VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-012-0056-3
Math. Slovaca **62** (2012), No. 5, 937–948

DETERMINATION TEMPERATURE OF A BACKWARD HEAT EQUATION WITH TIME-DEPENDENT COEFFICIENTS

Nguyen Huy Tuan* — Ngo Van Hoa

(*Communicated by Peter Takac*)

ABSTRACT. We introduce the truncation method for solving a backward heat conduction problem with time-dependent coefficients. For this method, we give the stability analysis with new error estimates. Meanwhile, we investigate the roles of regularization parameters in these two methods. These estimates prove that our method is effective.

1. Introduction

A classical inverse problem in connection with heat equation is the *backward heat equation*, which models the problem of determining the temperature in the past from observation of the present distribution. In general, the solution of ill-posed problems always leads to numerical problems. Therefore, to solve such problems numerically, it is essential to find approximate solution of ill-posed problems which are called *regularization methods*. The basic idea of regularization is to replace the original equation by a *close* equation involving a small parameter ε, such that the changed equation can be solved in a stable way and its solution is close to the solution of the original equation when ε is small.

2010 Mathematics Subject Classification: Primary 35K05, 35K99; Secondary 47J06, 47H10.
Keywords: backward heat problem, Ill-posed problem, nonhomogeneous heat, contraction principle.
This project was supported by National Foundation for Science and Technology Development (NAFOSTED).

In this paper, we consider the problem of finding the temperature $u(x,t)$, $(x,t) \in (0,\pi) \times [0,T]$, such that

$$\begin{cases} a(t)u_{xx} = u_t, & (x,t) \in (0,\pi) \times (0,T) \\ u(0,t) = u(\pi,t) = 0, & t \in (0,T) \\ u(x,T) = g(x), & x \in (0,\pi) \end{cases} \tag{1.1}$$

where $a(t)$, $g(x)$ are given. The problem is called the backward heat problem with time-dependent coefficient. In the simple case $a(t) = 1$, the problem (1.1) is investigated in many papers, such as Clark and Oppenheimer [3], Denche and Bessila [5], Tautenhahn et al [24] Melnikova et al [15, 16], ChuLiFu [4, 10, 9], Tautenhahn[24], Trong et al [21, 22], B. Yildiz et al [25, 26]. Although there are many papers on the backward heat equation with the constant coefficient, there are rarely works considered the backward heat with the time-dependent coefficient, such as (1.1). A few works of analytical methods were presented for this problem, for example [18]. However, the authors in [18] only used numerical computation method and the stability theory with explicitly error estimate has not been generalized accordingly. One of the major object of this paper is to provide a regularization method to establish the Hölder estimates for (1.1). The truncated regularization method is a very simple and effective method for solving some ill-posed problems and it has been successfully applied to some inverse heat conduction problems [2, 7, 11]. However, in many earlier works, we find that only logarithmic type estimates in L^2-norm are available; and estimates of Hölder type on $[0,T]$ are very rare (see some remarks for more detail comparisons). In our method, corresponding to different levels of the smoothness of the exact solution, the convergence rates will be improved gradually.

2. The ill-posed problem

We suppose that $a(t)\colon [0,T] \to R$ is a continuous function on $[0,T]$ satisfying $0 < p \leqslant a(t) \leqslant q$, for all $t \in [0,T]$. Throughout this article, we denote by $\|\cdot\|$ the L^2-norm and $\langle \cdot,\cdot \rangle$ denote inner product on $L^2(0,\pi)$.

Theorem 2.1. *The problem* (1.1) *has a unique solution u if and only if*

$$\sum_{m=1}^{\infty} \exp\left\{2m^2 \int_0^T a(s)\,\mathrm{d}s\right\} \left|\left\langle g(x), \sin mx \right\rangle\right|^2 < \infty. \tag{2.1}$$

Proof. Suppose the Problem (1.1) has an exact solution $u \in C\big([0,T]; H_0^1(0,\pi)\big) \cap C^1\big((0,T); L^2(0,\pi)\big)$, then u can be formulated in the frequency domain

$$u(x,t) = \sum_{m=1}^{\infty} \exp\left(m^2 \int_t^T a(s)\,\mathrm{d}s \right) \big\langle g(x), \sin(mx) \big\rangle \sin(mx). \tag{2.2}$$

