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ASYMPTOTIC PROPERTIES FOR THE LOGLOG LAWS UNDER POSITIVE ASSOCIATION

Xiao-Rong Yang — Ke-Ang Fu

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ABSTRACT. Let $\{X_n : n \geq 1\}$ be a strictly stationary sequence of positively associated random variables with mean zero and finite variance. Set $S_n = \sum_{k=1}^{n} X_k$, $M_n = \max_{k\leq n} |S_k|$, $n \geq 1$. Suppose that $0 < \sigma^2 = EX_1^2 + 2\sum_{k=2}^{\infty} EX_1X_k < \infty$. In this paper, we prove that if $E|X_1|^{2+\delta} < \infty$ for some $\delta \in (0,1]$, and $\sum_{j=n+1}^{\infty} \operatorname{Cov}(X_1, X_j) = O(n^{-\alpha})$ for some $\alpha > 1$, then for any $b > -1/2$

$$\lim_{\varepsilon\searrow 0} \varepsilon^{2b+1} \sum_{n=1}^{\infty} \frac{(\log\log n)^{b-1/2}}{n^{3/2}\log n} E\Big\{M_n - \sigma\varepsilon\sqrt{2n\log\log n}\Big\}_+ = \frac{2^{-1/2-b}E|N|^{2(b+1)}}{(b+1)(2b+1)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2(b+1)}}$$

and

$$\lim_{\varepsilon\nearrow\infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^{b}}{n^{3/2}\log n} E\left\{\sigma\varepsilon\sqrt{\frac{\pi^2 n}{8\log\log n}} - M_n\right\}_+ = \frac{\Gamma(b+1/2)}{\sqrt{2}(b+1)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+2}},$$

where $x_+ = \max\{x, 0\}$, N is a standard normal random variable, and $\Gamma(\cdot)$ is a Gamma function.

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1. Introduction and main results

Let $\{X, X_n : n \geq 1\}$ be a sequence of random variables with common distribution, $EX_1 = 0$ and $0 < EX_1^2 < \infty$. Set $S_n = \sum_{k=1}^{n} X_k$, $M_n = \max_{k \leq n} |S_k|$, $n \geq 1$. Denote $\log t = \ln(t \vee \mathrm{e})$, $t \geq 0$, and $x_+ = \max\{x, 0\}$. When $\{X_n : n \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables, Chow [5] first discussed the complete moment convergence, and got the following result.

Theorem A. *Suppose that $EX = 0$. Assume $p \geq 1$, $\alpha > 1/2$, $p\alpha > 1$ and $E(|X|^p + |X| \log(1 + |X|)) < \infty$. Then for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{p\alpha - 2 - \alpha} E\Big\{\max_{j \leq n} |S_j| - \varepsilon n^{\alpha}\Big\}_+ < \infty.$$

Recently, Jiang and Zhang [9], established the following precise rates in the law of the iterated logarithm for the moment of i.i.d. random variables via strong approximation methods.

Theorem B. *Let $\{X, X_n : n \geq 1\}$ be a sequence of i.i.d. random variables with $EX = 0$, $EX^2 = \sigma^2 < \infty$. For $b > -1$, we have*

$$\begin{aligned}\lim_{\varepsilon \searrow 0} \varepsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n^{3/2} \log n} E\Big\{|M_n| - \sigma\varepsilon\sqrt{2n \log\log n}\Big\}_+ \\ = \frac{2^{-b}}{(b+1)(2b+3)} E|N|^{2b+3} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+3}},\end{aligned}$$

where N is a standard normal random variable.

Inspired by Chow [5] and Jiang and Zhang [9], here we consider the exact moment convergence rates in the law of the iterated logarithm and Chung-type law of the iterated logarithm for positively associated (PA) random variables including partial sums and the maximum of the partial sums. First, we shall give the definition of positively associated random variables:

Definition 1. A finite sequence of random variables $\{X_k : 1 \leq k \leq n\}$ is said to be positively associated (PA), if for any finite subsets A and B of $\{1, 2, \ldots, n\}$, we have

$$\mathrm{Cov}\{f(X_i;\, i \in A), g(X_j;\, j \in B)\} \geq 0,$$

whenever f and g are coordinatewise increasing and the covariance exists. An infinite sequence of random variables is PA if every finite subsequence is PA.

The notation of PA was first introduced by Esary et al. [6]. Because of its wide application in multivariate statistical analysis and system reliability, it has received considerable attention in the past two decades. Under some covariance restrictions, a number of limit theorems have been obtained for PA sequences. We refer to Newman [10] for the central limit theorem, Newman and Wright [11] for the functional central limit theorem, Yu [14, 15] for the law of the iterated logarithm and the strong invariance principle, Wood [13] and Birkel [3, 4] for the Berry-Esseen inequality and the moment equalities.

Now we are ready to state our main results. In what follows, let $\{X_n:\ n\ge 1\}$ be a sequence of strictly stationary PA random variables, $EX_1=0$, $0<EX_1^2<\infty$, and set $0<\sigma^2=EX_1^2+2\sum_{k=2}^{\infty}EX_1X_k<\infty$ unless it is mentioned otherwise.

