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CANTOR EXTENSION OF AN ABELIAN LATTICE ORDERED GROUP EQUIPPED WITH A WEAK RELATIVELY UNIFORM CONVERGENCE

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ABSTRACT. The notion of weak relatively uniform convergence (wru-convergence, for short) on an abelian lattice ordered group G has been investigated in a previous authors' article. In the present paper we deal with Cantor extension of G and completion of G with respect to a wru-convergence on G.

1. Introduction

The relatively uniform convergence in archimedean vector lattices and in archimedean lattice ordered groups has been studied in several papers; for references, cf. Černák [2] and Černák and Lihová [5].

We remark that each archimedean lattice ordered group is abelian, but not conversely.

Let G be an abelian lattice ordered group and $\mathcal{A}(G)$ the set of all archimedean elements of G. Assume that $\emptyset \neq M \subseteq \mathcal{A}(G)$ and that M is closed with respect to the addition. The weak relatively uniform convergence (wru-convergence, for short) on G with respect to the set M has been defined and investigated in the authors' paper [3]. This convergence is denoted by $\alpha(M)$; the detailed definition

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is recalled in Section 2 below. If G is archimedean and $M = G^+$, then the convergence $\alpha(M)$ coincides with the relatively uniform convergence on G.

Cantor extension of an archimedean lattice ordered group with respect to the relatively uniform convergence has been studied by Černák and Lihová [4]. In the present paper we deal with a Cantor extension of an abelian lattice ordered group G with respect to the convergence $\alpha(M)$.

The relatively uniform completion (ru-completion, for short) of an archimedean lattice ordered group has been investigated by Hager and Martinez [8], Ball and Hager [1], Martinez [12], Černák and Lihová [5], and Jakubík and Černák [11]. The ru-completion of an archimedean lattice ordered group G is denoted by G_{ω_1}. It can be obtained by ω_1 steps; the application of Cantor extension of archimedean lattice ordered groups is essential in this construction.

The definition of weak relatively uniform completion (wru-completion, for short) of an abelian lattice ordered group G is analogous to that of ru-completion of an archimedean lattice ordered group. The corresponding definitions are recalled in §2.

In §3, the relation between wru-convergence of an abelian lattice ordered group G and wru-completion of its archimedean kernel is described. This connection has been considered in [3], Proposition 3.5. In the present paper, a different method is applied that seems to be lucider and brings more light into the considered situation.

In §4, a Cantor extension of an abelian lattice ordered group G with respect to wru-convergence $\alpha(M)$ is constructed. Having a Cantor extension of G, a wru-completion of G with respect to convergence $\alpha(M)$ can be constructed by the same method as in the case of archimedean lattice ordered groups.

Further, it is shown that the Cantor extension and the wru-completion of G with respect to considered convewrgence $\alpha(M)$ are uniquely determined up to isomorphisms over G.

In §5, a wru-completion of convex ℓ-subgroups of G are considered.

2. Preliminaries

For lattice ordered groups we apply the notation and the terminology as in Glass [7] with the distinction that the group operation is written additively.

All lattice ordered groups considered in the present paper are assumed to be abelian.

For the sake of completeness, we recall the basic definitions concerning wru-convergence in a lattice ordered group (cf. [3]).

Let G be a lattice ordered group. An element $a \in G^+$ will be said to be *archimedean* if, whenever $0 \leq b \in G$ and $nb \leq a$ for each $n \in \mathbb{N}$, then $b = 0$. If all elements of G^+ are archimedean, then G is an *archimedean lattice ordered group.*

Assume that M is a nonempty set of archimedean elements of G and that M is closed with respect to the addition.

Let (x_n) be a sequence in G and $x \in G$. We say that this sequence $\alpha(M)$-*converges* to x and we write

$$x_n \to_{\alpha(M)} x, \tag{1}$$

if there exists $b \in M$ such that for each $k \in \mathbb{N}$ there is $n_0(b,k) \in \mathbb{N}$ such that the relation

$$k|x_n - x| \leq b$$

is valid whenever $n \in \mathbb{N}$, $n \geq n_0(b,k)$.

This type of convergence is said to be a *weak relatively uniform convergence* (*wru-convergence*, for short) corresponding to the system M of regulators.

Sometimes we write $\alpha(G,M)$ instead of $\alpha(M)$, if the role of G is to be emphasized.

The assumption that the set M is closed under the addition enables to prove the basic properties of $\alpha(M)$-convergence presented in [3].

The ru-convergence in an archimedean lattice ordered group G is a particular case of $\alpha(M)$ convergence (for the case $M = G^+$). If $x_n \to_{\alpha(G,G^+)} x$, we also write $x_n \to_{ru} x$.

