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# EXISTENCE OF HOMOCLINIC SOLUTIONS FOR A CLASS OF SECOND-ORDER NON-AUTONOMOUS HAMILTONIAN SYSTEMS

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ABSTRACT. By using the variant version of Mountain Pass Theorem, the existence of homoclinic solutions for a class of second-order Hamiltonian systems is obtained. The result obtained generalizes and improves some known works.



## 1. Introduction

Consider the second-order non-autonomous Hamiltonian system

$$\ddot{u}(t) + \nabla F(t, u(t)) = 0, \tag{1.1}$$

where $t \in \mathbb{R}$, $u \in \mathbb{R}^n$, $F \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. As usual, we say that a solution $u$ of (1.1) is a nontrivial homoclinic (to 0) if $u \neq 0$, $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $t \to \pm\infty$.

The existence of homoclinic solutions for Hamiltonian systems is a classical problem and its importance in the study of the behavior of dynamical systems has been recognized by Poincaré [15]. Up to the year of 1990, a few of isolated results can be found, and the only method for dealing with such problem was the small perturbation technique of Melnikov.

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Assuming that $F(t,x) = -\frac{1}{2}(L(t)x,x) + W(t,x)$, where $L(t)$ is a positive definite symmetric matrix-valued function for all $t \in \mathbb{R}$ and $W$ is superquadratic at infinity, many authors investigated the existence and multiplicity of homoclinic solutions for Hamiltonian systems by critical point theory. For example, see [1, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 16, 17] for second-order systems and [2, 22] for first order systems. When $L(t)$ and $W(t,x)$ are either independent of $t$ or periodic in $t$, Rabinowitz [16] had proved the existence of homoclinic orbits as the limit of solutions of a certain sequence of boundary value problems. By the same method, several results for general Hamiltonian systems were obtained by Izydorek and Janczewska [9], Ding and Lee [7], Tang and Xiao [18, 19, 20].

If $L(t)$ and $W(t,x)$ are neither autonomous nor periodic in $t$, the problem of existence of homoclinic solutions for (1.1) is quite different from the periodic systems, because of the lack of compactness of the Sobolev embedding. In [17], Rabinowitz and Tanaka studied the existence of homoclinic solutions for (1.1) without periodicity assumption both for $L$ and $W$. More precisely, they assumed that the smallest eigenvalue of $L(t)$ tends to $+\infty$ as $|t| \to \infty$, and using a variant of the Mountain Pass Theorem without (PS) condition, obtained the following theorem of the existence of a nontrivial homoclinic solution for (1.1).

**Theorem 1.1.** (See [17].) *Suppose that $L$ and $W$ satisfy the following conditions:*

(L) *$L(t)$ is positive definite symmetric matrix for all $t \in \mathbb{R}$ and there exists $l \in C(\mathbb{R}, (0,+\infty))$ such that $l(t) \to +\infty$ as $|t| \to \infty$ and*

$$(L(t)x,x) \geq l(t)|x|^2 \qquad \text{for all} \quad t \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}^n;$$

(W1) *$W(t,x) \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and there exists a constant $\theta > 2$ such that*

$$0 < \theta W(t,x) \leq (\nabla W(t,x), x) \qquad \text{for all} \quad t \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}^n \backslash \{0\};$$

(W2) *$|\nabla W(t,x)| = o(|x|)$ as $|x| \to 0$ uniformly with respect to $t \in \mathbb{R}$;*

(W3) *There exists $\overline{W} \in C(\mathbb{R}^n, \mathbb{R})$ such that*

$$|W(t,x)| + |\nabla W(t,x)| \leq |\overline{W}(x)| \qquad \text{for all} \quad t \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}^n.$$

*Then problem (1.1) has a nontrivial homoclinic solution.*

Recently, inspired by the papers [13, 17], Wan and Tang [21] considered the case that $F(t,x) = -K(t,x) + W(t,x)$ and obtained the following theorem which generalized the corresponding results in [13] and [17].