Theorem 1. *Assume that* $E|X_1|^{2+\delta}<\infty$ *for some* $\delta\in(0,1]$, *and* $u(n):=\sum_{j=n+1}^{\infty}\operatorname{Cov}(X_1,X_j)=O(n^{-\alpha})$ *for some* $\alpha>1$. *Then for any* $b>-1/2$, *we have*

$$\lim_{\varepsilon\searrow 0}\varepsilon^{2b+1}\sum_{n=1}^{\infty}\frac{(\log\log n)^{b-1/2}}{n^{3/2}\log n}E\Big\{M_n-\sigma\varepsilon\sqrt{2n\log\log n}\Big\}_+ =\frac{2^{-1/2-b}E|N|^{2(b+1)}}{(b+1)(2b+1)}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{2(b+1)}}$$

and

$$\lim_{\varepsilon\searrow 0}\varepsilon^{2b+1}\sum_{n=1}^{\infty}\frac{(\log\log n)^{b-1/2}}{n^{3/2}\log n}E\Big\{|S_n|-\sigma\varepsilon\sqrt{2n\log\log n}\Big\}_+ =\frac{2^{-1/2-b}}{(b+1)(2b+1)}E|N|^{2(b+1)}. \tag{1}$$

Theorem 2. *Assume that* $E|X_1|^{2+\delta}<\infty$ *for some* $\delta\in(0,1]$, *and* $u(n):=\sum_{j=n+1}^{\infty}\operatorname{Cov}(X_1,X_j)=O(n^{-\alpha})$ *for some* $\alpha>1$. *Then for any* $b>-1/2$, *we have*

$$\lim_{\varepsilon\nearrow\infty}\varepsilon^{-2(b+1)}\sum_{n=1}^{\infty}\frac{(\log\log n)^{b}}{n^{3/2}\log n}E\left\{\sigma\varepsilon\sqrt{\frac{\pi^2 n}{8\log\log n}}-M_n\right\}_+ =\frac{\Gamma(b+1/2)}{\sqrt{2}(b+1)}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{2b+2}}, \tag{2}$$

where $\Gamma(\cdot)$ *is a Gamma function.*

Remark 1.1. It is well known that Theorems A and B investigated the complete moment convergence and exact moment convergence rates for i.i.d. random variables, respectively. While our main results extend them to PA random variables, and further we obtain the exact moment convergence rates of the maximum of the partial sums by universal law of the iterated logarithm and Chung's law of the iterated logarithm, extending the results of Fu [7] where only probability convergence rates of the universal law of logarithm is considered.

2. The proof of Theorem 1

From this section on, we begin to prove the theorems, and in the sequel, let M, C, etc. denote positive constants whose values possibly vary from place to place. The notation $a_n \sim b_n$ means that $\frac{a_n}{b_n} \to 1$ as $n \to \infty$, and $[x]$ denotes the greatest integer part of x. We first proceed with some useful lemmas.

Lemma 2.1. ([2]) *Let $\{W(t):\ t \geq 0\}$ be a standard Wiener process, and let N be a standard normal random variable. Then for any $x > 0$*

$$\begin{aligned}P\Big\{\sup_{0\leq s\leq 1}|W(s)| \geq x\Big\} &= 1 - \sum_{k=-\infty}^{\infty}(-1)^k P\big\{(2k-1)x \leq N \leq (2k+1)x\big\}\\ &= 4\sum_{k=0}^{\infty}(-1)^k P\{N \geq (2k+1)x\}\\ &= 2\sum_{k=0}^{\infty}(-1)^k P\{|N| \geq (2k+1)x\}.\end{aligned}$$

In particular,

$$P\Big\{\sup_{0\leq s\leq 1} W(s) \geq x\Big\} \sim 2P(N \geq x) \sim \frac{2}{\sqrt{2\pi}x}\exp\left(-\frac{x^2}{2}\right) \qquad as \quad x \to \infty.$$

Also, for any $x > 0$,

$$P\Big(\sup_{0\leq s\leq 1}|W(s)| \leq x\Big) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{2k+1}\exp\left\{-\frac{\pi^2(2k+1)^2}{8x^2}\right\}$$

and

$$P\Big(\sup_{0\leq s\leq 1}|W(s)| \leq x\Big) \sim \frac{4}{\pi}\exp\left(-\frac{\pi^2}{8x^2}\right) \qquad as \quad x \to 0.$$

Remark 2.1. In fact, from Lemma 2.1, it follows that for any $x > 0$,

$$P(|N| \leq x) \leq C\exp\big(-\pi^2/8x^2\big),$$

since $W(1) = N$.

Lemma 2.2. ([11]) *Under the conditions of* Lemma 2.1 *above, we have*

$$\frac{M_n}{\sigma\sqrt{n}} \to \sup_{0\le s\le 1} |W(s)| \quad \textit{and} \quad \frac{S_n}{\sigma\sqrt{n}} \to N \qquad \textit{in distribution.}$$

Lemma 2.3. ([10]) *Suppose that* λ_1 *and* λ_2 *satisfy*

$$0<\lambda_1<\lambda_2 \qquad \textit{and} \qquad (\lambda_2-\lambda_1)^2 \ge \sigma_n^2,$$

where $\sigma_n^2 = ES_n^2$. *Then we have*

$$P(M_n \ge \lambda_2) \le \big(1-\sigma_n^2/(\lambda_2-\lambda_1)^2\big)^{-1} P(|S_n|\ge \lambda_1).$$

Set

$$b(\varepsilon)=\exp\big(\exp(M/\varepsilon^2)\big), \qquad M>4, \quad 0<\varepsilon<1/4$$

and

$$\beta = \min\big(1/4, \delta\alpha/\big(2(2+\delta+(1+\delta)\alpha)\big)\big).$$

Without loss of generality, assume $\sigma = 1$.

