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CONVERGENCE RATES IN THE COMPLETE MOMENT OF MOVING-AVERAGE PROCESSES

Qing-pei Zang

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ABSTRACT. In this paper, we discuss precise asymptotics for a new kind of moment convergence of the moving-average process $X_k = \sum\limits_{i=-\infty}^{\infty} a_{i+k}\varepsilon_i$, $k \geq 1$, where $\{\varepsilon_i : -\infty < i < \infty\}$ is a doubly infinite sequence of independent identically distributed random variables with mean zero and the finiteness of variance, $\{a_i : -\infty < i < \infty\}$ is an absolutely summable sequence of real numbers, i.e., $\sum\limits_{i=-\infty}^{\infty} |a_i| < \infty$.

1. Introduction and main results

We assume that $\{\varepsilon, \varepsilon_i : -\infty < i < \infty\}$ is a doubly infinite sequence of identically distributed variables. Let $\{a_i : -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers and $X_k = \sum\limits_{i=-\infty}^{\infty} a_{i+k}\varepsilon_i$, $k \geq 1$, also set $S_n = \sum\limits_{k=1}^{n} X_k$.

Many limiting results have been obtained for moving-average processes $\{X_k : k \geq 1\}$. For example, Burton and Dehling [1] have obtained a large deviation principle for $\{X_k : k \geq 1\}$ assuming $E \exp t\varepsilon_1 < \infty$ for all t, Ibragimov [5] has established the central limit theorem for $\{X_k : k \geq 1\}$, Li, et al. [6]

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derived convergence rates of moderate deviations and the precise asymptotics in the law of the iterated logarithm, Dong, et al. [3] also obtained moderate deviation principles for moving-average processes of real stationary sequences.

On the other hand, Gut and Spătaru [4] proved the precise asymptotics of i.i.d random variables. One of their results is as follows.

Theorem 1.1. *Suppose that* $\{Y_k :\ k \geq 1\}$ *is a sequence of i.i.d random variables with* $EY_1 = 0$ *and* $EY_1^2 = \sigma^2 < \infty$. *Then*

$$\lim_{\varepsilon \searrow 0} \varepsilon^2 \sum_{n=1}^{\infty} \frac{1}{n \log n} P\left(\Big|\sum_{k=1}^{n} Y_k\Big| \geq \varepsilon\sqrt{n \log\log n}\right) = \sigma^2.$$

Chow [2] discussed the complete moment convergence of i.i.d random variables. He got the following result:

Theorem 1.2. *Let* $\{Y, Y_k :\ k \geq 1\}$ *be a sequence of i.i.d random variables with* $EY_1 = 0$. *Suppose that* $p \geq 1$, $\alpha > \frac{1}{2}$, $p\alpha > 1$, $E\big\{|Y|^p + |Y|\log(1 + |Y|)\big\} < \infty$. *Then for any* $\varepsilon > 0$, *we have*

$$\sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} E\left\{\max_{j\leq n}\Big|\sum_{k=1}^{j} Y_k\Big| - \varepsilon n^{\alpha}\right\}_+ < \infty.$$

Recently, Liu and Lin [9] derived precise asymptotics for a new kind of complete moment convergence, one of their results is as follows:

Theorem 1.3. *Suppose that* $\{X, X_n :\ n \geq 1\}$ *is a sequence of i.i.d. random variables,* $0 < \delta \leq 1$,

$$EX = 0, \quad EX^2 = \sigma^2 \quad \text{and} \quad EX^2\left(\log^+|X|\right)^{\delta} < \infty.$$

Then we have

$$\lim_{\varepsilon \searrow 0} \frac{1}{-\log\varepsilon} \sum_{n=1}^{\infty} \frac{1}{n^2} ES_n^2 I\left\{|S_n| \geq \varepsilon n\right\} = 2\sigma^2.$$

Inspired by the above results, we will extend this kind of results to moving-average processes. Now we state our results as follows.

Theorem 1.4. *Suppose that* $\{\varepsilon, \varepsilon_n :\ n \geq 1\}$ *is defined as above, and*

$$E\varepsilon = 0, \quad E\varepsilon^2 = \sigma^2, \quad E\varepsilon^3 < \infty. \tag{1.1}$$

If $\sum\limits_{i=-\infty}^{\infty} a_i \neq 0$, *we have*

$$\lim_{\varepsilon \searrow 0} \frac{1}{-\log\varepsilon} \sum_{n=1}^{\infty} \frac{1}{n^2} ES_n^2 I\left\{|S_n| \geq \varepsilon n\right\} = 2\tau^2, \tag{1.2}$$

where $\tau^2 = \sigma^2\Big(\sum\limits_{i=-\infty}^{\infty} a_i\Big)^2$.