**Theorem 1.2.** (See [21].) *Suppose that* (W1)–(W3) *hold and the following conditions hold:*

(H1) $K \in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ *and there exists a positive constant* $\lambda$ *such that*

$$\frac{1}{2}(L(t)x,x) \le K(t,x) \le \frac{\lambda}{2}(L(t)x,x) \qquad \textit{for all} \quad t\in\mathbb{R} \quad \textit{and} \quad x\in\mathbb{R}^n,$$

*where* $L(t)$ *is a positive definite symmetric matrix-valued function for all* $t \in \mathbb{R}$*;*

(H2) $\frac{K(t,x)}{|x|^2} \to +\infty$ *as* $|t|\to\infty$ *uniformly in* $x\in\mathbb{R}^n\backslash\{0\}$*;*

(H3) *There exists a constant* $C_0>0$ *such that*

$$0 \le 2K(t,x) - (\nabla K(t,x),x) \le C_0|x|^2 \qquad \textit{for all} \quad t\in\mathbb{R} \quad \textit{and} \quad x\in\mathbb{R}^n.$$

*Then problem* (1.1) *has at least one nontrivial homoclinic solution.*

Motivated mainly by the ideas of [17, 21], we will consider the case

$$F(t,x) = -K(t,x) + W(t,x)$$

and further study the existence of homoclinic solutions for (1.1) under more general conditions. Here is our main result.

**Theorem 1.3.** *Suppose that* (H2), (H3) *hold and the following conditions hold:*

(W2)' $\nabla W(t,x)\to 0$ *as* $|x|\to 0$ *uniformly with respect to* $t\in\mathbb{R}$*;*

(H1)' $K \in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ *and there exist positive constants* $a$ *and* $b$ *such that*

$$a(L(t)x,x) \le K(t,x) \le b(L(t)x,x) \qquad \textit{for all} \quad t\in\mathbb{R} \quad \textit{and} \quad x\in\mathbb{R}^n,$$

*where* $L(t)$ *is a positive definite symmetric matrix-valued function for all* $t\in\mathbb{R}$*;*

(H4) *There exist constant* $c>0$, $d\in L^1(\mathbb{R},\mathbb{R}^+)$ *and* $\nu>2$ *such that*

$$W(t,x) \le c|x|^\nu + d(t) \qquad \textit{for all} \quad t\in\mathbb{R} \quad \textit{and} \quad x\in\mathbb{R}^n;$$

(H5) *There exist* $\alpha>0$, $\beta\in L^1(\mathbb{R},\mathbb{R}^+)$ *and* $\mu>\nu-2$ *such that*

$$(\nabla W(t,x),x) - 2W(t,x) \ge \alpha|x|^\mu - \beta(t) \qquad \textit{for all} \quad t\in\mathbb{R} \quad \textit{and} \quad x\in\mathbb{R}^n.$$

*Then problem* (1.1) *has at least one nontrivial homoclinic solution.*

**Remark 1.1.** As pointed out in [21], there are functions which can not be written in the form $F(t,x) = -\frac{1}{2}(L(t)x,x)+W(t,x)$, then the result here is different. It is also remarked that the function $K(t,x)$ is not necessarily homogeneous of degree 2 with respect to $x$ and so $\big[\int_{\mathbb{R}}(|\dot{x}(t)|^2 + K(t,x))\,\mathrm{d}t\big]^{\frac{1}{2}}$ is not a norm in general. From this point, our result is different from the previous ones.

**Remark 1.2.** It is easy to see that (H1)' is more general than (H1) and (W2)' is weaker than (W2). As is known, (W1) is the so-called global Ambrosetti-Rabinowitz condition on $W$ which implies that $W(t,x)$ is superquadratic growth at infinity, i.e.,

$$\lim_{|x|\to\infty} \frac{W(t,x)}{|x|^2} = +\infty \qquad \text{uniformly in} \quad t \in \mathbb{R}.$$

This kind of superquadratic condition is very important in many proofs, however, this condition is somewhat restrictive. Here we use other weaker superquadratic conditions (H4) and (H5) instead of (W1). We also note that the condition (W3) is not necessary in our proof and we drop it. So we generalize and improve [21, Theorem 1.2].