A sequence (x_n) in a lattice ordered group G is a *Cauchy sequence* with respect to the convergence $\alpha(M)$ if there exists $b \in M$ such that for each $k \in \mathbb{N}$ there is $n_1(b,k) \in \mathbb{N}$ such that the relation

$$k|x_n - x_m| \leq b$$

is valid whenever $n \in \mathbb{N}$, $m \in \mathbb{N}$, $n \geq n_1(b,k)$ and $m \geq n_1(b,k)$.

The lattice ordered group G is *Cauchy complete* with respect to the convergence $\alpha(M)$ if, whenever (x_n) is a Cauchy sequence with respect to $\alpha(M)$, then there exists $x \in G$ such that the relation (1) is valid. We also say that G is *wru-complete* with respect to $\alpha(M)$.

If G is an archimedean lattice ordered group which is Cauchy complete with respect to the convergence $\alpha(G^+)$, then G is said to be *ru-complete.* (Cf. [4].)

We also recall the following definition (cf. [1], [11], [5]).

Definition 2.1. Assume that G is an archimedean lattice ordered group. Let K be an archimedean lattice ordered group with the following properties:

(a) G is an ℓ-subgroup of K;

(b) K is ru-complete;

(c) if H is an ℓ-subgroup of K, G is an ℓ-subgroup of H and H is ru-complete, then $H = K$.

Under these assumptions, K is said to be a *relatively uniform completion* (*ru-completion*, for short) of G.

Let H be a lattice ordered group. The symbol $\mathcal{A}(H)$ will denote the set of all archimedean elements of H.

DEFINITION 2.2. Assume that G is a lattice ordered group and that M is a nonempty subset of $\mathcal{A}(G)$ which is closed with respect to the addition. Let K_1 be a lattice ordered group with the following properties:

(a) G is an ℓ-subgroup of K_1;

(b) $M \subseteq \mathcal{A}(K_1)$;

(c) K_1 is Cauchy complete with respect to the convergence $\alpha(K_1, M)$;

(d) if H is an ℓ-subgroup of K_1, G is an ℓ-subgroup of H and H is Cauchy complete with respect to the convergence $\alpha(H, M)$, then $H = K_1$.

Under these assumptions, K_1 is said to be a *weak relatively uniform completion* (*wru-completion*, for short) of G with respect to the convergence $\alpha(K_1, M)$.

We recall that when we modify the definition of the convergence $\alpha(G, M)$ in such a way that the elements of M belong to G^+ but need not be archimedean, then the limits under such definition of convergence need not be uniquely determined. In view of this fact we consider only archimedean elements as belong to the set M.

We will apply the basic properties of the convergence $\alpha(G, M)$ which have been proved in [3].

3. Archimedean kernel of a lattice ordered group

Again, let G be a lattice ordered group, $\mathcal{A}(G)$ the set of all archimedean elements of G.

LEMMA 3.1. (Cf. [9].) *Let $A(G)$ be the ℓ-subgroup of G generated by the set $\mathcal{A}(G)$. Then*

(i) *$A(G)$ is a convex ℓ-subgroup of G;*

(ii) *$A(G)$ is an archimedean lattice ordered group;*

(iii) *if H is a convex ℓ-subgroup of G and if H is archimedean, then $H \subseteq A(G)$.*

We say that $A(G)$ is the *archimedean kernel* of G.

LEMMA 3.2. *Let $p, q \in G$ and suppose that $p + A(G) \neq q + A(G)$. Then the element $|p - q|$ fails to be archimedean.*

P r o o f. In view of (i), $A(G)$ is an ℓ-ideal of G. From the relation $p + A(G) \neq q + A(G)$, we get that the elements p and q are distinct. Hence $|p - q| > 0$.

We put

$$p \wedge q = r, \quad p \vee q = v, \quad p_1 = p - r, \quad q_1 = q - r.$$

Then we have

$$v - p = q_1, \quad v - q = p_1, \quad |p - q| = v - r = p_1 + q_1,$$

$$0 \le p_1 \le |p - q|, \quad 0 \le q_1 \le |p - q|, \quad q = p + q_1 - p_1.$$

By way of contradiction, assume that both p_1 and q_1 are archimedean elements of G. Then $q_1 - p_1 \in A(G)$, yielding that $q \in p + A(G)$, whence $p + A(G) = q + A(G)$, which is impossible. Therefore, either p_1 or q_1 fails to be archimedean. Then $|p - q|$ cannot be archimedean. □

Let M be as in Section 2; consider the convergence $\alpha(M)$.