Theorem 1.5. *Suppose that* $\{\varepsilon, \varepsilon_n : n \geq 1\}$ *is defined as above, and*

$$E\varepsilon = 0, \quad E\varepsilon^2 = \sigma^2, \quad E\varepsilon^3 < \infty. \tag{1.3}$$

If $\sum_{i=-\infty}^{\infty} a_i \neq 0$, *then, for* $1 \leq p < 2$, $1 + \frac{p}{2} < r < 2$, $0 \leq q < \frac{2(r-p)}{2-p}$, *we have*

$$\begin{aligned}\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p} - q} \sum_{n=1}^{\infty} n^{\frac{r}{p} - 2 - \frac{q}{p}} E|S_n|^q I\left\{|S_n| \geq \varepsilon n^{\frac{1}{p}}\right\} \\ = \frac{2p}{2r - 2p - 2q + pq} E|Z|^{\frac{2(r-p)}{2-p}},\end{aligned} \tag{1.4}$$

where Z *has a normal distribution with mean* 0 *and variance* $\tau^2 = \sigma^2 \Big(\sum_{i=-\infty}^{\infty} a_i\Big)^2$.

2. Some lemmas

First, we give some lemmas which will be used in the proofs. Lemmas 3.1 and 3.2 are from Burton and Dehling [1], Yang [10] respectively.

Lemma 2.1. *Let* $\sum_{i=-\infty}^{\infty} a_i$ *be an absolutely convergent series of real numbers with* $a = \sum_{i=-\infty}^{\infty} a_i$ *and* $k \geq 1$. *Then*

$$\lim_{n \longrightarrow \infty} \frac{1}{n} \sum_{i=-\infty}^{\infty} \Bigg| \sum_{j=i+1}^{i+n} a_j \Bigg|^k = |a|^k.$$

Lemma 2.2. *Let* $\{\varepsilon_i : -\infty < i < \infty\}$ *be a sequence of random variables with* $E\varepsilon_i = 0$, $0 < E\varepsilon_i^2 < \infty$, *and* $\{X_i : i \geq 1\}$ *is defined as above, where* $\{\varepsilon_i : -\infty < i < \infty\}$ *is a sequence of real numbers with* $\sum_{i=-\infty}^{\infty} |a_i| < \infty$. *Then the moving-average process* $\{X_k\}$ *fulfills the CLT, that is,*

$$\frac{S_n}{\tau\sqrt{n}} \xrightarrow{\mathcal{D}} N(0,1) \qquad \text{where} \quad \tau = \sigma \sum_{i=-\infty}^{\infty} a_i.$$

Lemma 2.3 (Rosenthal inequality). ([8]) *Assume* $\{Y_i : i \geq 1\}$ *is a sequence of independent random variables,* $EY_i = 0$, $E|Y_i|^p < \infty$, *for some* $p \geq 2$ *and every* $i \in \mathbb{R}$. *Then there exists* $C = C(p)$, *such that*

$$E\Bigg|\sum_{i \in \mathbb{R}} Y_i\Bigg|^p \leq C\Bigg\{\sum_{i \in \mathbb{R}} E|Y_i|^p + \Bigg(\sum_{i \in \mathbb{R}} |EY_i|^2\Bigg)^{\frac{p}{2}}\Bigg\}.$$

Throughout the sequel, N represents standard normal variable and Z has a normal distribution with mean 0 and variance $\tau^2 = \sigma^2\Big(\sum\limits_{i=-\infty}^{\infty} a_i\Big)^2$. C will denote a positive constant although its value may change from one appearance to the next and let $[x]$ indicate the maximum integer not larger than x. Without loss of generality, we assume $\tau = 1$ in the sequel.