## 2. Proof of Theorem 1.3

Let

$$E = \Big\{ u \in H^1(\mathbb{R}, \mathbb{R}^n) : \int_{\mathbb{R}} \big[ |\dot{u}(t)|^2 + (L(t)u(t), u(t)) \big] \, \mathrm{d}t < +\infty \Big\}.$$

Then $E$ is a Hilbert space with the norm given by

$$\|u\| = \bigg( \int_{\mathbb{R}} |\dot{u}(t)|^2 \, \mathrm{d}t + \int_{\mathbb{R}} (L(t)u(t), u(t)) \, \mathrm{d}t \bigg)^{\frac{1}{2}}.$$

Further, we denote by $C_j$ positive constants. Since that $E \subset L^p(\mathbb{R}, \mathbb{R}^n)$ for all $p \in [2, \infty]$ with the imbedding being continuous, there exist positive constants $C_1$ and $C_2$ such that

$$\|u\|_\infty \le C_1 \|u\|, \quad \|u\|_{L^2} \le C_2 \|u\|, \qquad \text{for all} \quad u \in E. \tag{2.1}$$

Here $L^p(\mathbb{R}, \mathbb{R}^n)$ $(2 \le p < \infty)$ denotes the Banach spaces of functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the norm

$$\|u\|_{L^p} = \bigg( \int_{\mathbb{R}} |\dot{u}(t)|^p \, \mathrm{d}t \bigg)^{\frac{1}{p}}.$$

$L^\infty(\mathbb{R}, \mathbb{R}^n)$ is the Banach space of essentially bounded functions from $\mathbb{R}$ into $\mathbb{R}^n$ equipped with the norm

$$\|u\|_\infty = \operatorname{ess\,sup}\big\{ |u(t)| : t \in \mathbb{R} \big\}.$$

For any $u \in E$, let

$$I(u) = \frac{1}{2}\int_{\mathbb{R}} |\dot{u}(t)|^2 \,\mathrm{d}t + \int_{\mathbb{R}} [K(t,u(t)) - W(t,u(t))] \,\mathrm{d}t. \tag{2.2}$$

Then one can easily check that $I \in C^1(E,\mathbb{R})$ and

$$\langle I'(u), v\rangle = \int_{\mathbb{R}} \big[(\dot{u}(t),\dot{v}(t)) + (\nabla K(t,u(t)), v(t)) - (\nabla W(t,u(t)), v(t))\big] \,\mathrm{d}t \tag{2.3}$$

for $u, v \in E$. It is well known that the critical points of $I$ are classical solutions of (1.1). The following lemma is useful in our proof.

**Lemma 2.1.** (See [8].) *Let $E$ be a real Banach space with its dual space $E^*$ and suppose that $I \in C^1(E,\mathbb{R})$ satisfies*

$$\max\{I(0), I(e)\} \le \eta_0 < \eta \le \inf_{\|u\|=\rho} I(u),$$

*for some $\eta_0 < \eta$, $\rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $C \ge \eta$ be characterized by*

$$C = \inf_{\Upsilon\in\Gamma} \max_{0\le\tau\le 1} I(\Upsilon(\tau)),$$

*where $\Gamma = \big\{\Upsilon \in C([0,1],E) : \ \Upsilon(0) = 0, \ \Upsilon(1) = e\big\}$ is the set of continuous paths joining $0$ to $e$, then there exists $\{u_n\}_{n\in\mathbb{N}} \subset E$ such that*

$$I(u_n) \to C \quad \text{and} \quad (1 + \|u_n\|)\|I'(u_n)\|_{E^*} \to 0 \qquad \text{as} \quad n \to \infty.$$

Proof.

*Step 1.*

From (W2)', there exists $\rho_0 > 0$ such that

$$|\nabla W(t,x)| \le \frac{C_3}{C_2^2}|x| \qquad \text{for all} \quad t \in \mathbb{R} \quad \text{and} \quad |x| \le \rho_0, \tag{2.4}$$

where $C_3 = \min\{\frac{1}{2}, a\}$. From (2.4), we have

$$\begin{aligned} |W(t,x)| &= \Big|\int_0^1 (\nabla W(t,sx), x)\,\mathrm{d}s\Big| \\ &\le \int_0^1 |\nabla W(t,sx)||x|\,\mathrm{d}s \\ &\le \int_0^1 \frac{C_3}{C_2^2}|x|^2 s\,\mathrm{d}s = \frac{C_3}{2C_2^2}|x|^2 \qquad \text{for all} \quad t \in \mathbb{R} \quad \text{and} \quad |x| \le \rho_0. \end{aligned} \tag{2.5}$$

