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REALIZING COHOMOLOGY CLASSES AS EULER CLASSES

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ABSTRACT. For a space X, let $E_k(X)$, $E_k^s(X)$ and $E_k^\circ(X)$ denote respectively the set of Euler classes of oriented k-plane bundles over X, the set of Euler classes of stably trivial k-plane bundles over X and the spherical classes in $H^k(X;\mathbb{Z})$. We prove some general facts about the sets $E_k(X)$, $E_k^s(X)$ and $E_k^\circ(X)$. We also compute these sets in the cases where X is a projective space, the Dold manifold $P(m,1)$ and obtain partial computations in the case that X is a product of spheres.

1. Introduction

Given a topological space X, we address the question of which classes $x \in H^k(X;\mathbb{Z})$ can be realized as the Euler class $e(\xi)$ of an oriented k-plane bundle ξ over X. We also look at the question of which integral cohomology classes are spherical. It is convenient to make the following definition.

Definition 1.1. For a space X and an integer $k \geq 1$, the sets $E_k(X)$, $E_k^s(X)$ and $E_k^\circ(X)$ are defined to be

$$E_k(X) = \big\{e(\xi) \in H^k(X;\mathbb{Z}) \mid \xi \text{ is an oriented } k\text{-plane bundle over } X\big\}$$
$$E_k^s(X) = \big\{e(\xi) \in H^k(X;\mathbb{Z}) \mid \xi \text{ is a stably trivial } k\text{-plane bundle over } X\big\}$$
$$E_k^\circ(X) = \big\{x \in H^k(X;\mathbb{Z}) \mid \text{there exists } f\colon S^k \longrightarrow X \text{ with } f^*(x) \neq 0\big\}.$$

Note that we always have an inclusion $E_k^s(X) \subseteq E_k(X)$ and the inclusion can be strict. The classes in $E_k^\circ(X)$ are called spherical classes. Clearly, spherical

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classes are of infinite order and are indecomposable as elements of the integral cohomology ring.

In recent times there has been some interest in understanding the sets E_k (see [5], [6], [14]). Besides being a natural question to study, part of the motivation for studying the sets E_k and E_k° comes from the following result.

Theorem 1.2. ([5, Theorem 1.3, p. 378]) *Let ξ be an oriented k-plane bundle over a CW-complex X. If the Euler class $e(\xi)$ is spherical, then the holonomy group of any Riemannian connection on ξ acts transitively on the sphere bundle $S(\xi)$ of ξ.*

There are isolated results in the literature about the set $E_k(X)$ (see [5], [6], [14]). The most general result about realizing cohomology classes as Euler classes seems to be the following theorem of Guijarro, Schick and Walschap [5].

Theorem 1.3. ([5, Theorem 1.6, p. 379]) *Given $k, n \in \mathbb{N}$ with k even, there is an integer $N(k,n) > 0$ such that for every CW-complex X of dimension n and every cohomology class $x \in H^k(X;\mathbb{Z})$, there is an oriented k-plane bundle ξ over X with $e(\xi) = 2N(k,n) \cdot x$.*

Thus, in our notation, under the hypothesis of the above theorem

$$2N(k,n)H^k(X;\mathbb{Z}) \subseteq E_k(X).$$

Note that, with the hypothesis as in the above theorem, we are guaranteed that (for k even) if $E_k^\circ(X) \neq \emptyset$, then $E_k^\circ(X) \cap E_k(X) \neq \emptyset$. Thus the existence of a spherical class implies that some spherical class is also the Euler class of some oriented k-plane bundle. In this paper we shall show that the hypothesis that k is even is essential (see Theorem 1.5 below).

In this paper we shall compute the sets $E_k(X)$, $E_k^\circ(X)$ and $E_k^s(X)$ for the cases when X is a projective space, a product $S^m \times S^n$ of spheres for certain values of k, m, n (see Theorem 1.4 below) and the Dold manifold $P(m,1)$ with $m > 1$. The paper is organized as follows.

In Section 2 we discuss some general properties of the sets E_k, E_k° and E_k^s.

Section 3 contains the computational part of the paper. We first describe the sets $E_k(X)$, $E_k^\circ(X)$ and $E_k^s(X)$ when X is a projective space. The description of these sets when X is the real projective space is arrived at by looking at certain canonical bundles over X. The case when X is a complex projective space follows from a general result that we prove about spaces whose cohomology ring is generated by the second cohomology (see Proposition 3.2).

Section 3.3 deals with the computation of the sets $E_k(X)$, $E_k^\circ(X)$ and $E_k^s(X)$ when $X = S^m \times S^n$ is a product of two spheres with restrictions on k, m, n (see Theorem 1.4 below for the precise statement). It is a classical result of Milnor [9] and Atiyah-Hirzebruch [1] that if $n \neq 2, 4, 8$ is even, then $E_n(S^n) = 2H^n(S^n;\mathbb{Z})$ (see also [14, Theorem 1.2] for a geometric proof of this fact). In particular, in these cases a generator of $H^n(S^n;\mathbb{Z})$ is never an Euler class. The main theorem of this section is the following.

Theorem 1.4. *Let $X = S^m \times S^n$.*

(1) *If $m, n \equiv 3 \pmod 8$, then $E_k(X) = 0$ for $1 \le k < m+n$ and $E^s_{m+n}(X) = E_{m+n}(X) = 2H^{m+n}(X;\mathbb{Z})$.*

(2) *If n is even, and $n \neq m$, then*
$$E_n(X) = \begin{cases} 2H^n(X;\mathbb{Z}) & \text{if } n \neq 2,4,8 \\ H^n(X;\mathbb{Z}) & \text{if } n = 2,4,8 \end{cases}$$

(3) *If $m = 1$ and $n \equiv 5 \pmod 8$, then $E^s_{n+1}(X) = E_{n+1}(X) = 2H^{n+1}(X;\mathbb{Z})$.*

Note that the conclusion $2H^{m+n}(X;\mathbb{Z}) = E_{m+n}(X)$ in the cases (1) and (3) in the above theorem are not true in general. Indeed, if $X = S^3 \times S^5$, then the inclusion $2H^8(X;\mathbb{Z}) \subsetneq E_8(X)$ is strict. We shall make a more general observation later (see Example 2.10 below).

Finally, in Section 3.9 we discuss the computation of the sets E_k, E°_k and E^s_k when X is the Dold manifold $P(m,1)$ with $m > 1$. We give a complete description of the sets $E_k(P(m,1))$ except in the case when m is even and $k = m+2$ (see Proposition 3.13). The computations depend upon the existence of certain canonical bundles over the Dold manifolds (see [12], [13]). It follows from our computations that the assumption that k is even in Theorem 1.3 is essential. The main theorem of this section is the following.

