langle [0,a]; +,^\flat, 0, a\rangle$ and $\langle [0,a']; +,^\natural, 0, a'\rangle$, where $x^\flat = a - x$ and $x^\natural = a' - x$.*

Proof. In view of Lemma 4.3, the map ψ is a well-defined bijection because, for each $x \in A$, $x \wedge a$ and $x \wedge a'$ exist, and $x = (x \wedge a) + (x \wedge a')$. Clearly, $\psi(0) = \langle 0, 0\rangle$ and $\psi(1) = \langle a, a'\rangle$.

Let $x, y \in A$. If $x + y$ is defined in A, then $x + y = (x \wedge a) + (x \wedge a') + (y \wedge a)$ $+ (y \wedge a') = (x \wedge a) + (y \wedge a) + (x \wedge a') + (y \wedge a')$. Since $(x \wedge a) + (y \wedge a) \le a$ and $(x \wedge a') + (y \wedge a') \le a'$, it follows that $(x + y) \wedge a = (x \wedge a) + (y \wedge a)$ and $(x \wedge y) \wedge a' = (x \wedge a') + (y \wedge a')$ by Lemma 4.3. Thus $\psi(x) + \psi(y) = \langle x \wedge a, x \wedge a'\rangle + \langle y \wedge a, y \wedge a'\rangle = \langle (x \wedge a) + (y \wedge a), (x \wedge a') + (y \wedge a')\rangle$ is defined in $[0,a] \times [0,a']$. Conversely, let $\psi(x) + \psi(y)$ be defined. Then $(x \wedge a) + (y \wedge a)$ and $(x \wedge a') + (y \wedge a')$ are defined in $[0,a]$ and in $[0,a']$, respectively. But then also $(x \wedge a) + (y \wedge a) + (x \wedge a') + (y \wedge a') = (x \wedge a) + (x \wedge a') + (y \wedge a) + (y \wedge a') = x + y$ is defined in A. In either case, we have $\psi(x+y) = \langle (x+y) \wedge a, (x+y) \wedge a'\rangle = \psi(x) + \psi(y)$.

It remains to show that $\psi(x') = \psi(x)'$, i.e. $\langle x' \wedge a, x' \wedge a'\rangle = \langle (x \wedge a)^\flat, (x \wedge a')^\natural\rangle$, for all $x \in A$. We have $x' = (x' \wedge a) + (x' \wedge a')$ and $x = (x \wedge a) + (x \wedge a')$, hence by Lemma 4.7, $x' \wedge a = a - (x \wedge a) = (x \wedge a)^\flat$ and $x' \wedge a' = a' - (x \wedge a') = (x \wedge a')^\natural$. □

Thus there is a one-to-one correspondence between direct product decompositions and central elements.

Corollary 4.9. *Let $a \in A$ be central. Then $a \vee x$ and $a \wedge x$ exist for all $x \in A$. Moreover, a is a distributive element in $\langle A; \le\rangle$, i.e. $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$ for all $x, y \in A$ for which $x \wedge y$ exists.*

Proof. We can represent $a, x, y \in A$ respectively as $\langle a, 0\rangle, \langle x_1, x_2\rangle, \langle y_1, y_2\rangle \in [0,a] \times [0,a']$. Then $\langle a, x_2\rangle = \langle a, 0\rangle \vee \langle x_1, x_2\rangle$ and $\langle x_1, 0\rangle = \langle a, 0\rangle \wedge \langle x_1, x_2\rangle$. If $\langle x_1, x_2\rangle \wedge \langle y_1, y_2\rangle$ exists, then it is equal to $\langle x_1 \wedge y_1, x_2 \wedge y_2\rangle$ and $\langle a, 0\rangle \vee (\langle x_1, x_2\rangle \wedge \langle y_1, y_2\rangle) = \langle a, 0\rangle \vee \langle x_1 \wedge y_1, x_2 \wedge y_2\rangle = \langle a, x_2 \wedge y_2\rangle = \langle a, x_2\rangle \wedge \langle a, y_2\rangle = (\langle a, 0\rangle \vee \langle x_1, x_2\rangle) \wedge (\langle a, 0\rangle \vee \langle y_1, y_2\rangle)$. □

Lemma 4.10. *Let $a_1, \dots, a_n$ be principal elements of A such that $a_1 + \dots + a_n$ exists. Assume that for every $x \in A$ there exist $x_1, \dots, x_n \in A$ such that $x_i \le a_i$ (for $i = 1, \dots, n$) and $x = x_1 + \dots + x_n$. Then every element of the form $a_{i_1} + \dots + a_{i_k}$, where the indices $i_1, \dots, i_k \in \{1, \dots, n\}$ are mutually distinct, is central.*