Let $\rho = \frac{\rho_0}{C_1}$ and $S = \{u \in E : \|u\| = \rho\}$, then we have

$$\|u\|_\infty \le \rho_0, \quad \|u\|_{L^2} \le C_2\rho \qquad \text{for all} \quad u \in S,$$

which together with (2.5) and (H1)' implies that

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}} |\dot{u}(t)|^2\,\mathrm{d}t + \int_{\mathbb{R}} K(t,u(t))\,\mathrm{d}t - \int_{\mathbb{R}} W(t,u(t))\,\mathrm{d}t \\ &\ge \frac{1}{2}\int_{\mathbb{R}} |\dot{u}(t)|^2\,\mathrm{d}t + \int_{\mathbb{R}} a(L(t)u(t),u(t))\,\mathrm{d}t - \int_{\mathbb{R}} W(t,u(t))\,\mathrm{d}t \\ &\ge \min\left\{\frac{1}{2}, a\right\}\|u\|^2 - \frac{C_3}{2C_2^2}\|u\|_{L^2}^2 \\ &\ge C_3\|u\|^2 - \frac{C_3}{2}\|u\|^2 = \frac{C_3}{2}\|u\|^2 = \alpha_1. \end{aligned}$$

*Step 2.*
From (H1)', we have

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}} |\dot{u}(t)|^2\,\mathrm{d}t + \int_{\mathbb{R}} K(t,u(t))\,\mathrm{d}t - \int_{\mathbb{R}} W(t,u(t))\,\mathrm{d}t \\ &\le \frac{1}{2}\int_{\mathbb{R}} |\dot{u}(t)|^2\,\mathrm{d}t + \int_{\mathbb{R}} b(L(t)u(t),u(t))\,\mathrm{d}t - \int_{\mathbb{R}} W(t,u(t))\,\mathrm{d}t \\ &\le \max\left\{\frac{1}{2}, b\right\}\int_{\mathbb{R}} [|\dot{u}(t)|^2 + (L(t)u(t),u(t))]\,\mathrm{d}t - \int_{\mathbb{R}} W(t,u(t))\,\mathrm{d}t \\ &= C_4\|u\|^2 - \int_{\mathbb{R}} W(t,u(t))\,\mathrm{d}t. \end{aligned} \tag{2.6}$$

Set $\varphi(s) = s^{-2}W(t,sx)$ for $s > 0$. Then it follows from (H5) that

$$\begin{aligned} \varphi'(s) &= s^{-3}\big[-2W(t,sx) + (\nabla W(t,sx), sx)\big] \\ &\ge \alpha s^{-3}|sx|^\mu - \beta(t)s^{-3} \\ &= \alpha s^{\mu-3}|x|^\mu - \beta(t)s^{-3} \qquad \text{for all} \quad (t,x) \in \mathbb{R}\times\mathbb{R}^n \quad \text{and} \quad s > 0. \end{aligned}$$

Integrating the above inequality from 1 to $\xi > 1$, we have

$$W(t,\xi x) \ge \xi^2 W(t,x) + \frac{\alpha}{\mu-2}(\xi^\mu - \xi^2)|x|^\mu - \frac{\beta(t)}{2}(\xi^2 - 1). \tag{2.7}$$

We have by Lagrange mean-value theorem:

$$\frac{1}{\mu-2}(\xi^\mu - \xi^2) = \xi^{\theta_1}\ln\xi, \qquad \theta_1 \in (2,\mu). \tag{2.8}$$