Theorem 1.5. *Let $m > 1$ be an odd integer. Then we have*
$$E^\circ_m(P(m,1)) \neq \emptyset, \quad E_m(P(m,1)) \neq 0 \quad \text{and} \quad E^\circ_m(P(m,1)) \cap E_m(P(m,1)) = \emptyset.$$

2. Generalities

In this section we prove some general facts about the sets E_k, E^s_k and E°_k. Throughout, we follow the notations in [10].

Recall that if ξ is an oriented k-plane bundle over X, then its Euler class $e(\xi)$ is an element of $H^k(X;\mathbb{Z})$. Let $u_\xi \in H^k(E(\xi), E(\xi)_0;\mathbb{Z})$ be the Thom class of ξ and $\varphi\colon H^k(X;\mathbb{Z}) \longrightarrow H^{2k}(E(\xi), E(\xi)_0;\mathbb{Z})$ be the Thom isomorphism, then the Euler class $e(\xi)$ of ξ is by definition $e(\xi) := \varphi^{-1}(u_\xi \smile u_\xi)$. If k is odd, then the (graded) commutativity of the cup product shows that $2e(\xi) = 0$. For an oriented k-plane bundle ξ, the mod 2 reduction of the Euler class $e(\xi)$ equals the top Stiefel-Whitney class $w_k(\xi)$ of ξ.

Given a space X, we have inclusions $E^s_k(X) \subseteq E_k(X) \subseteq H^k(X;\mathbb{Z})$ and in general all the inclusions can be strict. The set $E_k(X)$ is inverse closed as changing the orientation changes the sign of the Euler class. Also, $0 \in E_k(X)$

and hence the set $E_k(X)$ is non empty. On the other hand, as $E_k^\circ(X)$ consists of spherical classes, $0 \notin E_k^\circ(X)$. It is well known that a real line bundle is orientable if and only if it is trivial. Thus for any space X the set $E_1(X)$ is trivial (meaning $E_1(X) = \{0\}$, which will simply be denoted by $E_1(X) = 0$).

We first note what is known about the set $E_k(S^n)$. We begin by recalling the following result.

Theorem 2.1. ([1], [9], [14, Theorem 1.2]) *If $n \neq 2, 4, 8$ is even, then the Euler class of any n-plane bundle over S^n is an even multiple of a generator of $H^n(S^n; \mathbb{Z})$.*

In particular, the above theorem implies that if n is even and $n \neq 2, 4, 8$, then $E_n(S^n) \subseteq 2H^n(S^n; \mathbb{Z})$. The set $E_k(S^n)$ for the spheres can now be described completely.

Example 2.2. If n is odd then clearly $E_n(S^n) = 0$. It is known that Euler classes (of the underlying real bundle) of the canonical (complex, quaternionic and octonionic) line bundles over $S^2 = \mathbb{CP}^1$, $S^4 = \mathbb{HP}^1$ and $S^8 = \mathbb{OP}^1$ are generators of $H^2(S^2; \mathbb{Z})$, $H^4(S^4; \mathbb{Z})$, and $H^8(S^8; \mathbb{Z})$ respectively. Since there are maps $f\colon S^n \longrightarrow S^n$ of arbitrary degrees and as the Euler class is natural, it follows that $E_n(S^n) = H^n(S^n; \mathbb{Z})$ if $n = 2, 4, 8$. It is well known (see [10]) that if n is even, then the Euler class of the tangent bundle of S^n is twice a generator. Thus if n is even, then every element of $2H^n(S^n; \mathbb{Z})$ is the Euler class of some n-plane bundle over S^n. In other words, $2H^n(S^n; \mathbb{Z}) \subseteq E_n(S^n)$. Together with Theorem 2.1, this implies that $E_n(S^n) = 2H^n(S^n; \mathbb{Z})$ if n is even and $n \neq 2, 4, 8$. Thus, in this case, the generator of $H^n(S^n; \mathbb{Z})$ does not occur as an Euler class of some bundle over S^n.

We begin by making some easy observations. First note that if $f\colon X \longrightarrow Y$ is a continuous map, then as the Euler class is natural it follows that $f^*(E_k(Y)) \subseteq E_k(X)$ and $f^*(E_k^s(Y)) \subseteq E_k^s(X)$.

Lemma 2.3. *Let X be a topological space and A a retract of X.*

(1) *If X is paracompact, then we have $E_1(X) = 0$ and $E_2(X) = H^2(X; \mathbb{Z})$.*

(2) *If $E_k(A) \neq 0$, then $E_k(X) \neq 0$.*

Proof. As noted before, the equivalence of orientability and triviality for line bundles implies $E_1(X) = 0$. Let $\mathrm{Vect}^1_{\mathbb{C}}(X)$ denote the group of isomorphism classes of complex line bundles over X. That $E_2(X) = H^2(X; \mathbb{Z})$ follows from the fact that the first Chern class map

$$c_1\colon \mathrm{Vect}^1_{\mathbb{C}}(X) \longrightarrow H^2(X; \mathbb{Z})$$

is an isomorphism. This completes the proof of (1).

To proof of (2) is routine and is therefore omitted. $\square$

Part (2) of the above lemma remains true with E_k replaced by E_k^s and E_k°.

The nature of the sets $E_n(S^n)$ of the spheres forces similar restrictions on the sets $E_n(X)$ of n-manifolds and more generally on that of n-dimensional CW-complexes.

Recall ([7, Corollary 3.28, p. 238]) that if X is a connected closed orientable n-manifold, then $H_{n-1}(X;\mathbb{Z})$ is torsion free and hence the top dimensional integral cohomology is infinite cyclic, whereas if X is nonorientable, then $H_n(X;\mathbb{Z}) = 0$ and the torsion subgroup of $H_{n-1}(X;\mathbb{Z})$ is cyclic of order two and hence the top dimensional integral cohomology is also cyclic of order two.