Lemma 3.3. *Let (x_n) be a Cauchy sequence with respect to the convergence $\alpha(M)$. Then there exists $n_0 \in \mathbb{N}$ such that $x_{n(1)} + A(G) = x_{n(2)} + A(G)$ whenever $n(1), n(2) \in \mathbb{N}$ and $n(1) \ge n_0$, $n(2) \ge n_0$.*

P r o o f. By way of contradiction, suppose that there does not exist any $n_0 \in \mathbb{N}$ having the mentioned property. Thus for each $n \in \mathbb{N}$ there are $n(1)$ and $n(2)$ in $\mathbb{N}$ such that $n(1) \ge n$, $n(2) \ge n$ and

$$x_{n(1)} + A(G) \neq x_{n(2)} + A(G).$$

Thus in view of Lemma 3.2, the element $|x_{n(1)} - x_{n(2)}|$ fails to be archimedean.

Since (x_n) is a Cauchy sequence with respect to the convergence $\alpha(M)$, there exist $b \in M$ and $n_0 \in \mathbb{N}$ such that

$$|x_{m(1)} - x_{m(2)}| \le b$$

whenever $m(1) \ge n_0$ and $m(2) \ge n_0$.

Take $n(1)$ and $n(2)$ as above for $n = n_0$. Then $|x_{n(1)} - x_{n(2)}|$ fails to be archimedean. But as $|x_{n(1)} - x_{n(2)}| \le b$, b fails to be archimedean as well; we arrived at a contradiction. □

Proposition 3.4. *Let G and M be as above. Let (x_n) be a sequence in G. The following conditions are equivalent:*

(i) *the sequence (x_n) is Cauchy with respect to the convergence $\alpha(G, M)$;*

(ii) *there are $n_0 \in \mathbb{N}$, $x \in G$ and a sequence (y_n) in $A(G)$ which is Cauchy with respect to the convergence $\alpha(A(G), M)$ such that $x_n = x + y_n$ for each $n \ge n_0$.*

P r o o f. Let (i) be valid. In view of Lemma 3.3, there are $x \in G$ and $n_0 \in \mathbb{N}$ such that $x_n \in x + A(G)$ for each $n \ge n_0$. Put $y_n = x_n - x$ for each $n \in \mathbb{N}$ with $n \ge n_0$, and $y_n = 0$ otherwise. Then (y_n) is a sequence in $A(G)$ and whenever $n(1) \ge n_0$ and $n(2) \ge n_0$, then

$$|y_{n(1)} - y_{n(2)}| = |x_{n(1)} - x_{n(2)}| \tag{1}$$

is valid. From the fact that (x_n) is Cauchy with respect to $\alpha(G, M)$ and from relation (1), we conclude that (y_n) is Cauchy with respect to the convergence $\alpha(A(G), M)$.

Conversely, assume that (ii) holds. Again, if $n(1) \geq n_0$ and $n(2) \geq n_0$, then the relation (1) is valid. Then in view of the assumption concerning the sequence (y_n), we infer that the sequence (x_n) is Cauchy with respect to the convergence $\alpha(G, M)$. $\square$

Proposition 3.5. *Let G and M be as above. The following conditions are equivalent:*

(i) *the lattice ordered group G is wru-complete with respect to the convergence $\alpha(G, M)$;*

(ii) *the lattice ordered group $A(G)$ is wru-complete with respect to the convergence $\alpha(A(G), M)$.*

Proof. The implication (i) $\Longrightarrow$ (ii) is obvious. From Proposition 3.4 we conclude that the implication (ii) $\Longrightarrow$ (i) is valid. $\square$

4. Dedekind extension and Cantor extension

Let G be a lattice ordered group and M as in Section 2. We are going to prove that there exist a Cantor extension (without using Cantor's scheme) and a *wru*-completion of G with respect to considered convergence $\alpha(M)$ and that they are uniquely determined up to isomorphisms.

The notion of Dedekind extension $G^\wedge$ of a lattice ordered group G is well-known; its construction is described in detail, e.g., in Fuchs [6, Chapter V, Section 10]. Under a natural embedding, G is an ℓ-subgroup of $G^\wedge$.

The following lemma is easy to verify, the proof will be omitted.

Lemma 4.1. *Let G be a lattice ordered group and let H be a convex ℓ-subgroup of G. Let H_0 be the set of all elements $x \in G^\wedge$ such that there exist $y_1, y_2 \in H$ with $y_1 \leq x \leq y_2$. Then H_0 is a convex ℓ-subgroup of $G^\wedge$ and $H_0 = H^\wedge$.*

Also, it is well-known that if G is an archimedean lattice ordered group, then $G^\wedge$ is archimedean as well.