Let $u_0 \in E$ such that $|u_0(t)| \geq 1$ on a closed and non-empty interval $A \subset \mathbb{R}$. Then, from (2.6), (2.7) and (2.8), we have

$$\begin{aligned}
I(\xi u_0) &\leq C_4\|u_0\|^2\xi^2 - \int\limits_A W(t,\xi u_0(t))\,\mathrm{d}t \\
&\leq C_4\|u_0\|^2\xi^2 - \int\limits_A \xi^2 W(t,u_0(t))\,\mathrm{d}t \\
&- \int\limits_A \frac{\alpha}{\mu-2}(\xi^\mu - \xi^2)|u_0|^\mu\,\mathrm{d}t + \int\limits_A \frac{\beta(t)}{2}(\xi^2-1)\,\mathrm{d}t \\
&\leq C_4\|u_0\|^2\xi^2 - \int\limits_A \xi^2 W(t,u_0(t))\,\mathrm{d}t \\
&- \frac{\alpha}{\mu-2}(\xi^\mu - \xi^2)\,\mathrm{meas}(A) + C_5(\xi^2-1) \\
&\leq C_4\|u_0\|^2\xi^2 - \xi^2 C_6 - \frac{\alpha}{\mu-2}(\xi^\mu - \xi^2)\,\mathrm{meas}(A) + C_5(\xi^2-1) \\
&= \xi^2(C_4\|u_0\|^2 - C_6 + C_5) - \alpha\,\mathrm{meas}(A)\xi^{\theta_1}\ln\xi - C_5,
\end{aligned}$$

where $C_5$ is a positive constant and $C_6 = \min \int\limits_A W(t,u_0(t))\,\mathrm{d}t$. Since $\mu > 2$, $\theta_1 \in (2,\mu)$, we can choose $\xi_0 > \rho_1$ sufficiently large such that

$$I(\xi u_0) \leq 0 \qquad \text{for} \quad \|\xi u_0\| > \xi_0.$$

*Step 3.*

From Step 1, Step 2 and Lemma 2.1, we know that there is a sequence $\{u_n\}_{n\in\mathbb{N}} \subset E$ such that

$$I(u_n) \to C \quad \text{and} \quad (1+\|u_n\|)\|I'(u_n)\|_{E^*} \to 0 \qquad \text{as} \quad n \to \infty, \tag{2.9}$$

where $E^*$ is the dual space of $E$. In the following, we will prove that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $E$. It follows from (2.2), (2.3), (2.9), (H3) and (H5) that

$$\begin{aligned}
&C_7 \geq 2I(u_n) - \langle I'(u_n), u_n\rangle \\
&= \int\limits_{\mathbb{R}} \big[(\nabla W(t,u_n(t)), u_n(t)) - 2W(t,u_n(t))\big]\,\mathrm{d}t \\
&\quad + \int\limits_{\mathbb{R}} \big[2K(t,u_n(t)) - (\nabla K(t,u_n(t)), u_n(t))\big] \\
&\geq \int\limits_{\mathbb{R}} (\alpha|u_n(t)|^\mu - \beta(t))\,\mathrm{d}t = \int\limits_{\mathbb{R}} \alpha|u_n(t)|^\mu\,\mathrm{d}t - C_8.
\end{aligned} \tag{2.10}$$

From (2.1), (2.2), (2.10) and (H4), we have

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}} |\dot{u}_n(t)|^2\,\mathrm{d}t &+ \int_{\mathbb{R}} K(t,u_n(t))\,\mathrm{d}t \\
&= I(u_n) + \int_{\mathbb{R}} W(t,u_n(t))\,\mathrm{d}t \\
&\leq C_9 + \int_{\mathbb{R}} c|u_n(t)|^{\nu}\,\mathrm{d}t + \int_{\mathbb{R}} d(t)\,\mathrm{d}t \\
&\leq C_9 + C_{10} + c\|u_n\|_{\infty}^{\nu-\mu} \int_{\mathbb{R}} |u_n(t)|^{\mu}\,\mathrm{d}t \\
&\leq C_9 + C_{10} + \frac{cC_1^{\nu-\mu}(C_7+C_8)}{\alpha}\|u_n\|^{\nu-\mu}.
\end{aligned} \tag{2.11}$$