Theorem 2.4. *Let X be a connected, closed n-manifold.*

(1) *Let X be orientable. Then $E_n(X) = 0$ if n is odd. If $n \in \{2,4,8\}$ then $E_n(X) = H^n(X;\mathbb{Z})$. If n is even and $n \neq 2,4,8$, then $2H^n(X;\mathbb{Z}) \subseteq E_n(X)$, and in this case, equality holds if and only if for any oriented n-plane bundle ξ over X, the top Stiefel-Whitney class $w_n(\xi) = 0$.*

(2) *If X is nonorientable, then, $E_n(X) = H^n(X;\mathbb{Z})$ if and only if there exists an orientable n-plane bundle ξ over X with $w_n(\xi) \neq 0$.*

Proof. Assuming X is orientable, $H^n(X;\mathbb{Z}) = \mathbb{Z}$. If n is odd, it is clear that $E_n(X) = 0$ as $H^n(X;\mathbb{Z})$ is torsion free. So assume that n is even. Let $f\colon X \longrightarrow S^n$ be a degree 1 map, then the homomorphism $f_*\colon H_n(X;\mathbb{Z}) \longrightarrow H_n(S^n;\mathbb{Z})$ is an isomorphism. As $H_{n-1}(X;\mathbb{Z})$ is torsion free the top dimensional cohomology is the dual of the top dimensional homology and hence the homomorphism $f^*\colon H^n(S^n;\mathbb{Z}) \longrightarrow H^n(X;\mathbb{Z})$ in cohomology is also an isomorphism. In view of Example 2.2, if $n \in \{2,4,8\}$ it follows that

$$E_n(X) \supseteq f^*(E_n(S^n)) = f^*(H^n(S^n;\mathbb{Z})) = H^n(X;\mathbb{Z}).$$

If n is even and $n \notin \{2,4,8\}$, then

$$E_n(X) \supseteq f^*(E_n(S^n)) = f^*(2H^n(S^n;\mathbb{Z})) = 2H^n(X;\mathbb{Z})$$

as f^* is an isomorphism. Finally, it is clear that $2H^n(X;\mathbb{Z}) = E_n(X)$ if and only if for any oriented n-plane bundle ξ, $w_n(\xi) = 0$. This proves (1).

Finally if X is a closed, nonorientable n-manifold, then $H^n(X;\mathbb{Z}) \cong \mathbb{Z}_2 \cong H^n(X;\mathbb{Z}_2)$. Thus, $E_n(X) = H^n(X;\mathbb{Z})$ if and only if there exists an orientable n-plane bundle ξ with $w_n(\xi) \neq 0$. This completes the proof of the theorem. $\square$

A similar statement holds in the case when X is an even dimensional CW-complex. The proof is an application of the classical Hopf classification theorem.

Proposition 2.5. *Suppose X is an n-dimensional CW-complex with n even. If $n \in \{2,4,8\}$, then $E_n(X) = H^n(X;\mathbb{Z})$. In all other cases $2H^n(X;\mathbb{Z}) \subseteq E_n(X)$.*

Proof. It is well known, by the Hopf classification theorem, for example, that there is an bijection $[X, S^n] \leftrightarrow H^n(X;\mathbb{Z})$ given by $[f] \mapsto f^*(u)$ where $u \in H^n(S^n;\mathbb{Z})$ is a generator. Thus, given a cohomology class $e \in H^n(X;\mathbb{Z})$, there exists a map $f_e \colon X \longrightarrow S^n$ with $f_e^*(u) = e$. If $n \in \{2,4,8\}$, then $u = e(\xi)$ for some orientable n-plane bundle over S^n and hence $e = f_e^*(u) = f_e^*(e(\xi)) = e(f_e^*(\xi))$. Consequently, $E_n(X) = H^n(X;\mathbb{Z})$.

Now suppose that $n \notin \{2,4,8\}$ is even. Let $e \in H^n(X;\mathbb{Z})$. As $2H^n(S^n;\mathbb{Z}) = E_n(S^n)$, it follows that $2u \in E_n(S^n)$ and hence $2e = f_e^*(2u) \in E_n(X)$. Thus $2H^n(X;\mathbb{Z}) \subseteq E_n(X)$. This completes the proof. □

Remark 2.6. The inclusions $2H^n(X;\mathbb{Z}) \subseteq E_n(X)$ above can be strict. For example, if X is a smooth, closed oriented n-manifold with the top Stiefel-Whitney class $w_n(X) \neq 0$, then the inclusion $2H^n(X;\mathbb{Z}) \subseteq E_n(X)$ is always strict in view of Theorem 2.4(1). As a concrete example consider $X = \mathbb{CP}^n$ with n even. Then $w_{2n}(X) \neq 0$ and hence the inclusion $2H^{2n}(X;\mathbb{Z}) \subseteq E_{2n}(X)$ is strict.

Recall that a k-plane bundle ξ is stably trivial if the Whitney sum $\xi \oplus \epsilon^n$ is trivial for some n. Here ϵ^n denotes the trivial n-plane bundle.

Proposition 2.7. *Let X be a topological space.*

1. *Let ξ be a stably trivial k-plane bundle over X. Then $e(\xi) \in 2H^k(X;\mathbb{Z})$. Thus, if every orientable k-plane bundle ξ over X is stably trivial, then $E_k^s(X) = E_k(X) \subseteq 2H^k(X;\mathbb{Z})$.*
2. *If X is compact and $\widetilde{KO}(X) = 0$, then $E_k^s(X) = E_k(X) \subseteq 2H^k(X;\mathbb{Z})$ for all $k \geq 1$.*

Proof. As ξ is stably trivial the Whitney sum $\xi \oplus \epsilon^n$ is trivial for some n. Thus, by the Whitney product theorem, the total Stiefel-Whitney class $w(\xi) = 1$. Hence the top Stiefel-Whitney class $w_k(\xi) = 0$. As the Euler class $e(\xi)$ reduced mod 2 is $w_k(\xi)$, it follows that $e(\xi) \in 2H^k(X;\mathbb{Z})$. This proves (1).

Now if X is compact, then the stable equivalence classes of vector bundles over X can be identified with the reduced (real) K-theory $\widetilde{KO}(X)$ of X [8]. Thus if $\widetilde{KO}(X) = 0$, then every vector bundle over X is stably trivial and (2) follows from (1). □

Remark 2.8. We remark that the converse to Proposition 2.7(2) is not true. Let $X = S^{16}$. Note that $E_k(X) \subseteq 2H^k(X;\mathbb{Z})$ for all $k \geq 1$. However, $\widetilde{KO}(X) \cong \mathbb{Z}$. Lemma 2.9(3) below shows that $E_k^s(X) = E_k(X)$.

Lemma 2.9. *Let X be a space. Then*

(1) $E^s_k(X) \subseteq 2H^k(X;\mathbb{Z})$.

(2) *If X is a closed, connected, nonorientable n-manifold, then $E^s_n(X) = 0$.*

(3) *If n is even and X is a connected, closed, oriented n-manifold, then $E^s_n(X) = 2H^n(X;\mathbb{Z})$.*

Proof. (1) follows from Proposition 2.7. Part (2) follows from (1) since for such a manifold $H^n(X;\mathbb{Z}) \cong \mathbb{Z}_2$.