3. Proof of Theorem 1.4

In this section, we set $a(\varepsilon) = [\varepsilon^{-2}]$ for $0 < \varepsilon < 1$. Note that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} ES_n^2 I\left\{|S_n| \geq \varepsilon n\right\} = \varepsilon^2 \sum_{n=1}^{\infty} P\left(|S_n| \geq \varepsilon n\right) + \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\varepsilon n}^{\infty} 2xP\left(|S_n| \geq x\right)\, \mathrm{d}x.$$

When taking $p = 1$ and $r = 2$ in Li [7], we have

$$\lim_{\varepsilon \searrow 0} \varepsilon^2 \sum_{n=1}^{\infty} P\left(|S_n| \geq \varepsilon n\right) = 1.$$

Thus, in order to prove (1.2), it suffices to show

$$\lim_{\varepsilon \searrow 0} \frac{1}{-\log \varepsilon} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\varepsilon n}^{\infty} 2xP\left(|S_n| \geq x\right)\, \mathrm{d}x = 2. \tag{3.1}$$

This will be proved by the following propositions.

Proposition 3.1. *One has*

$$\lim_{\varepsilon \searrow 0} \frac{1}{-\log \varepsilon} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\varepsilon n}^{\infty} 2xP\left(|N| \geq x/\sqrt{n}\,\right)\, \mathrm{d}x = 2.$$

Proof. See the proof of [9, Proposition 3.1]. □

Proposition 3.2. *One has*

$$\lim_{\varepsilon \searrow 0} \frac{1}{-\log \varepsilon} \sum_{n=1}^{a(\varepsilon)} \frac{1}{n^2} \left| \int_{\varepsilon n}^{\infty} 2xP\left(|S_n| \geq x\right)\, \mathrm{d}x - \int_{\varepsilon n}^{\infty} 2xP\left(|N| \geq x/\sqrt{n}\right)\, \mathrm{d}x \right| = 0. \tag{3.2}$$

Proof. Denote

$$\triangle_n = \sup_x \left| P\left(\frac{|S_n|}{\sqrt{n}} \geq x\right) - P(|N| \geq x)\right|,$$

it follows from Lemma 2 that $\triangle_n \longrightarrow 0$ as $n \longrightarrow \infty$. Obviously,

$$\begin{aligned}
&\sum_{n=1}^{a(\varepsilon)} \frac{1}{n^2}\left| \int_{\varepsilon n}^{\infty} 2xP\left(|S_n| \geq x\right) \mathrm{d}x - \int_{\varepsilon n}^{\infty} 2xP\big(|N| \geq x/\sqrt{n}\,\big)\,\mathrm{d}x \right| \\
&= \sum_{n=1}^{a(\varepsilon)}\left| \int_{0}^{\infty} 2(x+\varepsilon)P\left(|S_n| \geq n(x+\varepsilon)\right) \mathrm{d}x - \int_{0}^{\infty} 2(x+\varepsilon)P\left(|N| \geq \sqrt{n}(x+\varepsilon)\right)\,\mathrm{d}x \right| \\
&\leq \sum_{n=1}^{a(\varepsilon)} \frac{1}{n} \int_{0}^{\infty} 2n(x+\varepsilon)\left|P\left(|S_n| \geq n(x+\varepsilon)\right) - P\left(|N| \geq \sqrt{n}(x+\varepsilon)\right)\right|\,\mathrm{d}x \\
&\leq \sum_{n=1}^{a(\varepsilon)} \frac{1}{n}(\Delta_{n1} + \Delta_{n2} + \Delta_{n3}),
\end{aligned}$$

where

$$\Delta_{n1} = \int_{0}^{\frac{1}{\sqrt{n}\Delta_n^{1/4}}} 2n(x+\varepsilon)\left|P\big(|S_n| \geq n(x+\varepsilon)\big) - P\left(|N| \geq \sqrt{n}(x+\varepsilon)\right)\right|\,\mathrm{d}x;$$

$$\Delta_{n2} = \int_{\frac{1}{\sqrt{n}\Delta_n^{1/4}}}^{\infty} 2n(x+\varepsilon)P\big(|S_n| \geq n(x+\varepsilon)\big)\,\mathrm{d}x;$$

$$\Delta_{n3} = \int_{\frac{1}{\sqrt{n}\Delta_n^{1/4}}}^{\infty} 2n(x+\varepsilon)P\left(|N| \geq \sqrt{n}(x+\varepsilon)\right)\,\mathrm{d}x.$$