Under the notation as in Section 3, consider the convergence $\alpha(G, M)$ on G; as above, we speak about *ru*-convergence on G if dealing with $\alpha(G, G^+)$.

Definition 4.2. (Cf. [5, Definition 1.11].) Let G be an archimedean lattice ordered group. Let H be an archimedean lattice ordered group with the following properties:

(i) G is an ℓ-subgroup of H.

(ii) For every Cauchy sequence (x_n) in G there exists $x \in H$ such that $x_n \to_{ru} x$ in H.

(iii) For each $x \in H$ there is a Cauchy sequence (x_n) in G such that $x_n \to_{ru} x$ in H.

Then H will be called a *Cantor extension* of G.

Theorem 4.3. (Cf. [5, Theorem 1.13].) *Let G be an archimedean lattice ordered group. Then a Cantor extension H of G exists and is uniquely determined up to isomorphisms over G.*

The following definition is analogous to that of 4.2.

Definition 4.4. Let G be a lattice ordered group and let M be as in Section 2. Let H_1 be a lattice ordered group with the following properties:

(i) G is an ℓ-subgroup of H_1.

(ii) $M \subseteq \mathcal{A}(H_1)$.

(iii) If (x_n) is a sequence in G which is Cauchy with respect to the convergence $\alpha(G, M)$, then there exists $x \in H_1$ such that $x_n \to_{\alpha(H_1,M)} x$.

(iv) If $x \in H_1$, then there exists a sequence (x_n) in G such that $x_n \to_{\alpha(H_1,M)} x$.

Then H_1 will be called a *Cantor extension* of G with respect to the convergence $\alpha(H_1, M)$.

Remark that the sequence (x_n) in the condition (iv) is Cauchy in G. This is a consequence of the condition (ii).

Consider the following construction. Let G be a lattice ordered group. Then the archimedean kernel $A(G)$ is a convex ℓ-subgroup of G. Hence according to Lemma 4.1, the lattice ordered group $(A(G))^\wedge$ is a convex ℓ-subgroup of $G^\wedge$.

Let M be a nonempty subset of $\mathcal{A}(G)$ which is closed with respect to the addition.

We have $\mathcal{A}(G) \subseteq \mathcal{A}(G^\wedge)$, thus we can consider the convergence $\alpha(G^\wedge, M)$. Let H be the set of all elements $x \in G^\wedge$ having the property that there exists a sequence (x_n) in G such that

$$x_n \to_{\alpha(G^\wedge,M)} x. \tag{1}$$

Lemma 4.5. *H is an ℓ-subgroup of $G^\wedge$ and G is an ℓ-subgroup of H.*

Proof. In view of the definition of H, we infer that H is closed with respect to the operations $+, \wedge$ and $\vee$ (taken in $G^\wedge$). Hence H is an ℓ-subgroup of $G^\wedge$.

Since both G and H are ℓ-subgroups of $G^\wedge$ and because of the relation $G \subseteq H$ we get that G is an ℓ-subgroup of H. $\square$

Proposition 4.6. *Let G be a lattice ordered group and let H be as in* Lemma 4.5. *Then H is the Cantor extension of G with respect to the convergence $\alpha(H, M)$.*

P r o o f.

a) According to Lemma 4.5, G is an ℓ-subgroup of H.

b) As observed above, $\mathcal{A}(G) \subseteq \mathcal{A}(G^\wedge)$. According to 4.5, H is an ℓ-subgroup of $G^\wedge$. Hence $M \subseteq \mathcal{A}(H)$.

c) Let (x_n) be a sequence in G which is Cauchy with respect to the convergence $\alpha(G, M)$. Then (x_n) is Cauchy also with respect to the convergence $\alpha(G^\wedge, M)$. According to [3], $G^\wedge$ is Cauchy complete with respect to convergence $\alpha(G^\wedge, M)$ (in [3], the symbol $D(G)$ is used instead of $G^\wedge$). Therefore there exists $x \in G^\wedge$ such that the relation (1) is valid. Thus the element x belongs to H. Moreover, the relation

$$x_n \to_{\alpha(H,M)} x \tag{2}$$

holds.

d) Let $x \in H$. Then there exists a sequence (x_n) in G such that (1) is satisfied. Then the relation (2) is valid as well. □

Now, we denote by G_0^a the Cantor extension of $A(G)$ with respect to the convergence $\alpha(G_0^a, M)$. Its existence follows from Proposition 4.6, taking $A(G)$ instead of G, just like the fact that G_0^a is an ℓ-subgroup of both $(A(G))^\wedge$ and H.