To prove (3), we first look at the case when $X = S^n$. By (1), $E^s_n(S^n) \subseteq 2H^n(S^n;\mathbb{Z})$. The tangent bundle τ of S^n is stably trivial and its Euler class $e(\tau)$ is twice a generator. Pulling back the tangent bundle by a self map of degree d gives us a stably trivial bundle over S^n with Euler class $2d$ times a generator. Thus for an even dimensional sphere S^n, $E^s_n(S^n) = 2H^n(S^n;\mathbb{Z})$.

Now suppose that X is a connected, closed, oriented n-manifold. Let $f\colon X \longrightarrow S^n$ be a degree 1 map. Then as $f^*\colon H^n(S^n;\mathbb{Z}) \longrightarrow H^n(X;\mathbb{Z})$ is an isomorphism we see that

$$2H^n(X;\mathbb{Z}) = f^*(2H^n(S^n;\mathbb{Z})) = f^*(E^s_n(S^n)) \subseteq E^s_n(X).$$

Thus, by (1), $E^s_n(X) = 2H^n(X;\mathbb{Z})$. This completes the proof. □

Note that if n is odd, then $E^s_n(S^n) = E_n(S^n) = 0$. The following example shows that the sets $E_k(X)$ and $E^s_k(X)$ need not be equal when k is even.

Example 2.10. Let X be a closed, connected orientable n manifold with $n \in \{2,4,8\}$. Then, by Theorem 2.4 and Lemma 2.9

$$E_n(X) = H^n(X;\mathbb{Z}) \neq 2H^n(X;\mathbb{Z}) = E^s_n(X).$$

We do not know of such an example in the cases $n \notin \{2,4,8\}$.

We shall consider computations of the sets E_k, E^s_k and E°_k in the next section. It turns out that whenever we have a complete description of the set E_k, it is always a subgroup of H^k. We do not know of a single example where it is not. In this context we mention the proposition below. Although the proof is elemental, the result is not needed in the paper, and therefore not proved.

Proposition 2.11. *Let X be a pointed space with a pointed co-multiplication $\mu\colon X \longrightarrow X \vee X$. Then, $E_k(X)$ is a subgroup of $H^k(X;\mathbb{Z})$ for all $k \geq 1$.*

It would be interesting to decide whether or not $E_k(X)$ is always a subgroup of $H^k(X;\mathbb{Z})$.

We end this section by proving some general results about spherical classes. Given a k-plane bundle ξ over X, let

$$S^{k-1} \hookrightarrow S(\xi) \longrightarrow X$$

denote the associated sphere bundle with fiber the sphere S^{k-1}. The following is implicit in the results of [5].

Proposition 2.12. *Let X be a n-dimensional CW-complex and $k \in \mathbb{N}$ a fixed even integer. If the homotopy group $\pi_k(X)$ is finite, then $E_k^\circ(X) = \emptyset$.*

Proof. If $x \in E_k^\circ(X)$, then by Theorem 1.3, $2N(k,n) \cdot x = e(\xi)$ for some oriented k-plane bundle ξ over X. Note that, in this case, $e(\xi)$ is also spherical. As $\pi_{k-1}(S^{k-1}) \cong \mathbb{Z}$, the connecting homomorphism in the exact homotopy sequence

$$\cdots \longrightarrow \pi_k(S^{k-1}) \longrightarrow \pi_k(S(\xi)) \longrightarrow \pi_k(X) \longrightarrow \pi_{k-1}(S^{k-1}) \longrightarrow \cdots$$

of the associated sphere bundle $S^{k-1} \hookrightarrow S(\xi) \longrightarrow X$ is clearly zero. Therefore, for any map $f\colon S^k \longrightarrow X$, we have $f^*(e(\xi)) = 0$. This contradiction completes the proof. □

Remark 2.13. The above idea is used in [5] to prove that whether k is even or odd, and whether $\pi_k(X)$ is finite or infinte, if the connecting homomorphism

$$\pi_k(X) \longrightarrow \pi_{k-1}(S^{k-1})$$

in the exact homotopy sequence of the associated sphere bundle of an oriented k-plane bundle ξ is zero, then $e(\xi)$ is not spherical. Thus, under these assumptions, we always have $E_k^\circ(X) \cap E_k(X) = \emptyset$. As an extreme situation consider $X = \mathbb{CP}^n$, where $E_{2n}^\circ(X) = \emptyset$ but $E_{2n}(X) = H^{2n}(X;\mathbb{Z})$ (see Proposition 3.2).

Lemma 2.14. *Let X be a space with finitely generated integral homology in dimensions k and $(k-1)$ for some $k \geq 1$.*

(1) *If $H_k(X;\mathbb{Z})$ is torsion, then $E_k^\circ(X) = \emptyset$,*

(2) *If the Hurewicz homomorphism*

$$\varphi\colon \pi_k(X) \longrightarrow H_k(X,\mathbb{Z})$$

is the zero homomorphism, then $E_k^\circ(X) = \emptyset$.

(3) *If $H_k(X;\mathbb{Z})$ is not torsion and the Hurewicz homomorphism*

$$\varphi\colon \pi_k(X) \longrightarrow H_k(X;\mathbb{Z})$$

has finite cokernel, then $E_k^\circ(X)$ equals the elements of infinite order in $H^k(X;\mathbb{Z})$.

Proof. If $H_k(X;\mathbb{Z})$ is torsion, then by the universal coefficient theorem, so is $H^k(X;\mathbb{Z})$. Since spherical classes are of infinite order (1) follows. Next, let $e \in H^k(X;\mathbb{Z})$ and let $f\colon S^k \longrightarrow X$ be any map. We compute

$$\langle f^*(e), \sigma\rangle = \langle e, f_*(\sigma)\rangle = \langle e, \varphi[f]\rangle = \langle e, 0\rangle = 0,$$

where $\sigma \in H^k(S^k;\mathbb{Z})$ is a generator. This shows that $f^*(e) = 0$. This proves (2). Finally, note that as $H_k(X;\mathbb{Z})$ is not torsion, $H^k(X;\mathbb{Z})$ has elements of infinite order. Given $e \in H^k(X;\mathbb{Z})$ of infinite order, find $x \in H_k(X;\mathbb{Z})$ with $\langle e, x\rangle \neq 0$. Let n be such that $nx \in \operatorname{im}(\varphi)$. If $nx = f_*(\sigma)$ for some $f\colon S^k \longrightarrow X$ then the computation

$$\langle f^*(e), \sigma\rangle = \langle e, f_*(\sigma)\rangle = \langle e, nx\rangle \neq 0$$

shows that $f^*(e) \neq 0$. This proves (3). □

In particular, if X is $(k-1)$-connected with $k \geq 2$, then the Hurewicz map $\varphi\colon \pi_k(X) \longrightarrow H_k(X;\mathbb{Z})$ is an isomorphism and if $\pi_k(X)$ is not torsion then $E_k^\circ(X)$ equals the elements of infinite order in $H^k(X;\mathbb{Z})$.