This implies that

$$\big\langle u(x,0), \sin(mx) \big\rangle = \exp\left(m^2 \int_0^T a(s)\,\mathrm{d}s \right) \big\langle g(x), \sin(mx) \big\rangle. \tag{2.3}$$

Then

$$\|u(\cdot,0)\|^2 = \sum_{m=1}^{\infty} \exp\left\{ 2m^2 \int_0^T a(s)\,\mathrm{d}s \right\} \big|\big\langle g(x), \sin mx \big\rangle\big|^2 < \infty.$$

If we get (2.1), then define $v(x)$ as the function

$$v(x) = \sum_{m=1}^{\infty} \exp\left\{ m^2 \int_0^T a(s)\,\mathrm{d}s \right\} \big\langle g(x), \sin mx \big\rangle \sin mx \in L^2(0,\pi).$$

Consider the problem

$$\begin{cases} u_t - a(t)u_{xx} = 0, & \\ u(0,t) = u(\pi,t) = 0, & t \in (0,T) \\ u(x,0) = v(x), & x \in (0,\pi) \end{cases} \tag{2.4}$$

It is clear to see that (2.4) is the direct problem so it has a unique solution u (See [6]). We have

$$u(x,t) = \sum_{m=1}^{\infty} \exp\left(-m^2 \int_0^t a(s)\,\mathrm{d}s \right) \big\langle v(x), \sin mx \big\rangle \sin mx. \tag{2.5}$$

Let $t = T$ in (2.5), we obtain

$$u(x,T) = \sum_{m=1}^{\infty} \big\langle g(x), \sin(mx) \big\rangle \sin mx = g(x).$$

Hence, u is the unique solution of (1.1). $\square$

Theorem 2.2. *The Problem* (1.1) *has at most one solution* $C\big([0,T];H_0^1(0,\pi)\big) \cap C^1\big((0,T);L^2(0,\pi)\big)$.

Proof. Let $u(x,t)$, $v(x,t)$ be two solutions of Problem (1.1) such that $u,v \in C\big([0,T];H_0^1(0,\pi)\big) \cap C^1\big((0,T);L^2(0,\pi)\big)$. Put $w(x,t)=u(x,t)-v(x,t)$. Then w satisfies the equation

$$\begin{aligned} w_t - a(t)w_{xx} &= 0 && (x,t)\in(0,\pi)\times(0,T),\\ w(0,t)=w(\pi,t) &= 0, && t\in(0,T)\\ w(x,T) &= 0, && x\in(0,\pi). \end{aligned}$$

Now, setting $G(t)=\int_0^\pi w^2(x,t)\,\mathrm{d}x$ $(0\le t\le T)$, and by direct computation we get

$$G'(t)=2\int_0^\pi w(x,t)w_t(x,t)\,\mathrm{d}x = 2a(t)\int_0^\pi w(x,t)w_{xx}(x,t)\,\mathrm{d}x.$$

Using Green formula, we obtain

$$G'(t) = -2a(t)\int_0^\pi w_x^2(x,t)\,\mathrm{d}x. \tag{2.6}$$

Taking the derivative of $G'(t)$ with respect to t, one has

$$G''(t) = -4a(t)\int_0^\pi w_x(x,t)w_{xt}(x,t)\,\mathrm{d}x.$$

By a simple calculation and using the integration by parts, we get

$$\begin{aligned} G''(t) &= 4a(t)\int_0^\pi w_{xx}(x,t)w_t(x,t)\,\mathrm{d}x\\ &= 4a^2(t)\int_0^\pi w_x^2(x,t)\,\mathrm{d}x. \end{aligned} \tag{2.7}$$

Now, from (2.6) and applying the Holder inequality, we have

$$\begin{aligned} \int_0^\pi w_x^2(x,t)\,\mathrm{d}x &= -\int_0^\pi w(x,t)w_{xx}(x,t)\,\mathrm{d}x\\ &\le \left(\int_0^\pi w^2(x,t)\,\mathrm{d}x\right)^{\frac12}\left(\int_0^\pi w_{xx}^2(x,t)\,\mathrm{d}x\right)^{\frac12}. \end{aligned} \tag{2.8}$$

Thus (2.6)–(2.7)–(2.8) imply

$$(G'(t))^2 \leq G(t)G''(t).$$

Hence by [6, Theorem 11, p. 65], which gives $G(t) = 0$. This implies that $u(x,t) = v(x,t)$. The proof is completed. $\square$