Lemma 2.4. *For any* $M>4$, *we have*

$$\lim_{\varepsilon\searrow 0} \varepsilon^{2b+1} \sum_{n\le b(\varepsilon)} \frac{(\log\log n)^{b-1/2}}{n\log n} \Big| n^{-1/2} E\Big\{M_n - \varepsilon\sqrt{2n\log\log n}\Big\}_+ - E\Big\{\sup_{0\le s\le 1}|W(s)| - \varepsilon\sqrt{2\log\log n}\Big\}_+\Big| = 0$$

and

$$\lim_{\varepsilon\searrow 0} \varepsilon^{2b+1} \sum_{n\le b(\varepsilon)} \frac{(\log\log n)^{b-1/2}}{n\log n} \Big| n^{-1/2} E\Big\{|S_n| - \varepsilon\sqrt{2n\log\log n}\Big\}_+ - E\Big\{|N| - \varepsilon\sqrt{2\log\log n}\Big\}_+\Big| = 0.$$

P r o o f. We only give the proof of the former one, since the proof of the latter one is similar.

Note that

$$
\begin{aligned}
&\varepsilon^{2b+1} \sum_{n\le b(\varepsilon)} \frac{(\log\log n)^{b-1/2}}{n\log n} \Big| n^{-1/2} E\Big\{ M_n - \varepsilon\sqrt{2n\log\log n}\Big\}_+ \\
&\quad - E\Big\{ \sup_{0\le s\le 1} |W(s)| - \varepsilon\sqrt{2\log\log n}\Big\}_+ \Big| \\
&= \varepsilon^{2b+1} \sum_{n\le b(\varepsilon)} \frac{(\log\log n)^{b-1/2}}{n\log n} \Big| n^{-1/2} \int_0^\infty P\Big(M_n \ge x + \varepsilon\sqrt{2n\log\log n}\Big)\,\mathrm{d}x \\
&\quad - \int_0^\infty P\Big(\sup_{0\le s\le 1} |W(s)| \ge x + \varepsilon\sqrt{2\log\log n}\Big)\,\mathrm{d}x \Big| \\
&= \varepsilon^{2b+1} \sum_{n\le b(\varepsilon)} \frac{(\log\log n)^{b}}{n\log n} \Big| \int_0^\infty P\Big(M_n \ge (x + \varepsilon)\sqrt{2n\log\log n}\Big)\,\mathrm{d}x \\
&\quad - \int_0^\infty P\Big(\sup_{0\le s\le 1} |W(s)| \ge (x + \varepsilon)\sqrt{2\log\log n}\Big)\,\mathrm{d}x \Big| \\
&\le \sqrt{2}\varepsilon^{2b+1} \sum_{n\le b(\varepsilon)} \frac{(\log\log n)^{b}}{n\log n} \int_0^\infty \Big| P\Big(M_n \ge (x + \varepsilon)\sqrt{2n\log\log n}\Big) \\
&\quad - P\Big(\sup_{0\le s\le 1} |W(s)| \ge (x + \varepsilon)\sqrt{\log\log n}\Big) \Big|\,\mathrm{d}x \\
&=: \sqrt{2}\big((I) + (II)\big),
\end{aligned}
$$

where

$$
\begin{aligned}
(I) := \varepsilon^{2b+1} \sum_{n\le b(\varepsilon)} \frac{(\log\log n)^{b}}{n\log n} \int_0^{\Gamma_n} &\Big| P\Big(M_n \ge (x + \varepsilon)\sqrt{2n\log\log n}\Big) \\
&- P\Big(\sup_{0\le s\le 1} |W(s)| \ge (x + \varepsilon)\sqrt{2\log\log n}\Big) \Big|\,\mathrm{d}x,
\end{aligned}
$$

$$
\begin{aligned}
(II) := \varepsilon^{2b+1} \sum_{n\le b(\varepsilon)} \frac{(\log\log n)^{b}}{n\log n} \int_{\Gamma_n}^\infty &\Big| P\Big(M_n \ge (x + \varepsilon)\sqrt{2n\log\log n}\Big) \\
&- P\Big(\sup_{0\le s\le 1} |W(s)| \ge (x + \varepsilon)\sqrt{2\log\log n}\Big) \Big|\,\mathrm{d}x,
\end{aligned}
$$

$$\Gamma_n = (\log\log n)^{-1/2}\Delta_n^{-1/2} \qquad \text{and}$$
$$\Delta_n = \sup_x \Big| P\Big(M_n \geq x\sqrt{n}\Big) - P\Big(\sup_{0\leq s\leq 1}|W(s)| \geq x\Big)\Big|.$$