Since $n \leq b(\varepsilon)$ implies $\varepsilon\sqrt{n} \leq 1$, we have

$$\begin{aligned}
\Delta_{n1} \leq \int_{0}^{\frac{1}{\sqrt{n}\Delta_n^{1/4}}} 2n(x+\varepsilon)\Delta_n\,\mathrm{d}x &\leq n\Delta_n\left(\frac{1}{\sqrt{n}\Delta_n^{1/4}} + \varepsilon\right)^2 \\
&\leq \left(\Delta_n^{1/4} + \Delta_n^{1/2}\right)^2 \to 0, \qquad \text{as} \quad n \to \infty.
\end{aligned} \tag{3.3}$$

Next, observe that

$$\sum_{k=1}^{n} X_k = \sum_{i=-\infty}^{\infty}\sum_{k=1}^{n} a_{k+i}\varepsilon_i.$$

Set $a_{ni} = \sum_{k=1}^{n} a_{k+i}$. Then

$$\sum_{k=1}^{n} X_k = \sum_{i=-\infty}^{\infty} a_{ni}\varepsilon_i =: \sum_{i=-\infty}^{\infty} Y_i.$$

From Lemma 1, we can assume, without loss of generality, that

$$\sum_{i=-\infty}^{\infty} |a_{ni}|^k \le Cn, \quad n \ge 1, \quad k \ge 1 \qquad \text{and} \qquad \sum_{i=-\infty}^{\infty} |a_i| \le 1.$$

And then, by Lemma 3 (rosenthal's inequality), we get

$$E|S_n|^3 \le C\left(\sum_{i=-\infty}^{\infty} E|a_{ni}\varepsilon_i|^3 + \left(\sum_{i=-\infty}^{\infty} E|a_{ni}\varepsilon_i|^2\right)^{\frac{3}{2}}\right) \le Cn^{\frac{3}{2}}. \tag{3.4}$$

Thus, via Markov's inequality, we have

$$\begin{aligned} \Delta_{n2} &\le C \int_{\frac{1}{\sqrt{n}\Delta_n^{1/4}}}^{\infty} n(x+\varepsilon)\frac{n^{\frac{3}{2}}}{n^3(x+\varepsilon)^3}\,\mathrm{d}x \\ &\le C \int_{\frac{1}{\sqrt{n}\Delta_n^{1/4}}}^{\infty} \frac{1}{\sqrt{n}(x+\varepsilon)^2}\,\mathrm{d}x \\ &\le C\triangle_n^{\frac{1}{4}} \to 0, \qquad \text{as} \quad n \to \infty. \end{aligned} \tag{3.5}$$

Now, we estimate Δ_{n3}. By Markov's inequality, we have

$$\Delta_{n3} \le Cn \int_{\frac{1}{\sqrt{n}\Delta_n^{1/4}}}^{\infty} (x+\varepsilon)\frac{1}{n^2(x+\varepsilon)^4}\,\mathrm{d}x \le C\Delta_n^{\frac{1}{2}} \to 0, \qquad \text{as} \quad n \to \infty. \tag{3.6}$$

From (3.3), (3.5), (3.6), the proof of this proposition is derived. □

Proposition 3.3. *One has*

$$\lim_{\varepsilon \searrow 0} \frac{1}{-\log\varepsilon} \sum_{n=a(\varepsilon)+1}^{\infty} \frac{1}{n^2}\left| \int_{\varepsilon n}^{\infty} 2xP\left(|S_n| \ge x\right)\,\mathrm{d}x - \int_{\varepsilon n}^{\infty} 2xP\left(|N| \ge x/\sqrt{n}\right)\,\mathrm{d}x \right| = 0. \tag{3.7}$$

Proof.

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\frac{1}{-\log\varepsilon}\sum_{n=a(\varepsilon)+1}^{\infty}\frac{1}{n^2}\left|\int_{\varepsilon n}^{\infty}2xP\left(|S_n|\geq x\right)\,\mathrm{d}x-\int_{\varepsilon n}^{\infty}2xP\left(|N|\geq x/\sqrt{n}\right)\,\mathrm{d}x\right|\\
\leq\ &\lim_{\varepsilon\searrow 0}\frac{1}{-\log\varepsilon}\sum_{n=a(\varepsilon)+1}^{\infty}\int_{0}^{\infty}2(x+\varepsilon)P\left(|S_n|\geq n(x+\varepsilon)\right)\,\mathrm{d}x\\
&+\frac{1}{-\log\varepsilon}\sum_{n=b(\varepsilon)+1}^{\infty}\int_{0}^{\infty}2(x+\varepsilon)P\left(|N|\geq\sqrt{n}(x+\varepsilon)\right)\,\mathrm{d}x\\
=:\ &I_1+I_2.
\end{aligned}$$