3. Error estimates with the main results

Let p, q be positive numbers. Let $a\colon [0,T] \to R$ be a continuous function such that $0 < p \leqslant a(t) \leqslant q$. Let g^ε be a measured data which satisfies $\|g^\varepsilon - g\| \leqslant \varepsilon$. Suppose the Problem (1.1) has an exact solution $u \in C([0,T]; H_0^1(0,\pi)) \cap C^1((0,T); L^2(0,\pi))$, then from (2.5) and Theorem 2.2, the solution u can be formulated in the frequency domain

$$u(x,t) = \sum_{m=1}^{\infty} \exp\left(m^2 \int_t^T a(s)\,\mathrm{d}s\right) \langle g(x), \sin(mx)\rangle \sin(mx) \tag{3.1}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(0,\pi)$. From (3.1), we note that the term $\exp\left(m^2 \int_t^T a(s)\,\mathrm{d}s\right)$ tends to infinity as m tends to infinity, then in order to guarantee the convergence of solution u given by (3.1), the coefficient $\langle u, \sin mx\rangle$ must decay rapidly. Usually such a decay is not likely to occur for the measured data g^ε. Therefore, a natural way to obtain a stable approximation solution u is to eliminate the high frequencies and consider the solution u for $m < N$, where N is a positive integer. We define the truncation regularized solution as follows

$$u^\varepsilon(x,t) = \sum_{m=1}^{N} \exp\left(m^2 \int_t^T a(s)\,\mathrm{d}s\right) \langle g(x), \sin(mx)\rangle \sin(mx) \tag{3.2}$$

and

$$v^\varepsilon(x,t) = \sum_{m=1}^{N} \exp\left(m^2 \int_t^T a(s)\,\mathrm{d}s\right) \langle g^\varepsilon(x), \sin(mx)\rangle \sin(mx) \tag{3.3}$$

where the positive integer N plays the role of the regularization parameter.

Definition 1. Let $0 \leqslant r < \infty$. By $H^r(0,\pi)$ we denote the space of all functions $g \in L^2(0,\pi)$ with the property

$$\sum_{m=1}^{\infty} (1+m^2)^r \left|\langle g(x), \sin(mx)\rangle\right|^2 < \infty, \qquad \text{where} \quad g_m = \frac{2}{\pi}\int_0^\pi g(x)\sin(mx)\,\mathrm{d}x.$$

We also define the norm of $H^r(0,\pi)$ as follows

$$\|g\|^2_{H^r(0,\pi)} = \sum_{m=1}^{\infty} (1+m^2)^r \big|\langle g(x), \sin(mx)\rangle\big|^2.$$

Noting that if $r = 0$ then $H^r(0,\pi)$ is also $L^2(0,\pi)$.

Theorem 3.1. *The solution u^ε given in* (3.2) *depends continuously on g in $L^2(0,\pi)$. Furthermore, we have*

$$\|v^\varepsilon(x,t) - u^\varepsilon(x,t)\| \leqslant \exp\big\{q(T-t)N^2\big\}\varepsilon.$$

Proof. Let u^ε and w^ε be two solutions of (3.2) corresponding to the final values g and h. From (3.2), we have

$$u^\varepsilon(x,t) = \sum_{m=1}^{N} \exp\left(m^2 \int_t^T a(s)\,\mathrm{d}s\right) \langle g(x), \sin(mx)\rangle \sin(mx), \tag{3.4}$$

$$w^\varepsilon(x,t) = \sum_{m=1}^{N} \exp\left(m^2 \int_t^T a(s)\,\mathrm{d}s\right) \langle h(x), \sin(mx)\rangle \sin(mx), \tag{3.5}$$

for all $0 \leqslant t \leqslant T$. This follows that

$$\begin{aligned} \|u^\varepsilon(\cdot,t) - w^\varepsilon(\cdot,t)\|^2 &= \frac{\pi}{2} \sum_{m=1}^{N} \left|\exp\left(m^2 \int_t^T a(s)\,\mathrm{d}s\right) \langle g(x) - h(x), \sin mx\rangle\right|^2 \\ &\leqslant \frac{\pi}{2} \exp\left\{2N^2 \int_t^T a(s)\,\mathrm{d}s\right\} \sum_{m=1}^{\infty} \big|\langle g(x) - h(x), \sin mx\rangle\big|^2 \\ &= \exp\big\{2q(T-t)N^2\big\}\|g-h\|^2. \end{aligned} \tag{3.6}$$