Proof. Due to commutativity, it is sufficient to prove that $a = a_1 + \dots + a_k$ with $k \le n$ is a central element. We first notice that any $x \le a$ can be written as $x = x_1 + \dots + x_k$ for some $x_i \le a_i$ $(i = 1, \dots, k)$. Indeed, by our hypothesis, $x = x_1 + \dots + x_n$ where $x_i \le a_i$ for $i = 1, \dots, n$. Since $x \le a = a_1 + \dots + a_k$, it follows that $x + a_{k+1} + \dots + a_n$ is defined and equals $x_1 + \dots + x_k + x_{k+1} + a_{k+1} + \dots + x_n + a_n$. But $x_j + a_j \le a_j$ for $j = k+1, \dots, n$, since a_j's are principal, and this is possible only if $x_j = 0$. Hence $x = x_1 + \dots + x_k$ as claimed.

Now, we show that a is principal. Let $x, y \le a$ and let $x + y$ be defined. We can write $x = x_1 + \dots + x_k$ and $y = y_1 + \dots + y_k$ where $x_i, y_i \le a_i$ $(i = 1, \dots, k)$. Thus $x + y = x_1 + y_1 + \dots + x_k + y_k \le a_1 + \dots + a_k = a$.

Next, we show that $a' = a_{k+1} + \dots + a_n$. We have $a' = z_1 + \dots + z_n$ for some $z_i \le a_i$ $(i = 1, \dots, n)$. Then $1 = a + a' = a_1 + z_1 + \dots + a_k + z_k + z_{k+1} + \dots + z_n$ and the same argument as before yields $z_1 = \dots = z_k = 0$, so $a' = z_{k+1} + \dots + z_n$. On the other hand, it is clear that $1 = a_1 + \dots + a_n = a + a_{k+1} + \dots + a_n$, and hence $a' = 1 - a \ge a_{k+1} + \dots + a_n \ge z_{k+1} + \dots + z_n = a'$. Thus $a' = a_{k+1} + \dots + a_n$. This also proves that a' is principal.

Now, let $x \in A$ be arbitrary. There exist $x_i \le a_i$ $(i = 1, \dots, n)$ such that $x = x_1 + \dots + x_n$. If we put $y = x_1 + \dots + x_k$ and $z = x_{k+1} + \dots + x_n$, then obviously $y \le a$, $z \le a'$ and $x = y + z$, which proves that a is a central element. □

By the *center* of a pre-effect algebra we mean the set of its central elements.

Theorem 4.11. *The center of a pre-effect algebra is a subalgebra which is a Boolean algebra in its own right.*

Proof. Let $\langle A; +, ', 0, 1\rangle$ be a pre-effect algebra and B its center. B is a subalgebra iff $0, 1 \in B$, $a' \in B$ for each $a \in B$, and $a + b \in B$ whenever $a, b \in B$ and $a + b$ is defined. We only have to check the last property.

Let $a, b \in B$ and put $c_1 = a \wedge b$, $c_2 = a \wedge b'$, $c_3 = a' \wedge b$ and $c_4 = a' \wedge b'$. The elements c_i exist by Corollary 4.9 and they are principal (c_i's are meets of principal elements). Moreover, we have $c_1 + c_2 = a$, $c_1 + c_3 = b$, $c_3 + c_4 = a'$ and $c_2 + c_4 = b'$. We show that the c_i's meet the conditions of Lemma 4.10. To this end, let $x \in A$ be arbitrary. Since $a \in B$, we can write $x = y + z$ for some $y \le a$ and $z \le a'$. At the same time, since $b \in B$, there exist $x_1, x_3 \le b$ and

$x_2, x_4 \leq b'$ such that $y = x_1 + x_2$ and $z = x_3 + x_4$. Then $x = x_1 + x_2 + x_3 + x_4$ where $x_1, x_2 \leq y \leq a$ and $x_3, x_4 \leq z \leq a'$, thus $x_i \leq c_i$ for $i = 1, 2, 3, 4$.

Now, if $a + b$ is defined, then $a \leq b'$ and hence $c_2 = a$ and $c_3 = b$. Thus $a + b = c_2 + c_3$ which is a central element by Lemma 4.10. This proves that B is a subalgebra. In view of Corollary 4.9 and Lemma 4.2, $\langle B; \leq \rangle$ is a Boolean lattice. $\square$

Acknowledgement. The authors would like to thank the referee for her/his valuable comments on the manuscript.

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Received 30. 3. 2011
Accepted 20. 10. 2011

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