For I_1, by (3.2) and Markov's inequality, we have

$$\begin{aligned}
I_1&\leq\lim_{\varepsilon\searrow 0}\frac{1}{-\log\varepsilon}\sum_{n=a(\varepsilon)+1}^{\infty}\frac{1}{n^2}\int_{\varepsilon n}^{\infty}\frac{n^{\frac{3}{2}}}{x^2}\,\mathrm{d}x\\
&\leq\frac{C}{-\log\varepsilon}\sum_{n=a(\varepsilon)+1}^{\infty}\frac{1}{\varepsilon n^{\frac{3}{2}}}\\
&\leq\frac{C}{-\log\varepsilon}\frac{1}{\sqrt{a(\varepsilon)+1}}\to 0,\qquad\text{as}\quad\varepsilon\searrow 0.
\end{aligned}$$

For I_2, in view of Markov's inequality, we have

$$\begin{aligned}
I_2&\leq\lim_{\varepsilon\searrow 0}\frac{C}{-\log\varepsilon}\sum_{n=a(\varepsilon)+1}^{\infty}\frac{1}{n^2}\int_{\varepsilon n}^{\infty}\frac{n^2}{x^3}\,\mathrm{d}x\\
&\leq\frac{C}{-\log\varepsilon}\sum_{n=a(\varepsilon)+1}^{\infty}\frac{1}{\varepsilon^2 n^2}\\
&\leq\frac{C}{-\varepsilon^2\log\varepsilon}\frac{1}{a(\varepsilon)+1}\to 0,\qquad\text{as}\quad\varepsilon\searrow 0.
\end{aligned}$$

Thus, we complete the proof of the proposition. □

Now, Theorem 1.4 follows from the propositions.

4. Proof of Theorem 1.5

In this section, we set $b(\varepsilon) = \left[\varepsilon^{\frac{-2p}{2-p}}\right]$ for $0 < \varepsilon < 1$, and $1 \le p < 2$. When taking $q = 0$, we have

$$\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n \ge 1} n^{\frac{r}{p}-2} P\big(|S_n| \ge \varepsilon n^{\frac{1}{p}}\big) = \frac{p}{r-p} E|Z|^{\frac{2(r-p)}{2-p}}.$$

This is the result of Li [7]. Thus we only discuss the case $0 < p < 2$. Note that

$$\begin{aligned}&\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{q}{p}} E|S_n|^q I\left\{|S_n| \ge \varepsilon n^{\frac{1}{p}}\right\} \\ &= \varepsilon^q \sum_{n=1}^{\infty} n^{\frac{r}{p}-2} P\left(|S_n| \ge \varepsilon n^{\frac{1}{p}}\right) + \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{q}{p}} \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} q x^{q-1} P\big(|S_n| \ge x\big)\,\mathrm{d}x.\end{aligned}$$

Via the result of Li [7]. Thus, in order to prove (1.4), it suffices to show

$$\begin{aligned}\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}-q} \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{q}{p}} \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} q x^{q-1} P\big(|S_n| \ge x\big)\,\mathrm{d}x \\ = \frac{pq(2-p)}{(r-p)(2r-2p-2q+pq)} E|N|^{\frac{2(r-p)}{2-p}}.\end{aligned} \tag{4.1}$$

This will be proved by the following propositions.

Proposition 4.1. *One has*

$$\begin{aligned}\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}-q} \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{q}{p}} \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} q x^{q-1} P\left(|N| \ge \frac{x}{\sqrt{n}}\right)\mathrm{d}x \\ = \frac{2p}{2r-2p-2q+pq} E|Z|^{\frac{2(r-p)}{2-p}} E|N|^{\frac{2(r-p)}{2-p}}.\end{aligned}$$