Example 2.15. Let $k > 1$ be an odd integer. It is easy to check that the Hurewicz homomorphism

$$\varphi\colon \pi_k(\mathbb{RP}^k) \longrightarrow H_k(\mathbb{RP}^k;\mathbb{Z})$$

is multiplication by 2 between two infinite cyclic groups. Thus, by part (3) of the above proposition, $E_k^\circ(\mathbb{RP}^k)$ equals the nonzero elements in the group $H^k(\mathbb{RP}^k;\mathbb{Z}) \cong \mathbb{Z}$.

Example 2.16. Let $V_k(\mathbb{R}^{2k})$ denote the Stiefel manifold of orthonormal k-frames in $\mathbb{R}^{2k}$. Let k be even, then by [8, Proposition 11.2, p. 91], $V_k(\mathbb{R}^{2k})$ is $(k-1)$-connected and $\pi_k(V_k(\mathbb{R}^{2k})) \cong \mathbb{Z}$. Hence by the Hurewicz isomorphism theorem, the Hurewicz homomorphism

$$\varphi\colon \pi_k(V_k(\mathbb{R}^{2k})) \cong \mathbb{Z} \longrightarrow H_k(V_k(\mathbb{R}^{2k});\mathbb{Z})$$

is an isomorphism.

Thus, $E_k^\circ(V_k(\mathbb{R}^{2k}))$ equals the nonzero elements in $H^k(V_k(\mathbb{R}^{2k});\mathbb{Z}) \cong \mathbb{Z}$. Walschap [14] has shown that a generator of $H^k(V_k(\mathbb{R}^{2k});\mathbb{Z}) \cong \mathbb{Z}$ (with k even and $k \neq 2, 4, 8$) cannot be the Euler class of any oriented k-plane bundle over $V_k(\mathbb{R}^{2k})$.

3. Computations

In this section we shall determine the E_k, E_k^s and E_k° of real and complex projective spaces and the Dold manifolds $P(m,1)$. We begin with the real projective spaces.

Proposition 3.1. *Let $n \in \mathbb{N} \cup \{\infty\}$, then for real projective space $\mathbb{RP}^n$,*

(1) *if n is even, then $E_k(\mathbb{RP}^n) = H^k(\mathbb{RP}^n;Z)$, $E_k^s(\mathbb{RP}^n) = 0$ and $E_k^\circ(\mathbb{RP}^n) = \emptyset$ for all $k \geq 1$,*

(2) *if n is odd, then $E_k(\mathbb{RP}^n) = H^k(\mathbb{RP}^n;\mathbb{Z})$ and $E_k^\circ(\mathbb{RP}^n) = \emptyset$ for $1 \leq k < n$, $E_n(\mathbb{RP}^n) = 0$, $E_k^s(\mathbb{RP}^n) = 0$, and $E_n^\circ(\mathbb{RP}^n)$ equals the non trivial elements of $H^n(\mathbb{RP}^n;\mathbb{Z})$.*

(3) *if $n = \infty$, then $E_k(\mathbb{RP}^\infty) = H^k(\mathbb{RP}^\infty;\mathbb{Z})$, $E_k^s(\mathbb{RP}^\infty) = 0$ and $E_k^\circ(\mathbb{RP}^\infty) = \emptyset$ for all $k \geq 1$.*

P r o o f. Suppose n is even. Then the integral cohomology groups of the projective space are given by

$$H^k(\mathbb{RP}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}_2 & \text{if } k \text{ is even and } 1 < k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Thus if k is odd, then clearly $H^k(\mathbb{RP}^n;\mathbb{Z})$=0, so that the assertions about E_k, E_k^s and E_k° are obvious. So assume that k is even. Let γ_n^1 denote the canonical line bundle over $\mathbb{RP}^n$ and $k\gamma_n^1$ denote the k-fold Whitney sum of γ_n^1 with itself. Then clearly $k\gamma_n^1$ is an orientable k-plane bundle over $\mathbb{RP}^n$ and

$$w_k(k\gamma_n^1) = a^k \neq 0$$

if $k \leq n$. Here $a \in H^1(\mathbb{RP}^n\mathbb{Z}_2) \cong \mathbb{Z}_2$ is the unique nonzero element. Thus $e(k\gamma_n^1) \neq 0$. By Lemma 2.9, $E_k^s(\mathbb{RP}^n) = 0$ for all $k \geq 1$. Since all the cohomology groups are finite, $E_k^\circ(\mathbb{RP}^n) = \emptyset$ for all $k \geq 1$. This proves (1).

Now assume that n is odd. Then

$$H^k(\mathbb{RP}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0, n \\ \mathbb{Z}_2 & \text{if } k \text{ is even and } 1 < k < n \\ 0 & \text{otherwise.} \end{cases}$$

The proofs in this case are similar to the case (1). That $E_n(\mathbb{RP}^n) = 0$ follows from the fact $H^n(\mathbb{RP}^n;\mathbb{Z}) \cong \mathbb{Z}$. Example 2.15 shows that $E_n^\circ(\mathbb{RP}^n)$ has the required form. This proves (2).

Finally, for the infinite real projective space we have

$$H^k(\mathbb{RP}^\infty;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}_2 & \text{if } k \text{ is even and } k > 1 \\ 0 & \text{otherwise.} \end{cases}$$

The proof now is similar to the cases above. Indeed, we now work with the canonical line bundle $\gamma^1 = \gamma^1(\mathbb{R}^\infty)$ over $G_1(\mathbb{R}^\infty) = \mathbb{RP}^\infty$. □

We now prove a general fact about $E_k(X)$ and $E_k^\circ(X)$ for spaces X whose cohomology ring is generated by the second cohomology.

Proposition 3.2. *Let X be a simply connected space with $H^2(X;\mathbb{Z})$ finitely generated. If the integral cohomology ring $H^*(X;\mathbb{Z})$ is generated by $H^2(X;\mathbb{Z})$, then*

(1) $E_k(X) = H^k(X;\mathbb{Z})$ *for all* $k \geq 1$*;*

(2) $E_k^\circ(X) = \emptyset$ *for* $k > 2$ *and* $E_2^\circ(X)$ *equals the elements of infinite order in* $H^2(X;\mathbb{Z})$.

Proof. Assume that $k = 2r$ is even and let $c \in H^k(X;\mathbb{Z})$. Let $c = x_1 \cdots x_r$ where $x_i \in H^2(X;\mathbb{Z})$. Let L_i, $1 \le i \le r$ be complex line bundles over X with $c_1(L_i) = x_i$. Then it is clear that

$$c = e\big((L_1 \oplus \cdots \oplus L_r)_{\mathbb{R}}\big).$$

This proves (1).