Hence

$$\|u^\varepsilon(\cdot,t) - w^\varepsilon(\cdot,t)\| \leqslant \exp\big\{q(T-t)N^2\big\}\|g-h\|. \tag{3.7}$$

Since (3.7) and the condition $\|g^\varepsilon - g\| \leqslant \varepsilon$, we have

$$\|v^\varepsilon(x,t) - u^\varepsilon(x,t)\| \leqslant \exp\big\{q(T-t)N^2\big\}\varepsilon. \tag{3.8}$$

□

Theorem 3.2. *Assume that there exists the positive number I such that $\|u(\cdot,0)\| \leqslant I$. Let $N = [k]$ where $[\cdot]$ denotes the largest integer part of a real number with $k = \sqrt{\frac{1}{qT}\ln(\frac{1}{\varepsilon})}$, then the following convergence estimate holds for every $t \in [0,T]$*

$$\|v^{\varepsilon}(x,t) - u(x,t)\| \leqslant (I+1)\varepsilon^{\frac{pt}{qT}}. \tag{3.9}$$

where v^{ε} is defined in (3.3).

Proof. Since (3.2), we have

$$\begin{aligned} u(x,t) - u^{\varepsilon}(x,t) &= \sum_{m=N}^{\infty} \exp\left(m^2 \int_t^T a(s)\,\mathrm{d}s\right) \langle g(x), \sin(mx)\rangle \sin(mx) \\ &= \sum_{m=N}^{\infty} \langle u(x,t), \sin mx\rangle \sin mx. \end{aligned}$$

Thus, using the inequality $(a+b)^2 \leqslant 2(a^2+b^2)$ and Holder inequality, we have

$$\begin{aligned} \|u(\cdot,t) - u^{\varepsilon}(\cdot,t)\|^2 &= \frac{\pi}{2}\sum_{m=N}^{\infty} \exp\left\{2m^2 \int_t^T a(s)\,\mathrm{d}s\right\} \left|\langle g(x), \sin mx\rangle\right|^2 \\ &= \frac{\pi}{2}\sum_{m=N}^{\infty} \exp\left\{-2m^2 \int_0^t a(s)\,\mathrm{d}s \left|\langle u(x,0), \sin mx\rangle\right|^2\right\} \\ &\leqslant \exp\left\{-2ptN^2\right\}\|u(\cdot,0)\|^2 \\ &\leqslant \exp\left\{-2tpN^2\right\} I^2. \end{aligned} \tag{3.10}$$

Combining (3.8) and (3.10) then

$$\begin{aligned} \|v^{\varepsilon}(x,t) - u(x,t)\| &\leqslant \|v^{\varepsilon}(\cdot,t) - u^{\varepsilon}(\cdot,t)\| + \|u^{\varepsilon}(\cdot,t) - u(\cdot,t)\| \\ &\leqslant \mathrm{e}^{q(T-t)N^2}\varepsilon + \mathrm{e}^{-tpN^2} I. \end{aligned}$$

From $N = \sqrt{\frac{1}{qT}\ln(\frac{1}{\varepsilon})}$ and $\varepsilon^{\frac{pt}{qT}} \geqslant \varepsilon^{\frac{t}{T}}$ we get the following convergence estimate

$$\begin{aligned} \|v^{\varepsilon}(x,t) - u(x,t)\| &\leqslant \varepsilon^{\frac{t}{T}} + \varepsilon^{\frac{pt}{qT}} I \\ &\leqslant \varepsilon^{\frac{pt}{qT}}\,(I+1)\,. \end{aligned}$$

□

Remark 1. From Theorem 2.1, we find that v^{ε} is an approximation of exact solution u. The approximation error depends continuously on the measurement error for fixed $0 < t \leqslant T$. However, as $t \to 0$ the accuracy of regularized solution becomes progressively lower. This is a common thing in the theory of ill-posed problems, if we do not have additional conditions on the smoothness of the solution. To retain the continuous dependence of the solution at $t = 0$, we introduce a stronger a priori assumption.