On the other hand, from (H1)', we have

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}} |\dot{u}_n(t)|^2\,\mathrm{d}t &+ \int_{\mathbb{R}} K(t,u_n(t))\,\mathrm{d}t \\
&\geq \frac{1}{2}\int_{\mathbb{R}} |\dot{u}_n(t)|^2\,\mathrm{d}t + \int_{\mathbb{R}} a(L(t)u_n(t),u_n(t))\,\mathrm{d}t \\
&\geq \min\left\{\frac{1}{2},a\right\}\left(\int_{\mathbb{R}} \big[|\dot{u}_n(t)|^2 + (L(t)u_n(t),u_n(t))\,\mathrm{d}t\big]\right) \\
&= C_3\|u_n\|^2.
\end{aligned} \tag{2.12}$$

It follows from (2.11) and (2.12) that

$$C_3\|u_n\|^2 \leq C_9 + +C_{10} + \frac{cC_1^{\nu-\mu}(C_7+C_8)}{\alpha}\|u_n\|^{\nu-\mu}. \tag{2.13}$$

Since $\mu > \nu - 2$, we obtain from (2.13) that $\|u_n\|$ is bounded in $E$.

Going to a subsequence if necessary, we may assume that there exists $u \in E$ such that $u_n \rightharpoonup u$ as $k \to \infty$. In order to prove our theorem, it is sufficient to show that $I'(u) = 0$. For $w \in C_0^{\infty}(\mathbb{R},\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}} (\dot{u}_n(t),\dot{w}(t))\,\mathrm{d}t \to \int_{\mathbb{R}} (\dot{u}(t),\dot{w}(t))\,\mathrm{d}t \qquad \text{as} \quad n \to \infty.$$

Noting that

$$\begin{gathered}
|(\nabla K(t,u_n),w)| \leq \sup(\nabla K(t,x))\|w\|_{\infty} \\
\text{for} \quad t \in \operatorname{supp} w \quad \text{and} \quad |x| \leq \sup_n \|u_n\|,
\end{gathered}$$

where $\operatorname{supp} w = \overline{\{t \in \mathbb{R} : \ w(t) \neq 0\}}$, it follows from (H1)' and Lebesgue dominated convergence theorem that

$$\int_{\mathbb{R}} (\nabla K(t, u_n(t)), w(t))\, \mathrm{d}t \to \int_{\mathbb{R}} (\nabla K(t, u(t)), w(t))\, \mathrm{d}t \qquad \text{as} \quad n \to \infty.$$

Similarly,

$$\int_{\mathbb{R}} (\nabla W(t, u_n(t)), w(t))\, \mathrm{d}t \to \int_{\mathbb{R}} (\nabla W(t, u(t)), w(t))\, \mathrm{d}t \qquad \text{as} \quad n \to \infty.$$

Hence, we have

$$\langle I'(u), w\rangle = \lim_{n\to\infty} \langle I'(u_n), w\rangle = 0.$$

Since $C_0^\infty(\mathbb{R}, \mathbb{R}^n)$ is dense in $E$, we get $I'(u) = 0$, i.e., $u$ is a critical point of $I$.

*Step 4.*

We will prove that $u$ is a nontrivial solution. Since $u_n \to u$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$, $u_n \to u$ in $L^2([-B, B], \mathbb{R}^n)$ for all $B < +\infty$. Hence, it is sufficient to show that there is $B > 0$ such that $u_n \not\to 0$ in $L^2([-B, B], \mathbb{R}^n)$. If not, we can assume that $u_n \to 0$ in $L^2([-B, B], \mathbb{R}^n)$ for all $B > 0$, then, there exists $\delta > 0$ independent of $B$ such that

$$\lim_{n\to\infty} \sup \|u_n\|^2_{L^2(\mathbb{R},\mathbb{R}^n)} \leq \frac{\delta}{\gamma(B)}, \tag{2.14}$$

where $\gamma(B) = \inf\limits_{|t|\geq B, x\neq 0} \frac{K(t,x)}{|x|^2} > 0$. Indeed, since $\{u_n\}$ is bounded, we have from (2.11) that $\int\limits_{\mathbb{R}} K(t, u_n)\, \mathrm{d}t$ is bounded. Let $\delta = \sup\limits_n \int\limits_{\mathbb{R}} K(t, u_n(t))\, \mathrm{d}t$. Hence, we have