Finally, as spherical cohomology classes cannot be the cup product of lower dimensional classes, we have $E_k^\circ(X) = \emptyset$ for $k > 2$. As X is simply connected, the Hurewicz homomorphism

$$\varphi\colon \pi_2(X) \longrightarrow H_2(X;\mathbb{Z})$$

is an isomorphism. Lemma 2.14 now completes the proof of (2). □

There is a large class of spaces which satisfy the conditions of the above proposition. Indeed, it is well known that complex projective spaces, flag manifolds and quasi-toric manifolds [2] satisfy the conditions of the above proposition.

3.3. Products of spheres

In this section we obtain partial computations of the sets E_k for products of spheres and give a proof of Theorem 1.4. As a consequence we show that cohomology generators of certain products of spheres cannot be realized as an Euler class.

As a motivation for the results of this section we begin by considering the following example.

Example 3.4. Suppose $m, n \ge 3$ are odd integers with $m \le n$. Let $X = S^m \times S^n$. Then clearly,

$$E_m^s(X) = E_m(X) = 0 = E_n(X) = E_n^s(X).$$

The integral cohomology ring structure of X shows that the cohomology group $H^{m+n}(X;\mathbb{Z})$ is generated by the cup product ab where $a \in H^m(X;\mathbb{Z})$ and $b \in H^n(X;\mathbb{Z})$ are generators. Hence $E_{m+n}^\circ(X) = \emptyset$. The sequence of maps

$$S^m \hookrightarrow S^m \vee S^n \hookrightarrow S^m \times S^n \xrightarrow{\pi_1} S^m,$$

for example, shows that $E_m^\circ(X)$ equals the elements of infinite order in $H^m(X;\mathbb{Z})$. Similarly for $E_n^\circ(X)$. Now, as $m+n$ is even, Theorem 2.4 implies that

$$2H^{m+n}(X;\mathbb{Z}) \subseteq E_{m+n}(X).$$

We shall show that this is actually an equality if $m, n \equiv 3 \pmod 8$.

Recall that if $f\colon X \longrightarrow Y$ is a map between spaces, then $f^*(E_k(Y)) \subseteq E_k(X)$ for all k. We first exhibit a sufficient condition for this inclusion to be an equality.

Lemma 3.5. *Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$ be maps between spaces. Assume that*

$$f^*\colon H^n(Y;\mathbb{Z}) \longrightarrow H^n(X;\mathbb{Z})$$

is an isomorphism for some $n \geq 1$ and $(f^*)^{-1} = g^*$. Then $f^*(E_n(Y)) = E_n(X)$.

Proof. Note that $g^*(E_n(X)) \subseteq E_n(Y)$. The sequence of inclusions

$$E_n(X) = f^*g^*(E_n(X)) \subseteq f^*(E_n(Y)) \subseteq E_n(X)$$

forces the equality $f^*(E_n(Y)) = E_n(X)$. □

Lemma 3.6. *Let n be an integer with $n \equiv 5 \pmod 8$ or $n \equiv 6 \pmod 8$. Then every orientable vector bundle over $S^1 \times S^n$ is stably trivial.*

Proof. A nonorientable vector bundle is never stably isomorphic to a trivial bundle. If $\pi_1\colon S^1 \times S^n \longrightarrow S^1$ denotes the projection, then the pullback $\pi_1^*(\gamma_1^1)$ is clearly a nonorientable bundle over $S^1 \times S^n$. Here γ_1^1 denotes the canonical line bundle over $S^1 = \mathbb{RP}^1$. The lemma now follows from the fact that $\widetilde{KO}(S^1 \times S^n) \cong \widetilde{KO}(S^1) \cong \mathbb{Z}_2$. □

We are now in a position to prove Theorem 1.4

Proof of Theorem 1.4. We first prove (1). Note that m and n are both odd. As $S^m \times S^n$ has cohomology only in dimensions 0, m, n and $m+n$ it follows that $E_k(S^m \times S^n) = 0$ if $1 \leq k < m+n$. As $(m+n) \equiv 6 \pmod 8$, Bott periodicity in real K-theory implies that

$$\widetilde{KO}(S^m) \cong \widetilde{KO}(S^n) \cong \widetilde{KO}(S^3) = 0$$

and

$$\widetilde{KO}(S^{m+n}) \cong \widetilde{KO}(S^6) = 0.$$

By Theorem 2.4,

$$2H^{m+n}(S^m \times S^n;\mathbb{Z}) \subseteq E_{m+n}(S^m \times S^n).$$

The fact that this is an equality will follow from Proposition 2.7 once we check that $\widetilde{KO}(S^m \times S^n) = 0$. The sequence of maps

$$S^m \vee S^n \hookrightarrow S^m \times S^n \longrightarrow S^{m+n} = (S^m \times S^n)/(S^m \vee S^n)$$

gives rise to an exact sequence (see [8])

$$\widetilde{KO}(S^{m+n}) \longrightarrow \widetilde{KO}(S^m \times S^n) \longrightarrow \widetilde{KO}(S^m \vee S^n)$$

in reduced K-theory. As noted earlier, $\widetilde{KO}(S^{m+n}) = 0$. There is an isomorphism $\widetilde{KO}(S^m \vee S^n) \longrightarrow \widetilde{KO}(S^m) \oplus \widetilde{KO}(S^n)$. This shows that $\widetilde{KO}(S^m \times S^n) = 0$. This completes the proof of (1).

We now prove (2). Fix a point in S^m and consider the corresponding inclusion $S^m \hookrightarrow S^m \times S^n$. This clearly induces an isomorphism in cohomology in dimension n. The inverse of this isomorphism in cohomology is induced on the space level by the projection $S^m \times S^n \longrightarrow S^n$. Lemma 3.5 together with Theorem 2.4 now completes the proof of (2).

Finally, (3) follows from Theorem 2.4, Proposition 2.7, and Lemma 3.6. This completes the proof of the theorem. □

Corollary 3.7.

(1) *If $m, n \equiv 3 \pmod 8$, then a generator of $H^{m+n}(S^m \times S^n; \mathbb{Z})$ cannot be realized as an Euler class of an oriented bundle over $S^m \times S^n$.*

(2) *If n is even, $n \neq 2, 4, 8$, and $n \neq m$, then a generator of $H^n(S^m \times S^n; \mathbb{Z})$ cannot be realized as an Euler class of an oriented bundle over $S^m \times S^n$.*

(3) *If $m = 1$ and $n \equiv 5 \pmod 8$, then a generator of $H^{n+1}(S^1 \times S^n; \mathbb{Z})$ cannot be realized as an Euler class of an oriented bundle over $S^1 \times S^n$.*

Example 3.8. Let $m, n \in \mathbb{N}$ be integers that are either both even or both odd. Then, by Lemma 2.9, $E^s_{m+n}(S^m \times S^n) = 2H^{m+n}(S^m \times S^n; \mathbb{Z})$.