Theorem 3.3. *Assume that there exist the positive numbers r, J such that*

$$\|u(\cdot,t)\|_{H^r(0,\pi)} < J,$$

for all $t \in [0,T]$. Let $N = [k]$ where $[.]$ denotes the largest integer part of a real number with

$$k = \sqrt{\frac{1}{q(T+\alpha)} \ln\left(\frac{1}{\varepsilon}\right)}$$

for $\alpha > 0$. Then the following convergence estimate holds

$$\|v^\varepsilon(x,t) - u(x,t)\| \leqslant \sqrt{\frac{\pi}{2}} J \left(\frac{1}{q(T+\alpha)} \ln\left(\frac{1}{\varepsilon}\right)\right)^{-\frac{r}{2}} + \varepsilon^{\frac{t+\alpha}{T+\alpha}}. \tag{3.11}$$

for every $t \in [0,T]$ and where v^ε is defined in (3.3)*.*

P r o o f. We have

$$\begin{aligned}
\|u(\cdot,t) - u^\varepsilon(\cdot,t)\|^2 &= \frac{\pi}{2} \sum_{m=N}^{\infty} m^{-2r} m^{2r} \big|\langle u(x,t), \sin(mx)\rangle\big|^2 \\
&\leqslant N^{-2r} \frac{\pi}{2} \sum_{m=1}^{\infty} m^{2r} \big|\langle u(x,t), \sin(mx)\rangle\big|^2 \\
&\leqslant N^{-2r} \frac{\pi}{2} \sum_{m=1}^{\infty} (1+m^2)^r \big|\langle u(x,t), \sin(mx)\rangle\big|^2 \\
&\leqslant N^{-2r} \frac{\pi}{2} J^2. \qquad (3.12)
\end{aligned}$$

Combining (3.8) and (3.12) then

$$\begin{aligned}
\|v^\varepsilon(x,t) - u(x,t)\| &\leqslant \|v^\varepsilon(\cdot,t) - u^\varepsilon(\cdot,t)\| + \|u^\varepsilon(\cdot,t) - u(\cdot,t)\| \\
&\leqslant \sqrt{\frac{\pi}{2}} N^{-r} J + \mathrm{e}^{q(T-t)N^2} \varepsilon.
\end{aligned}$$

From the definition of N in this theorem, the following convergence estimate holds

$$\|v^\varepsilon(x,t) - u(x,t)\| \leqslant \sqrt{\frac{\pi}{2}} \left(\frac{1}{T+\alpha} \ln\left(\frac{1}{\varepsilon}\right)\right)^{-\frac{r}{2}} J + \varepsilon^{\frac{t+\alpha}{T+\alpha}}.$$

□

Remark 2.

1. Denche and Bessila in [5], Trong and his group [22] gave the error estimates in the form
$$\|v^\varepsilon(\cdot,t)-u(\cdot,t)\| \leqslant \frac{C_1}{1+\ln\frac{T}{\varepsilon}}. \tag{3.13}$$
In recently, Chu-Li Fu and his group [4, 9, 10] gave the error estimates is of logarithmic order, which is similar to (3.13). If $r=2$, the error (3.11) is the same order as these above results.

2. By (3.11), the first term of the right hand side of (3.11) is the logarithmic form, and the second term is a power, so the order of (3.11) is also logarithmic order. Suppose that $E_\varepsilon = \|v^\varepsilon - u\|$ be the error of the exact solution and the approximate solution. In most of results concerning the backward heat, then optimal error between is of the logarithmic form. It means that $E_\varepsilon \leqslant C\left(\ln\frac{T}{\varepsilon}\right)^{-l}$ where $l>0$. The error order of logarithmic form is investigated in many recent papers, such as [3, 4, 5, 8, 9, 10, 21, 22, 23]. This often occurs in the boundary error estimate for ill-posed problems. To retain the Holder order in $[0,T]$, we introduce the following theorem with different priori assumption.

Theorem 3.4. *Assume that there exists the positive numbers β, L such that*
$$\frac{\pi}{2}\sum_{m=1}^{\infty} \mathrm{e}^{2\beta m^2} u_m^2(t) < L^2. \tag{3.14}$$
where $u_m(t)=\frac{2}{\pi}\int\limits_0^\pi u(x,t)\sin mx\,\mathrm{d}x$. Let $N=[k]$ where $[.]$ denotes the largest integer part of a real number with $k=\sqrt{\frac{1}{T+\beta}\ln\left(\frac{1}{\varepsilon}\right)}$ then the following convergence estimate holds
$$\|v^\varepsilon(x,t)-u(x,t)\| \leqslant \left(L+\varepsilon^{\frac{t}{T+\beta}}\right)\varepsilon^{\frac{\beta}{T+\beta}}. \tag{3.15}$$
for every $t\in[0,T]$ and where v^ε is defined in (3.3).