It is readily seen that $\Delta_n \to 0$ as $n\to\infty$ by Lemma 2.2. Thus, by applying the Toeplitz Lemma [12], we have

$$\frac{1}{(\log\log m)^{b+1/2}}\sum_{n=1}^{m}\frac{\Delta_n^{1/2}(\log\log n)^{b-1/2}}{n\log n} \to 0 \qquad \text{as} \quad m\to\infty.$$

Hence,

$$\begin{aligned}(I) &\leq \varepsilon^{2b+1}\sum_{n\leq b(\varepsilon)}\frac{\Delta_n^{1/2}(\log\log n)^{b-1/2}}{n\log n}\\ &= \varepsilon^{2b+1}(\log\log[b(\varepsilon)])^{b+1/2}\frac{1}{(\log\log[b(\varepsilon)])^{b+1/2}}\sum_{n\leq b(\varepsilon)}\frac{\Delta_n^{1/2}(\log\log n)^{b-1/2}}{n\log n}\\ &\leq M^{b+1/2}\frac{1}{(\log\log[b(\varepsilon)])^{b+1/2}}\sum_{n\leq b(\varepsilon)}\frac{\Delta_n^{1/2}(\log\log n)^{b-1/2}}{n\log n} \to 0 \qquad \text{as} \quad \varepsilon\searrow 0.\end{aligned}$$

As for (II),

$$\begin{aligned}(II) &\leq \varepsilon^{2b+1}\sum_{n\leq b(\varepsilon)}\frac{(\log\log n)^b}{n\log n}\int_{\Gamma_n}^{\infty}\Big(P\Big(M_n\geq (x+\varepsilon)\sqrt{2n\log\log n}\Big)\\ &\qquad + P\Big(\sup_{0\leq s\leq 1}|W(s)|\geq (x+\varepsilon)\sqrt{2\log\log n}\Big)\Big)\,\mathrm{d}x\\ &=: \varepsilon^{2b+1}\sum_{n\leq b(\varepsilon)}\frac{(\log\log n)^b}{n\log n}\int_{\Gamma_n}^{\infty}\big((II_1)+(II_2)\big)\,\mathrm{d}x.\end{aligned}$$

From Lemma 2.3 with

$$\lambda_1 = (x+\varepsilon)\sqrt{2n\log\log n}/2 \qquad \text{and} \qquad \lambda_2 = (x+\varepsilon)\sqrt{2n\log\log n}$$

and [4, Theorem 1], it follows that for some $\delta\in(0,1]$

$$\begin{aligned}&P\Big\{M_n\geq (x+\varepsilon)\sqrt{2n\log\log n}\Big\}\\ &\leq \Big(1-\frac{2\sigma_n^2}{(x+\varepsilon)^2 n\log\log n}\Big)^{-1}P\Big\{|S_n|\geq (x+\varepsilon)\sqrt{2n\log\log n}/2\Big\}\\ &\leq C\Big(1-\frac{2\sigma_n^2}{(x+\varepsilon)^2 n\log\log n}\Big)^{-1}(\log\log n)^{-((2+\delta)/2)}(x+\varepsilon)^{-(2+\delta)}\end{aligned}$$

$$\leq C\frac{n}{(x+\varepsilon)^2 n\log\log n - 2\sigma_n^2} \times \frac{1}{(x+\varepsilon)^\delta}$$

$$\leq C\frac{1}{(x+\varepsilon)^2 \log\log n - 2\sigma^2} \times \frac{1}{(x+\varepsilon)^\delta} \qquad \text{(by the definition of } \sigma^2\text{)}$$

$$\leq C\frac{1}{\frac{1}{2}(x+\varepsilon)^{2+\delta}\log\log n},$$

which guarantees that

$$\begin{aligned}\int_{\Gamma_n}^{\infty}(II_1)\,\mathrm{d}x &\leq C(\log\log n)^{-1}\int_{\Gamma_n}^{\infty}(x+\varepsilon)^{-(2+\delta)}\,\mathrm{d}x \\ &\leq C(\log\log n)^{-1}\int_{\Gamma_n}^{\infty}(x+\varepsilon)^{-2}\,\mathrm{d}x \\ &\leq C(\log\log n)^{-1/2}{\Delta_n}^{1/2}.\end{aligned}$$

As to (II_2), it follows from Lemma 2.1 that, for any $m \geq 1$ and $x > 0$,

$$\begin{aligned}2\sum_{k=0}^{2m+1}(-1)^k P\{|N| \geq (2k+1)x\} &\leq P\Big\{\sup_{0\leq s\leq 1}|W(s)| \geq x\Big\} \\ &\leq 2\sum_{k=0}^{2m}(-1)^k P\big\{|N| \geq (2k+1)x\big\}, \qquad (3)\end{aligned}$$

and then we have

$$\begin{aligned}\int_{\Gamma_n}^{\infty}(II_2)\,\mathrm{d}x &\leq \int_{\Gamma_n}^{\infty}P\Big(\sup_{0\leq s\leq 1}|W(s)| \geq (x+\varepsilon)\sqrt{2\log\log n}\Big)\,\mathrm{d}x \\ &\leq 2\sum_{k=0}^{2m}(-1)^k\int_{\Gamma_n}^{\infty}P\Big(|N| \geq (2k+1)(x+\varepsilon)\sqrt{2\log\log n}\Big)\,\mathrm{d}x \\ &\leq 2C\sum_{k=0}^{2m}\frac{1}{(2k+1)^2}(\log\log n)^{-1}\int_{\Gamma_n}^{\infty}(x+\varepsilon)^{-2}\,\mathrm{d}x \\ &\leq C(\log\log n)^{-1/2}{\Delta_n}^{1/2}.\end{aligned}$$