3.9. Dold manifolds

Recall that the Dold manifold $P(m, n)$ is a $(m + 2n)$-dimensional manifold defined as the quotient of $S^m \times \mathbb{CP}^n$ by the fixed point free involution $(x, z) \mapsto (-x, \bar{z})$. This gives rise to a two-fold covering

$$\mathbb{Z}_2 \hookrightarrow S^m \times \mathbb{CP}^n \longrightarrow P(m, n),$$

and via the projection $S^m \times \mathbb{CP}^n \longrightarrow S^m$, a fiber bundle

$$\mathbb{CP}^n \hookrightarrow P(m, n) \longrightarrow \mathbb{RP}^m$$

with fiber $\mathbb{CP}^n$ and structure group $\mathbb{Z}_2$. In particular, for $n = 1$, we have a fiber bundle

$$S^2 \hookrightarrow P(m, 1) \longrightarrow \mathbb{RP}^m.$$

The mod-2 cohomology ring of $P(m, n)$ is given by [3]

$$H^*(P(m, n); \mathbb{Z}_2) = \mathbb{Z}_2[c, d]/(c^{m+1} = d^{n+1} = 0)$$

where $c \in H^1(P(m, n); \mathbb{Z}_2)$ and $d \in H^2(P(m, n); \mathbb{Z}_2)$. The total Stiefel-Whitney class $w(P(m, n))$ of $P(m, n)$ is given by (see [13, Theorem 1.5])

$$w(P(m, n)) = (1 + c)^m (1 + c + d)^{n+1}.$$

Thus the Dold manifold $P(m, n)$ is orientable if and only if $m + n + 1$ is even.

The integral cohomology groups of the Dold manifolds are also well understood [4]. We concentrate on Dold manifolds $P(m, 1)$ below with $m > 1$. The integral cohomology of $P(m, 1)$ is described by the following cases:

- If m is even, then

$$H^k(P(m,1);\mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0, m+2 \\ 0 & k = 1 \\ \mathbb{Z}_2 & 2 \leq k \leq m+1 \end{cases}$$

- If m is odd and $m \neq 1$, then

$$H^k(P(m,1);\mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k = 1 \\ \mathbb{Z}_2 & 2 \leq k \leq m-1 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & k = m \\ 0 & k = m+1 \\ \mathbb{Z}_2 & k = m+2 \end{cases}$$

Remark 3.10. The paper of Fujii ([4, Proposition 1.6]) describing the integral cohomology of $P(m,n)$ has a typographical error. When m is odd, the indexing set for i should be $i = 1, 2, \ldots, [\frac{m}{2}] + 1$ and not just $i = 1, 2, \ldots, [\frac{m}{2}]$.

We now describe the sets E_k, E_k^s and E_k° for the Dold manifold $P(m,1)$. We shall make use of the fact that over $P(m,n)$ there exist certain canonical vector bundles.

Proposition 3.11. ([12, p. 86], [13, Proposition 1.4]) *Over $P(m,n)$,*

(1) *there exists a line bundle ξ with total Stiefel-Whitney class $w(\xi) = 1 + c$;*

(2) *there exists a 2-plane bundle η with total Stiefel-Whitney class $w(\eta) = 1 + c + d$.*

Note that all the bundles in the above proposition are nonorientable.

Remark 3.12. The Dold manifold $X = P(1,1)$ is a compact nonorientable 3-manifold. By the remarks preceeding Theorem 2.4, it follows that $H^3(X;\mathbb{Z}) = \mathbb{Z}_2$. By Lemma 2.3, $E_1(X) = 0$ and $E_2(X) = H^2(X;\mathbb{Z})$. Let ξ and η be the bundles over X as in Proposition 3.11. Then, as $w_1(\xi \oplus \eta) = 2c = 0$, the Whitney sum $\xi \oplus \eta$ is an orientable 3-plane bundle over X and $w_3(\xi \oplus \eta) = cd \neq 0$. Hence, by Theorem 2.4(2), $E_3(X) = H^3(X;\mathbb{Z})$.

Convention. *Throughout the rest of the paper m will be assumed to be greater than* 1.

Proposition 3.13. *For the Dold manifold $P(m,1)$,*

(1) *if m is even, then for all k, $1 \leq k \leq m+1$, we have $E_k(P(m,1)) = H^k(P(m,1);\mathbb{Z})$;*

(2) *if m is odd, then $E_k(P(m,1)) = H^k(P(m,1);\mathbb{Z})$ for $1 \leq k \leq m+2$ and $k \neq m$. When $k = m$, $E_k(P(m,1)) = \mathbb{Z}_2$;*

(3) *if m is even, then* $E^s_k(P(m,1)) = 0$ *for* $k \neq m+2$ *and* $E^s_{m+2}(P(m,1)) = 2H^{m+2}(P(m,1);\mathbb{Z})$*;*

(4) *if m is odd, then* $E^s_k(P(m,1)) = 0$ *for all* k*,* $1 \leq k \leq m+2$*.*

Proof. Let m be even. For $k = 1,2$, the claim follows from Lemma 2.3. Since in the range $2 \leq k \leq m+1$ the cohomology groups $H^k(P(m,1);\mathbb{Z})$ are of order 2, we only have to exhibit an orientable k-plane bundle over $P(m,1)$ with nonzero Euler class. Let ξ and η be the bundles in Proposition 3.11. Clearly, both ξ and η are nonorientable. If k is even and $2 \leq k \leq m$, then the k-fold Whitney sum $k\xi$ of ξ with itself is clearly orientable and $w(k\xi) = (1+c)^k$. In particular,

$$w_k(k\xi) = c^k \neq 0.$$

This forces the Euler class $e(k\xi)$ of the orientable bundle $k\xi$ to be nonzero. Finally, if $k = 2t+1$ is odd, then the k-plane bundle $(2t-1)\xi \oplus \eta$ is orientable and clearly

$$w_k((2t-1)\xi \oplus \eta) = c^{2t-1}d \neq 0.$$

In this case too the Euler class $e((2t-1)\xi \oplus \eta) \neq 0$. This proves (1).