Proof. Since
$$u(x,t)-u^\varepsilon(x,t) = \sum_{m=N}^{\infty}\exp\left(m^2\int\limits_t^T a(s)\,\mathrm{d}s\right)\big\langle g(x),\sin(mx)\big\rangle\sin(mx)$$
$$=\sum_{m=N}^{\infty}\big\langle u(x,t),\sin mx\big\rangle\sin mx.$$
we have
$$\|u(\cdot,t)-u^\varepsilon(\cdot,t)\|^2 = \frac{\pi}{2}\sum_{m=N}^{\infty}\exp\{-2\beta m^2\}\exp\{2\beta m^2\}\big|\big\langle u(x,t),\sin mx\big\rangle\big|^2$$

$$
\begin{aligned}
&\leqslant \frac{\pi}{2}\exp\{-2\beta N^2\}\sum_{m=N}^{\infty}\exp\{2\beta m^2\}\big|\langle u(x,t),\sin mx\rangle\big|^2\\
&\leqslant \exp\{-2\beta N^2\}\frac{\pi}{2}\sum_{m=1}^{\infty}\exp\{2\beta m^2\}\big|\langle u(x,t),\sin mx\rangle\big|^2\\
&\leqslant \exp\{-2\beta N^2\}L^2. \qquad (3.16)
\end{aligned}
$$

Combining (3.8) and (3.16), we get

$$
\begin{aligned}
\|v^\varepsilon(x,t)-u(x,t)\| &\leqslant \|v^\varepsilon(\cdot,t)-u^\varepsilon(\cdot,t)\|+\|u^\varepsilon(\cdot,t)-u(\cdot,t)\|\\
&\leqslant \exp\{-\beta N^2\}L+\exp\{q(T-t)N^2\}\varepsilon.
\end{aligned}
$$

From

$$
N=\left[\sqrt{\frac{1}{q(T+\beta)}\ln\left(\frac{1}{\varepsilon}\right)}\,\right]
$$

then the following convergence estimate holds

$$
\|v^\varepsilon(x,t)-u(x,t)\|\leqslant \varepsilon^{\frac{\beta}{T+\beta}}L+\varepsilon^{\frac{t+\beta}{T+\beta}}=\varepsilon^{\frac{\beta}{T+\beta}}\left(L+\varepsilon^{\frac{t}{T+\beta}}\right).
$$

□

Remark 3. Note that the error (3.15) ($\beta > 0$) is the order of Holder type for all $t \in [0,T]$. It is easy to see that the convergence rate of ε^a $(0 < a)$ is more quickly than the logarithmic order $\left(\ln(\frac{1}{\varepsilon})\right)^{-b}$ $(b > 0)$ when $\varepsilon \to 0$. Comparing (3.15) with the results in [3, 4, 5, 8, 9, 10, 21, 22, 23], we can see that the method in this paper give the better approximation. This showed that our method is effective.

Acknowledgement. The authors would like to thank the editor and the reviewers for their valuable comments leading to the improvement of our manuscript. We would like to thank Dr. Nguyen Minh Quan for his most helpful comments on English grammar.

REFERENCES

[1] AMES, K. A.—HUGHES, R. J.: *Structural stability for Ill-posed problems in Banach space*, Semigroup Forum **70** (2005), 127–145.

[2] BERNTSSON, F.: *A spectral method for solving the sideways heat equation*, Inverse Problems **15** (1999), 891–906.

[3] CLARK, G. W.—OPPENHEIMER, S. F.: *Quasireversibility methods for non-well posed problems*, Electron. J. Differential Equations (1994), No. 8, 1–9.

[4] FU, C.-L.—QIAN, Z.—SHI. R.: *A modified method for a backward heat conduction problem*, Appl. Math. Comput. **185** (2007), 564–573.

[5] DENCHE, M.—BESSILA, K.: *A modified quasi-boundary value method for ill-posed problems*, J. Math. Anal. Appl. **301** (2005), 419–426.