Proof. Via the change of variation, we have

$$\begin{aligned}&\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}-q} \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{q}{p}} \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} q x^{q-1} P\left(|N| \ge \frac{x}{\sqrt{n}}\right)\mathrm{d}x \\ &= \lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}-q} \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{q}{p}+\frac{q}{2}} \int_{\varepsilon n^{\frac{1}{p}-\frac{1}{2}}}^{\infty} q x^{q-1} P\big(|N| \ge t\big)\,\mathrm{d}t\end{aligned}$$

$$= \lim_{\varepsilon\searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}-q} \int_1^\infty x^{\frac{r}{p}-2-\frac{q}{p}+\frac{q}{2}}\,\mathrm{d}x \int_{\varepsilon x^{\frac{1}{p}-\frac{1}{2}}}^\infty qx^{q-1}P\big(|N|\geq t\big)\,\mathrm{d}t$$

$$= \frac{2pq}{2-p}\int_0^\infty y^{\frac{2(r-p)}{2-p}-q-1}\,\mathrm{d}y \int_y^\infty t^{q-1}P\big(|N|\geq t\big)\,\mathrm{d}t$$

$$= \frac{2p}{2-p}\frac{q}{\frac{2(r-p)}{2-p}-q}\int_0^\infty t^{\frac{2(r-p)}{2-p}-1}P\big(|N|\geq t\big)\,\mathrm{d}t$$

$$= \frac{2p}{2r-2p-2q+pq}E|Z|^{\frac{2(r-p)}{2-p}}E|N|^{\frac{2(r-p)}{2-p}}.$$

This completes the proof of Proposition 4.1. □

Proposition 4.2. *For $M>1$, one has*

$$\lim_{\varepsilon\searrow 0}\varepsilon^{\frac{2(r-p)}{2-p}-q}\sum_{n=1}^{Mb(\varepsilon)} n^{\frac{r}{p}-2-\frac{q}{p}}\left|\int_{\varepsilon n^{\frac{1}{p}}}^\infty qx^{q-1}P\left(|S_n|\geq x\right)\mathrm{d}x - \int_{\varepsilon n^{\frac{1}{p}}}^\infty qx^{q-1}P\left(|N|\geq x/\sqrt{n}\right)\mathrm{d}x\right| = 0.$$

P r o o f. It is easy to see that

$$\sum_{n=1}^{Mb(\varepsilon)} n^{\frac{r}{p}-2-\frac{q}{p}}\left|\int_{\varepsilon n^{\frac{1}{p}}}^\infty qx^{q-1}P\left(|S_n|\geq x\right)\mathrm{d}x - \int_{\varepsilon n^{\frac{1}{p}}}^\infty qx^{q-1}p\left(|N|\geq x/\sqrt{n}\right)\mathrm{d}x\right|$$

$$\leq \sum_{n=1}^{Mb(\varepsilon)} n^{\frac{r}{p}-2}\int_0^\infty q(x+\varepsilon)^{q-1}\left|p\left(|S_n|\geq (x+\varepsilon)n^{\frac{1}{p}}\right) - p\left(|N|\geq (x+\varepsilon)n^{\frac{1}{p}-\frac{1}{2}}\right)\right|\mathrm{d}x$$

$$\leq \sum_{n=1}^{Mb(\varepsilon)} n^{\frac{r}{p}-2-\frac{q(2-p)}{2p}}(\Delta'_{n1}+\Delta'_{n2}),$$

where

$$\Delta'_{n1} = n^{\frac{q(2-p)}{2p}} \int_0^{\frac{1}{n^{\frac{2-p}{2p}}\Delta_n^{1/2q}}} q(x+\varepsilon)^{q-1} \Big| p\left(|S_n| \geq (x+\varepsilon)n^{\frac{1}{p}}\right) - p\left(|N| \geq (x+\varepsilon)n^{\frac{1}{p}-\frac{1}{2}}\right)\Big| \,\mathrm{d}x,$$

$$\Delta'_{n2} = n^{\frac{q(2-p)}{2p}} \int_{\frac{1}{n^{\frac{2-p}{2p}}\Delta_n^{1/2q}}}^{\infty} q(x+\varepsilon)^{q-1} \Big| P\left(|S_n| \geq (x+\varepsilon)n^{\frac{1}{p}}\right) - p\left(|N| \geq (x+\varepsilon)n^{\frac{1}{p}-\frac{1}{2}}\right)\Big| \,\mathrm{d}x.$$