The proof of (2) when $k \neq m$ is similar to the above case. When $m = 2t+1$ is odd, then for the m-plane bundle $(2t-1)\xi \oplus \eta$ we have, as before,

$$w_m((2t-1)\xi \oplus \eta) = c^{2t-1}d \neq 0.$$

The fact that m is odd and $H^m(P(m,1);\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2$ now implies that

$$0 \neq e((2t-1)\xi \oplus \eta) \in \mathbb{Z}_2 \subseteq \mathbb{Z} \oplus \mathbb{Z}_2,$$

which completes the proof of (2).

The cases (3) and (4) follow from Lemma 2.9 and the case (2). This completes the proof of the proposition. □

Although this result fully computes the set $E_{m+2}(P(m,1))$ for m odd, we do not have a complete description for even m. Note that when m is even, the Dold manifold $P(m,1)$ is orientable and if $(m+2) \neq 2,4,8$, then by Theorem 2.4, $2H^{m+2}(P(m,1);\mathbb{Z}) \subseteq E_{m+2}(P(m,1))$. We shall show that $E_{m+2}(P(m,1))$ is strictly larger than $2H^{m+2}(P(m,1);\mathbb{Z})$ when m is even. We first prove the following.

Proposition 3.14. *There exists an orientable $(m+2)$-plane bundle μ over $P(m,1)$ with top Stiefel-Whitney class $w_{m+2}(\mu) = c^m d \neq 0$.*

Proof. First assume that m is even. In this case $P(m,1)$ is an orientable $(m+2)$-dimensional manifold. Let τ denote the tangent bundle of $\mathbb{RP}^m$ and $\pi\colon P(m,1) \longrightarrow \mathbb{RP}^m$ the sphere bundle with fiber S^2. The total Stiefel-Whitney class of the pullback $\pi^*(\tau)$ is evidently

$$w(\pi^*(\tau)) = (1+c)^{m+1}.$$

Hence, for the Whitney sum $\mu = \pi^*(\tau) \oplus \eta$,

$$w_{m+2}(\theta) = w_{m+2}(\pi^*(\tau) \oplus \eta) = c^m d \neq 0.$$

Here η is the 2-plane bundle over $P(m,1)$ of Proposition 3.11 with $w(\eta) = 1 + c + d$. Note that the Whitney sum $\pi^*(\tau) \oplus \eta$ is orientable.

If m is odd, it suffices to consider the Whitney sum $\mu = m\xi \oplus \eta$. Clearly, this Whitney sum is orientable and

$$w_{m+2}(\mu) = w_{m+2}(m\xi \oplus \eta) = c^m d \neq 0.$$

This completes the proof. □

The following is now an easy consequence of Theorem 2.4 and Proposition 3.14.

Corollary 3.15. *Let m be even. Then the inclusion $2H^{m+2}(P(m,1);\mathbb{Z}) \subseteq E_{m+2}(P(m,1))$ is strict.*

Note that $P(2,1)$ and $P(6,1)$ are orientable 4 and 8 dimensional manifolds respectively, and hence by Theorem 2.4, the sets $E_4(P(2,1))$ and $E_8(P(6,1))$ equal the top dimensional cohomology groups. Finally, we describe the sets $E_k^\circ(P(m,1))$. We make use of the following fact about homotopy groups of the spheres.

Theorem 3.16. ([11]) *The homotopy groups $\pi_k(S^n)$ are all finite for $k \neq n, 2n{-}1$. If n is odd, then $\pi_{2n-1}(S^n)$ is also finite.*

Proposition 3.17. *For the Dold manifold $P(m,1)$,*

(1) *if m is even, then $E_k^\circ(P(m,1)) = \emptyset$ for all $k \geq 1$;*
(2) *if m is odd, then $E_m^\circ(P(m,1))$ equals the elements of infinite order in $H^m(P(m,1);\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ while $E_k^\circ(P(m,1)) = \emptyset$ if $k \neq m$.*

Proof. Assume that m is even. We only have to look at the case $k = m+2$. As there is a double covering

$$\mathbb{Z}_2 \hookrightarrow S^m \times S^2 \longrightarrow P(m,1),$$

it follows that

$$\pi_i(P(m,1)) \cong \pi_i(S^m \times S^2) \cong \pi_i(S^m) \times \pi_i(S^2)$$

for all $i \geq 2$. The groups $\pi_{m+2}(S^m)$ and $\pi_{m+2}(S^2)$ are both finite and hence $\pi_{m+2}(P(m,1))$ is also finite. We now appeal to Proposition 2.12 to conclude that $E_{m+2}^\circ(P(m,1)) = \emptyset$. This proves (1).

We now come to the proof of (2). First note that $H_m(P(m,1);\mathbb{Z}) \cong \mathbb{Z}$. This follows from the Universal coefficients formula and the description of the cohomology groups of $P(m,1)$ at the beginning of this section. By Lemma 2.14(3), it is enough to check that the Hurewicz homomorphism

$$\varphi\colon \pi_m(P(m,1)) \longrightarrow H_m(P(m,1);\mathbb{Z})$$

has finite cokernel. Note that

$$\pi_m(P(m,1)) \cong \pi_m(S^m) \times \pi_m(S^2)$$

is either a free abelian group of rank two if $m = 3$ or is a direct sum of an infinite cyclic group and a finite abelian group. It is therefore enough to check that the Hurewicz homomorphism above is nonzero. Let $\pi\colon S^m \times S^2 \longrightarrow P(m,1)$ be the double covering. Then

$$\pi_*\colon \pi_m(S^m \times S^2) \longrightarrow \pi_m(P(m,1))$$

is an isomorphism. It follows from the description of the CW-structure of $P(m,1)$ ([4, pp. 50]) in that the homomorphism

$$\pi_*\colon H_m(S^m \times S^2;\mathbb{Z}) \cong \mathbb{Z} \longrightarrow H_m(P(m,1);\mathbb{Z}) \cong \mathbb{Z}$$

is multiplication by 2. The naturality of the Hurewicz homomorphism implies that the Hurewicz homomorphism

$$\varphi\colon \pi_m(P(m,1)) \longrightarrow H_m(P(m,1);\mathbb{Z})$$

is nonzero if and only if the Hurewicz homomorphism

$$\varphi\colon \pi_m(S^m \times S^2) \longrightarrow H_m(S^m \times S^2;\mathbb{Z})$$

is nonzero. It is therefore enough to find a map $f\colon S^m \longrightarrow S^m \times S^2$ such that the homomorphism

$$f_*\colon H_m(S^m;\mathbb{Z}) \longrightarrow H_m(S^m \times S^2;\mathbb{Z})$$

is nonzero. The composition

$$S^m \hookrightarrow S^m \vee S^2 \hookrightarrow S^m \times S^2$$

is evidently such a map. This completes the proof. □

Proof of Theorem 1.5. This is now immediate from the above observations. □

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