Then using the Toeplitz Lemma [12] again provides

$$(II) \leq C\varepsilon^{2b+1}\sum_{n\leq b(\varepsilon)}\frac{(\log\log n)^{b-1/2}}{n\log n}\Delta_n^{1/2} \to 0 \qquad \text{as} \quad \varepsilon \to 0.$$

Thus the proof is ended. □

Lemma 2.5. *For $0 < \varepsilon < 1/4$, we have uniformly*

$$\lim_{M\to\infty} \varepsilon^{2b+1} \sum_{n>b(\varepsilon)} \frac{(\log\log n)^{b-1/2}}{n\log n} E\Big\{ \sup_{0\le s\le 1} |W(s)| - \varepsilon\sqrt{2\log\log n}\Big\}_+ = 0.$$

P r o o f. Recalling Lemma 2.1, we have that for k large enough,

$$\begin{aligned}
&\varepsilon^{2b+1} \sum_{n>b(\varepsilon)} \frac{(\log\log n)^{b-1/2}}{n\log n} \int_0^\infty P\Big\{ \sup_{0\le s\le 1} |W(s)| \ge \varepsilon\sqrt{2\log\log n} + x\Big\}\,\mathrm{d}x \\
&\le C\varepsilon^{2b+1} \sum_{n>b(\varepsilon)} \frac{(\log\log n)^{b}}{n\log n} \int_0^\infty P\Big\{ N \ge (x+\varepsilon)\sqrt{2\log\log n}\Big\}\,\mathrm{d}x \\
&\le C\varepsilon^{2b+1} \sum_{n>b(\varepsilon)} \frac{(\log\log n)^{b}}{n\log n} \int_0^\infty \frac{E|N|^k}{(x+\varepsilon)^k(\log\log n)^{k/2}}\,\mathrm{d}x \\
&\le C\varepsilon^{2b+1} \sum_{n>b(\varepsilon)} \frac{(\log\log n)^{b-k/2}}{n\log n} \varepsilon^{-k+1} \\
&= CM^{b+1-k/2} \to 0,
\end{aligned}$$

when $M\to\infty$, uniformly for $0<\varepsilon<1/4$. □

Lemma 2.6. *For $b > -1$, if $E|X_1|^{2+\delta} < \infty$ $(0 < \delta \le 1)$, then*

$$\lim_{M\to\infty}\lim_{\varepsilon\searrow 0} \varepsilon^{2b+1} \sum_{n>b(\varepsilon)} \frac{(\log\log n)^{b-1/2}}{n^{3/2}\log n} E\Big\{ M_n - \varepsilon\sqrt{2n\log\log n}\Big\}_+ = 0.$$

P r o o f. Setting

$$\lambda_1 = (x+\varepsilon)\sqrt{2n\log\log n}/2 \qquad \text{and} \qquad \lambda_2 = (x+\varepsilon)\sqrt{2n\log\log n}$$

again, by the fact $\sigma_n^2 \sim n$, as $n\to\infty$, we have that $(\lambda_2-\lambda_1)^2 \ge \sigma_n^2$, whence $x > \Gamma_n$, $n > b(\varepsilon)$ and ε is small enough. By virtue of [4, Theorem 1] and Lemma 2.3, it leads to that for $0<\varepsilon<1/4$ and some $\delta\in(0,1]$

$$\begin{aligned}
P\Big\{ M_n \ge (x+\varepsilon)\sqrt{2n\log\log n}\Big\} &\le CP\Big\{ |S_n| \ge (x+\varepsilon)\sqrt{2n\log\log n}/2\Big\} \\
&\le C(\log\log n)^{-((2+\delta)/2)}(x+\varepsilon)^{-(2+\delta)} \\
&\le C(\log\log n)^{-1}(x+\varepsilon)^{-(2+\delta)}.
\end{aligned}$$

Thus it leads to

$$\begin{aligned}
&\varepsilon^{2(b+1)} \sum_{n>b(\varepsilon)} \frac{(\log\log n)^{b-1/2}}{n^{3/2}\log n} E\Big\{M_n - \varepsilon\sqrt{n\log n}\,\Big\}_+ \\
&\le C\varepsilon^{2b-\delta} \sum_{n>b(\varepsilon)} \frac{(\log\log n)^{b-3/2}}{n^{3/2}\log n} \\
&\le C\varepsilon^{2b-\delta}\big(\log\log b(\varepsilon)\big)^{-1-\delta/2} \sum_{n>b(\varepsilon)} \frac{(\log\log n)^{(2b+\delta-1)/2}}{n^{3/2}\log n} \\
&= C\varepsilon^{2b+2} M^{-1-\delta/2} \sum_{n>b(\varepsilon)} \frac{(\log\log n)^{(2b+\delta-1)/2}}{n^{3/2}\log n}.
\end{aligned}$$