Since $n \leq Mb(\varepsilon)$ implies $\varepsilon n^{\frac{2-p}{2p}} \leq M^{\frac{2-p}{2p}}$, one can easily obtain that

$$\Delta'_{n1} \leq \Delta_n n^{\frac{q(2-p)}{2p}} \left(\frac{1}{n^{\frac{2-p}{2p}}\Delta_n^{1/2q}} + \varepsilon\right)^q \leq \left(\Delta_n^{\frac{1}{2q}} + M^{\frac{2-p}{2p}}\Delta_n^{\frac{1}{q}}\right)^p. \tag{4.2}$$

By Markov's inequality, we have

$$\Delta'_{n2} \leq Cn^{\frac{q(2-p)}{2p}} \int_{\frac{1}{n^{\frac{2-p}{2p}}\Delta_n^{1/2q}}}^{\infty} \frac{1}{(x+\varepsilon)^{3-q}n^{\frac{2}{p}}} \,\mathrm{d}x \leq C\Delta_n^{\frac{1}{q}-\frac{1}{2}}. \tag{4.3}$$

Denote $\triangle'_n = \Delta'_{n1} + \Delta'_{n2}$, it follows that

$$m^{-\frac{r}{p}+1+\frac{q(2-p)}{2p}} \sum_{n=1}^{m} n^{\frac{r}{p}-2-\frac{q(2-p)}{2p}} \triangle'_n \to 0, \qquad \text{as} \quad m \to \infty.$$

We have

$$\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}-q} \sum_{n=1}^{Mb(\varepsilon)} n^{\frac{r}{p}-2-\frac{q}{p}} \Big| \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} qx^{q-1}P\left(|S_n| \geq x\right) \,\mathrm{d}x - \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} qx^{q-1}P\left(|N| \geq x/\sqrt{n}\right) \,\mathrm{d}x \Big|$$

$$= \lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}-q} \sum_{n=1}^{Mb(\varepsilon)} n^{\frac{r}{p}-2-\frac{q(2-p)}{2p}} \triangle'_n$$

$$\leq \lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}-q} [Mb(\varepsilon)]^{\frac{r}{p}-1-\frac{q(2-p)}{2p}} [Mb(\varepsilon)]^{-\frac{r}{p}+1+\frac{q(2-p)}{2p}} \sum_{n=1}^{Mb(\varepsilon)} n^{\frac{r}{p}-2-\frac{q(2-p)}{2p}} \triangle'_n = 0.$$

Then this proposition is proved. □

Proposition 4.3. *For $M > 1$, one has*

$$\lim_{M\to\infty} \lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}-q} \sum_{n>Mb(\varepsilon)} n^{\frac{r}{p}-2-\frac{q}{p}} \left| \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} q x^{q-1} P\big(|S_n| \geq x\big)\, \mathrm{d}x - \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} q x^{q-1} P\left(|N| \geq x/\sqrt{n}\right)\, \mathrm{d}x \right| = 0.$$

P r o o f. Observe that

$$E|S_n|^2 \leq CE\bigg(\sum_{i=-\infty}^{\infty} a_{ni}^2 \varepsilon_i^2\bigg) \leq Cn.$$

Thus, by Markov's inequality, we have

$$\begin{aligned}
&\varepsilon^{\frac{2(r-p)}{2-p}-q} \sum_{n>Mb(\varepsilon)} n^{\frac{r}{p}-2-\frac{q}{p}} \left| \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} q x^{q-1} P\left(|S_n| \geq x\right)\, \mathrm{d}x - \int_{\varepsilon n^{\frac{1}{p}}}^{\infty} q x^{q-1} P\left(|N| \geq x/\sqrt{n}\right)\, \mathrm{d}x \right| \\
&\leq C\varepsilon^{\frac{2(r-p)}{2-p}-q} \sum_{n>Mb(\varepsilon)} n^{\frac{r}{p}-2-\frac{q}{p}} \int_{\varepsilon n^{\frac{1}{p}}} \frac{n}{x^{3-q}}\, \mathrm{d}x \\
&\leq C\varepsilon^{\frac{2(r-p)}{2-p}-2} \sum_{n>Mb(\varepsilon)} n^{\frac{r}{p}-\frac{2}{p}-1} \\
&\leq CM^{\frac{r-2}{p}} \to 0, \qquad \text{as} \quad M \to \infty.
\end{aligned}$$

We complete the proof of this proposition. □

Our main result now follows from the propositions via triangle inequality.

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School of Mathematical science
Huaiyin normal university
Huaian 223300
CHINA
E-mail: zqphunhu@yahoo.com.cn