$$\begin{aligned}
\|u_n\|^2_{L^2(\mathbb{R},\mathbb{R}^n)} &= \int_{-B}^{B} |u_n(t)|^2\, \mathrm{d}t + \int_{\mathbb{R}\setminus[-B,B]} |u_n(t)|^2\, \mathrm{d}t \\
&\leq \int_{-B}^{B} |u_n(t)|^2\, \mathrm{d}t + \frac{1}{\gamma(B)} \int_{\mathbb{R}\setminus[-B,B]} K(t, u_n(t))\, \mathrm{d}t \\
&\leq \int_{-B}^{B} |u_n(t)|^2\, \mathrm{d}t + \frac{\delta}{\gamma(B)}.
\end{aligned}$$

Letting $n \to \infty$, then (2.14) holds. Let $M = \sup\limits_n \|u_n\|$, then from (W2)', there exists $C_{11} > 0$ such that

$$|\nabla W(t, x)| \leq C_{11}|x|, \quad |W(t, x)| \leq C_{11}|x|^2 \qquad \text{for all} \quad t \in \mathbb{R} \quad \text{and} \quad |x| \leq M.$$

Hence, we have

$$\begin{aligned}|(\nabla W(t,x),x)-2W(t,x)| &\le |\nabla W(t,x)||x|+2|\nabla W(t,x)|\\ &\le C_{11}|x|^2+2C_{11}|x|^2=3C_{11}|x|^2,\end{aligned}$$

which together with (2.2), (2.3) and (H3) implies that

$$\begin{aligned}2C&=\lim_{n\to\infty}\int_{\mathbb{R}}\big[2K(t,u_n)-(\nabla K(t,u_n),u_n)+(\nabla W(t,u_n),u_n)-2W(t,u_n)\big]\,\mathrm{d}t\\ &\le (C_0+3C_{11})\lim_{n\to\infty}\|u_n\|^2_{L^2(\mathbb{R},\mathbb{R}^n)}\le \frac{(C_0+3C_{11})\delta}{\gamma(B)}. \qquad (2.15)\end{aligned}$$

From (H2), we know that $\gamma(B)\to+\infty$ as $B\to+\infty$, but then (2.15) is a contradiction. Hence, there exists $B>0$ such that $u_n\not\to 0$ in $L^2([-B,B],\mathbb{R}^n)$, i.e., $u\neq 0$.

Finally, we prove that the critical points of $I$ satisfying $u(t)\to 0$ and $\dot u(t)\to 0$ as $|t|\to+\infty$. Let $C^0(\mathbb{R},\mathbb{R}^n)$ be the space of continuous functions $u$ on $\mathbb{R}$ such that $u(t)\to 0$ as $|t|\to+\infty$, then we have $E\subset C^0(\mathbb{R},\mathbb{R}^n)$. Moreover, it is easy to check that $\dot u(t)\to 0$ as $|t|\to+\infty$. The proof of Theorem 1.3 is complete. □

## 3. An example

Let

$$\begin{aligned}F(t,x)&=-\bigg(1+t^2+\frac{1}{|x|^2+1}\bigg)|x|^2+\mathrm{e}^{1-|t|}|x|^2\ln(1+|x|),\\ L(t)&=\operatorname{diag}(1+t^2,\ldots,1+t^2),\end{aligned}$$

where

$$K(t,x)=\bigg(1+t^2+\frac{1}{|x|^2+1}\bigg)|x|^2,\qquad W(t,x)=\mathrm{e}^{1-|t|}|x|^2\ln(1+|x|).$$

It is easy to check that $K$ satisfies conditions (H1)', (H2), (H3). An easy computation shows that $W(t,x)$ satisfies (W2)'. Since

$$W(t,x)\le \mathrm{e}|x|^3,\qquad (\nabla W(t,x),x)-2W(t,x)=\frac{\mathrm{e}^{1-|t|}|x|^3}{1+|x|},$$

then $W$ satisfies (H4) and (H5) with $\nu=3$ and $\mu=2$. Hence, $K(t,x)$ and $W(t,x)$ satisfy all the conditions of Theorem 1.3 and then problem (1.1) has at least one nontrivial homoclinic solution.

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