By letting $\varepsilon \to 0$ and $M \to \infty$, the conclusion follows, as desired. □

Proof of Theorem 1. First we prove the following propositions:

Proposition 2.1. *For any $b > -1/2$, we have*

$$\lim_{\varepsilon\searrow 0} \varepsilon^{2b+1} \sum_{n=1}^{\infty} \frac{(\log\log n)^{b-1/2}}{n\log n} E\Big\{\sup_{0\le s\le 1}|W(s)| - \varepsilon\sqrt{2\log\log n}\,\Big\}_+ = \frac{2^{-1/2-b}E|N|^{2(b+1)}}{(b+1)(2b+1)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2(b+1)}}$$

and

$$\lim_{\varepsilon\searrow 0} \varepsilon^{2b+1} \sum_{n=1}^{\infty} \frac{(\log\log n)^{b-1/2}}{n\log n} E\Big\{|N| - \varepsilon\sqrt{2\log\log n}\Big\}_+ = \frac{2^{-1/2-b}E|N|^{2(b+1)}}{(b+1)(2b+1)},$$

where N is the standard normal random variable.

Proof. Note that for any $m \ge 1$ and $x > 0$, (3) holds. Thus it follows that for any $t > 0$

$$\begin{aligned}
E\Big\{\sup_{0\le s\le 1}|W(s)| - t\Big\}_+ &= \int_0^{\infty} P\Big(\sup_{0\le s\le 1}|W(s)| \ge t + x\Big)\,\mathrm{d}x \\
&\le 2\sum_{k=0}^{2m}(-1)^k \int_0^{\infty} P\big(|N| \ge (2k+1)(t+x)\big)\,\mathrm{d}x
\end{aligned}$$

$$= 2\sum_{k=0}^{2m}\frac{(-1)^k}{2k+1}\int_0^\infty P\big(|N|\geq (2k+1)t+x\big)\,\mathrm{d}x$$
$$= 2\sum_{k=0}^{2m}\frac{(-1)^k}{2k+1}E\{|N|-(2k+1)t\}_+$$

and

$$E\Big\{\sup_{0\leq s\leq 1}|W(s)|-t\Big\}_+\geq 2\sum_{k=0}^{2m+1}\frac{(-1)^k}{2k+1}E\{|N|-(2k+1)t\}_+.$$

So, it suffices to show that for any $q\geq 1$ and $b>-1/2$,

$$\lim_{\varepsilon\searrow 0}\varepsilon^{2b+1}\sum_{n=1}^\infty\frac{(\log\log n)^{b-1/2}}{n\log n}E\Big\{|N|-q\varepsilon\sqrt{2\log\log n}\,\Big\}_+ = q^{-2b-1}\frac{2^{-1/2-b}E|N|^{2(b+1)}}{(b+1)(2b+1)}. \tag{4}$$

Obviously,

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\varepsilon^{2b+1}\sum_{n=1}^\infty\frac{(\log\log n)^{b-1/2}}{n\log n}E\Big\{|N|-q\varepsilon\sqrt{2\log\log n}\,\Big\}_+\\
=\;&\lim_{\varepsilon\searrow 0}\varepsilon^{2b+1}\int_{\mathrm{e}^{\mathrm{e}}}^\infty\frac{(\log\log y)^{b-1/2}}{y\log y}\int_{q\varepsilon\sqrt{2\log\log y}}^\infty P\{|N|\geq x\}\,\mathrm{d}x\,\mathrm{d}y\\
=\;&2^{1/2-b}q^{-2b-1}\lim_{\varepsilon\searrow 0}\int_{\sqrt{2}q\varepsilon}^\infty z^{2b}\int_z^\infty P\{|N|\geq x\}\,\mathrm{d}x\,\mathrm{d}z\\
=\;&q^{-2b-1}\frac{2^{-1/2-b}E|N|^{2(b+1)}}{(b+1)(2b+1)}.
\end{aligned}\tag{5}$$

Thus the Proposition follows by taking $q=2k+1$ and $q=1$, respectively. □

Proposition 2.2. *For any $b>-1/2$, if $E|X|^{2+\delta}$ $(0<\delta\leq 1)$, then we have*

$$\lim_{\varepsilon\searrow 0}\varepsilon^{2b+1}\sum_{n=1}^\infty\frac{(\log\log n)^{b-1/2}}{n\log n}\Big|n^{-1/2}E\Big\{M_n-\varepsilon\sqrt{2n\log\log n}\,\Big\}_+ -E\Big\{\sup_{0\leq s\leq 1}|W(s)|-\varepsilon\sqrt{2\log\log n}\,\Big\}_+\Big| = 0$$

and

$$\lim_{\varepsilon\searrow 0}\varepsilon^{2b+1}\sum_{n=1}^{\infty}\frac{(\log\log n)^{b-1/2}}{n\log n}\Big|n^{-1/2}E\Big\{|S_n|-\varepsilon\sqrt{2n\log\log n}\,\Big\}_+ \\ -E\{|N|-\varepsilon\sqrt{2\log\log n}\}_+\Big| = 0.$$