[6] EVANS, L. C.: *Partial Differential Equation*, American Mathematical Society, Providence, RI, 1997.

[7] ELDEN, L.—BERNTSSON, F.—REGINSKA, T.: *Wavelet and Fourier methods for solving the sideways heat equation*, SIAM J. Sci. Comput. **21** (2000), 2187–2205.

[8] EWING, R. E.: *The approximation of certain parabolic equations backward in time by Sobolev equations*, SIAM J. Math. Anal. **6** (1975), 283–294.

[9] FENG, X. L.—QIAN, Z.—FU, C. L.: *Numerical approximation of solution of nonhomogeneous backward heat conduction problem in bounded region*, Math. Comput. Simulation **79** (2008), 177–188.

[10] FU, C. L.—TUAN, X. X.—QIAN, Z.: *Fourier regularization for a backward heat equation*, J. Math. Anal. Appl. **331** (2007), 472–480.

[11] FU, C. L.: *Simplified Tikhonov and Fourier regularization methods on a general sideways parabolic equation*, J. Comput. Appl. Math. **167** (2004), 449–63.

[12] ISAKOV, V.: *Inverse Problems for Partial Differential Equations,* Springer-Verlag, New York, 1998.

[13] JOHN, F.: *Continuous dependence on data for solutions of partial differential equations with a prescribed bound*, Comm. Pure Appl. Math. **13** (1960), 551–585.

[14] LATTÈS, R.—LIONS, J.-L.: *Méthode de Quasi-réversibilité et Applications*, Dunod, Paris, 1967.

[15] MEL'NIKOVA, I. V.—BOCHKAREVA, S. V.: *C-semigroups and regularization of an ill-posed Cauchy problem*, Dokl. Akad. Nauk **329** (1993), 270–273.

[16] MEL'NIKOVA, I. V.—FILINKOV, A. I.: *The Cauchy problem. Three approaches.* Chapman & Hall/CRC Monogr. Surv. Pure Appl. Math. 120, Chapman & Hall, London-New York, 2001.

[17] PAYNE, L. E.: *Some general remarks on improperly posed problems for partial differential equations.* In: Symposium on Non-Well Posed Problems and Logarithmic Convexity. Lecture Notes in Math. 316, Springer-Verlag, Berlin, 1973, pp. 1–30.

[18] SHIDFAR, A.—ZAKERI, A.: *A numerical technique for backward inverse heat conduction problems in one-dimensional space*, Appl. Math. Comput. **171** (2005), 1016–1024.

[19] SHOWALTER, R. E.: *The final value problem for evolution equations*, J. Math. Anal. Appl. **47** (1974), 563–572.

[20] SHOWALTER, R. E.: *Cauchy problem for hyper-parabolic partial differential equations.* In: Trends in the Theory and Practice of Non-Linear Analysis. North-Holland Math. Stud. 110, North-Holland, Amsterdam, 1985.

[21] TRONG, D. D.—TUAN, N. H.: *Regularization and error estimates for nonhomogeneous backward heat problems*, Electron. J. Differential Equations **2006** (2006), No. 04, 1–10.

[22] TRONG, D. D.—QUAN, P. H.—KHANH, T. V.—TUAN, N. H.: *A nonlinear case of the 1-D backward heat problem: Regularization and error estimate*, Z. Anal. Anwendungen **26** (2007), 231–245.

[23] TRONG, D. D.—TUAN, N. H.: *Regularization and error estimate for the nonlinear backward heat problem using a method of integral equation*, Nonlinear Anal. **71** (2009), 4167–4176.

[24] TAUTENHAHN, U.—SCHRÖTER, T.: *On optimal regularization methods for the backward heat equation*, Z. Anal. Anw. **15** (1996), 475–93.
[25] YILDIZ, B.—OZDEMIR, M.: *Stability of the solution of backwrad heat equation on a weak conpactum*, Appl. Math. Comput. **111** (2000), 1–6.
[26] YILDIZ, B.—YETIS, H.—SEVER, A.: *A stability estimate on the regularized solution of the backward heat problem*, Appl. Math. Comput. **135** (2003), 561–567.

Received 10. 6. 2010
Accepted 29. 4. 2011

* *Division of Applied Mathematics*
Ton Duc Thang University
Nguyen Huu Tho Street
District 7, Hochiminh City
VIETNAM
E-mail: tuanhuy_bs@yahoo.com