P r o o f. It is trivial via Lemmas 2.4–2.6. □

Theorem 1 is implied from the above Propositions 1.1 and 1.2. □

3. The proof of Theorem 2

Lemma 3.1. *For any $M > 4$, we have*

$$\lim_{\varepsilon\nearrow\infty}\varepsilon^{-2(b+1)}\sum_{n\le b(1/\varepsilon)}\frac{(\log\log n)^{b}}{n\log n}\bigg|n^{-1/2}E\bigg\{\varepsilon\sqrt{\frac{\pi^2 n}{8\log\log n}}-M_n\bigg\}_+ \\ -E\bigg\{\varepsilon\sqrt{\frac{\pi^2}{8\log\log n}}-\sup_{0\le s\le 1}|W(s)|\bigg\}_+\bigg| = 0.$$

P r o o f. Note that $\Delta_n=\sup_x\big|P\big(M_n\le x\sqrt{n}\big)-P\big(\sup_{0\le s\le 1}|W(s)|\le x\big)\big|\to 0$, as $n\to\infty$. Thus following the same lines of Lemma 2.4, an application of the Toeplitz Lemma [12] provides the result. □

Lemma 3.2. *For $\varepsilon > 0$ sufficiently large, we have*

$$\lim_{M\to\infty}\varepsilon^{-2(b+1)}\sum_{n>b(1/\varepsilon)}\frac{(\log\log n)^{b}}{n\log n}E\bigg\{\varepsilon\sqrt{\frac{\pi^2}{8\log\log n}}-\sup_{0\le s\le 1}|W(s)|\bigg\}_+ = 0,$$

uniformly in ε.

Proof. By Lemma 2.1, we have that

$$\begin{aligned}
&\varepsilon^{-2(b+1)} \sum_{n>b(1/\varepsilon)} \frac{(\log\log n)^b}{n\log n} E\left\{\varepsilon\sqrt{\frac{\pi^2}{8\log\log n}} - \sup_{0\le s\le 1}|W(s)|\right\}_+ \\
&= \varepsilon^{-2(b+1)} \sum_{n>b(1/\varepsilon)} \frac{(\log\log n)^b}{n\log n} \int_0^{\varepsilon\sqrt{\frac{\pi^2}{8\log\log n}}} P\Big\{\sup_{0\le s\le 1}|W(s)| \le t\Big\}\,\mathrm{d}t \\
&\le C\int_M^\infty y^{b-1/2}\mathrm{e}^{-y}\,\mathrm{d}y \to 0 \qquad \text{as} \quad M\to\infty.
\end{aligned}$$

□

Lemma 3.3. ([10]) *Let* $\sigma_n^2 = ES_n^2$*, and then we have*

$$\Lambda_n = \sup_x \big|P\big(S_n/\sigma_n \le x\big) - \Phi(x)\big| \le C\Big(n^{-1/4} + n^{-\frac{\delta\alpha}{2(2+\delta+(1+\delta)\alpha)}}\Big),$$

where C *is a constant.*

Lemma 3.4. *For* $b > -1/2$ *and* $\varepsilon > 0$ *sufficiently large, we have*

$$\lim_{M\to\infty} \varepsilon^{-2(b+1)} \sum_{n>b(1/\varepsilon)} \frac{(\log\log n)^b}{n^{3/2}\log n} E\left\{\varepsilon\sqrt{\frac{\pi^2 n}{8\log\log n}} - M_n\right\}_+ = 0,$$

uniformly in ε.

Proof. Since $\beta = \min\big(1/4, \delta\alpha/(2(2+\delta+(1+\delta)\alpha))\big)$, it follows from Lemma 3.3 that $\Lambda_n = O(n^{-\beta})$. Combined with Remark 1.1, it implies that for n large enough,

$$\begin{aligned}
P(M_n \le x) &\le P\big(|S_n| \le x\big) \\
&\le \big|P\big(|S_n/\sigma_n| \le x/\sigma_n\big) - P\big(|N| \le x/\sigma_n\big)\big| + P\big(|N| \le x/\sigma_n\big) \\
&\le Cn^{-\beta} + C\exp\big(-(\pi^2\sigma_n^2)/8x^2\big).
\end{aligned}$$

Then for any ε large enough (entailing n large enough) and $b > -1/2$, we have

$$\begin{aligned}
&\varepsilon^{-2(b+1)} \sum_{n>b(1/\varepsilon)} \frac{(\log\log n)^b}{n^{3/2}\log n} E\left\{\varepsilon\sqrt{\frac{\pi^2 n}{8\log\log n}} - M_n\right\}_+ \\
&= \varepsilon^{-2(b+1)} \sum_{n>b(1/\varepsilon)} \frac{(\log\log n)^b}{n^{3/2}\log n} \int_0^{\varepsilon\sqrt{\frac{\pi^2 n}{8\log\log n}}} P(M_n \le t)\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\le C\varepsilon^{-2b-1} \sum_{n>b(1/\varepsilon)} \frac{(\log\log n)^{b-1/2}}{n\log n} P\biggl(M_n \le \varepsilon\sqrt{\frac{\pi^2 n}{8\log\log n}}\biggr) \\
&\le C\varepsilon^{-2b-1} \sum_{n>b(1/\varepsilon)} \frac{(\log\log n)^{b-1/2}}{n^{1+\beta}\log n} \\
&\quad + C\varepsilon^{-2b-1} \sum_{n>b(1/\varepsilon)} \frac{(\log\log n)^{b-1/2}}{n\log n} \exp\Bigl(-\frac{\log\log n}{\varepsilon^2}\Bigr) \\
&\le C\varepsilon^{-2b-1} \sum_{n>b(1/\varepsilon)} \frac{(\log\log n)^{b-1/2}}{n^{1+\beta}\log n} + C\int_M^\infty y^{b-1/2}\mathrm{e}^{-y}\,\mathrm{d}y \longrightarrow 0,
\end{aligned}
$$

by letting $\varepsilon \to \infty$ and $M \to \infty$. □

P r o o f o f T h e o r e m 2. We need the following propositions:

Proposition 3.1. *For any $b > -1/2$, we have*

$$
\lim_{\varepsilon\nearrow\infty} \varepsilon^{-2(b+1)} \sum_{n=1}^\infty \frac{(\log\log n)^b}{n\log n} E\biggl\{\varepsilon\sqrt{\frac{\pi^2}{8\log\log n}} - \sup_{0\le s\le 1}|W(s)|\biggr\}_+ = \frac{\Gamma(b+1/2)}{\sqrt{2}(b+1)} \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^{2b+2}}.
$$

P r o o f. It follows from [8, Lemma 2.4] and Lemma 2.1 that

$$
\begin{aligned}
&\lim_{\varepsilon\nearrow\infty} \varepsilon^{-2(b+1)} \sum_{n=1}^\infty \frac{(\log\log n)^b}{n\log n} E\biggl\{\varepsilon\sqrt{\frac{\pi^2}{8\log\log n}} - \sup_{0\le s\le 1}|W(s)|\biggr\}_+ \\
&= \lim_{\varepsilon\nearrow\infty} \varepsilon^{-2(b+1)} \sum_{n=1}^\infty \frac{(\log\log n)^b}{n\log n} \int_0^{\varepsilon\sqrt{\frac{\pi^2}{8\log\log n}}} P\Bigl\{\sup_{0\le s\le 1}|W(s)| \le t\Bigr\}\,\mathrm{d}t \\
&= \lim_{\varepsilon\nearrow\infty} \varepsilon^{-2(b+1)} \sum_{n=1}^\infty \frac{(\log\log n)^b}{n\log n} \int_0^{\varepsilon\sqrt{\frac{\pi^2}{8\log\log n}}} \frac{4}{\pi}\sum_{k=0}^\infty \frac{(-1)^k}{2k+1}\exp\Bigl\{-\frac{\pi^2(2k+1)^2}{8t^2}\Bigr\}\,\mathrm{d}t \\
&= \frac{4}{\pi}\sum_{k=0}^\infty \frac{(-1)^k}{2k+1} \lim_{\varepsilon\nearrow\infty} \varepsilon^{-2(b+1)} \int_{\mathrm{e}^\mathrm{e}}^\infty \frac{(\log\log x)^b}{x\log x} \int_0^{\varepsilon\sqrt{\frac{\pi^2}{8\log\log x}}} \exp\Bigl\{-\frac{\pi^2(2k+1)^2}{8t^2}\Bigr\}\,\mathrm{d}t\,\mathrm{d}x
\end{aligned}
$$

$$= \frac{1}{\sqrt{2}} \sum_{k=0}^{\infty} (-1)^k \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \int_{\mathrm{e}^{\mathrm{e}}}^{\infty} \frac{(\log\log x)^b}{x \log x} \int_{(2k+1)^2 \log\log x/\varepsilon^2}^{\infty} y^{-3/2} \mathrm{e}^{-y} \, \mathrm{d}y \, \mathrm{d}x$$

$$= \frac{1}{\sqrt{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+2}} \lim_{\varepsilon \nearrow \infty} \int_{(2k+1)^2/\varepsilon^2}^{\infty} s^b \int_{s}^{\infty} y^{-3/2} \mathrm{e}^{-y} \, \mathrm{d}y \, \mathrm{d}s$$

$$= \frac{1}{\sqrt{2}(b+1)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+2}} \lim_{\varepsilon \nearrow \infty} \int_{(2k+1)^2/\varepsilon^2}^{\infty} y^{b-1/2} \mathrm{e}^{-y} \, \mathrm{d}y$$

$$= \frac{\Gamma(b+1/2)}{\sqrt{2}(b+1)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+2}}.$$

Thus we terminate the proof. □

Proposition 3.2. *For any $b > -1/2$, we have*

$$\limsup_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n \log n} \left| n^{-1/2} E\left\{ \varepsilon\sigma \sqrt{\frac{n\pi^2}{8 \log\log n}} - M_n \right\}_+ - E\left\{ \varepsilon \sqrt{\frac{\pi^2}{8 \log\log n}} - \sup_{0 \le s \le 1} |W(s)| \right\}_+ \right| = 0.$$

P r o o f. It can be readily seen via Lemmas 3.1, 3.2 and 3.4. □

Based on Propositions 3.1 and 3.2, Theorem 2 follows immediately. □

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School of Statistics and Mathematics
Zhejiang Gongshang University
Hangzhou, 310018
Zhejiang Province
P. R. CHINA
E-mail: yangxr110@gmail.com
fukeang@hotmail.com