Since both G and G_0^a are subsets of $G^\wedge$, we can consider the set $G + G_0^a$ (where the operation $+$ is taken as in $G^\wedge$).

Proposition 4.7. *The relation $G + G_0^a = H$ is valid.*

P r o o f. The inclusion $G + G_0^a \subseteqq H$ is evident.

Let $h \in H$. There exists a sequence (x_n) in G such that the relation

$$x_n \to_{\alpha(H,M)} h \tag{1'}$$

is valid. Thus (x_n) is a Cauchy sequence with respect to the convergence $\alpha(H, M)$. This yields that (x_n) is also a Cauchy sequence with respect to the convergence $\alpha(G, M)$. Applying Proposition 3.4, we get that there are $n_0 \in \mathbb{N}$, $x \in G$ and a sequence (y_n) in $A(G)$ which is Cauchy with respect to the convergence $\alpha(A(G), M)$ and $x_n = x + y_n$ for each $n \geq n_0$. Therefore, there is $y \in G_0^a$ with

$$y_n \to_{\alpha(G_0^a,M)} y.$$

Then the analogous result holds with respect to the convergence $\alpha(H, M)$, thus

$$x_n \to_{\alpha(H,M)} x + y.$$

Hence $h = x + y$ and thus $h \in G + G_0^a$. Summarizing, $G + G_0^a = H$. □

Corollary 4.8. *Let G be a lattice ordered group. Then $G + G_0^a$ is a Cantor extension of G.*

Let G, M and H be as above. Assume that H' is a latticed ordered group with $M \subseteq \mathcal{A}(H')$. Further, suppose that H' is a Cantor extension of G with respect to the convergence $\alpha(H', M)$.

A sequence (x_n) in G is a Cauchy sequence in H with respect to $\alpha(H, M)$ if and only if it is a Cauchy sequence in H' with respect to $\alpha(H', M)$. From this and from Definition 4.4 we easily obtain

Proposition 4.9. *Let us apply the notation as above. Then there exists an isomorphism $f \colon H \to H'$ leaving all elements of G fixed.*

Again, let G and M be as above. We have already verified that there exists a Cantor extension of G which turns out to be an ℓ-subgroup of $G^\wedge$; let us denote this Cantor extension by G^*.

If G' is a lattice ordered group such that G is an ℓ-subgroup of G' and G' is an ℓ-subgroup of $G^\wedge$, then we clearly have $(G')^\wedge \subseteq G^\wedge$, thus, under the notation analogous to that applied above, we have $(G')^* \subseteq G^\wedge$.

We will now use a construction which has been already applied for the case of archimedean lattice ordered groups (cf., e.g., [1]).

Let ω_1 be the first uncountable ordinal. For each ordinal $\lambda \le \omega_1$ we define a lattice ordered group G_λ as follows:

$$G_0 = G$$

$$G_\lambda = (G_{\lambda-1})^* \text{ if } \lambda < \omega_1 \text{ and } \lambda - 1 \text{ is a predecessor of } \lambda;$$

$$G_\lambda = \Big(\bigcup_{\beta<\lambda} G_\beta\Big)^* \text{ if } \lambda < \omega_1 \text{ and } \lambda \text{ is a limit ordinal;}$$

$$G_{\omega_1} = \bigcup_{\beta<\omega_1} G_\beta.$$

Analogously as in the archimedean case, we have

Proposition 4.10. *Let G and M be as above. Then the lattice ordered group G_{ω_1} is a wru-completion of G with respect to the convergence $\alpha(G_{\omega_1}, M)$.*

Applying Proposition 4.9, Proposition 4.10 and the obvious induction, we obtain

Proposition 4.11. *Let G and M be as above. Then the wru-completion H of G with respect to the convergence $\alpha(H, M)$ is determined uniquely up to isomorphisms leaving all elements of G fixed.*

Let L be a lattice ordered group and $\emptyset \ne M \subseteq \mathcal{A}(L)$ such that M is closed with respect to the addition. Then $M \subseteq \mathcal{A}(L')$ whenever L' is an ℓ-subgroup of L with $M \subseteq L'$. Hence besides convergence $\alpha(L, M)$ also convergence $\alpha(L', M)$ can be considered. In the next this fact will be used.

As observed above, $\mathcal{A}(G) \subseteq \mathcal{A}(G^\wedge)$. Then we can consider the convergence $\alpha(K, M)$ for each ℓ-subgroup K of $G^\wedge$ with $M \subseteq K$.

Proposition 4.12. *Let G be a lattice ordered group and let M be a nonempty subset of $\mathcal{A}(G)$ which is closed with respect to the addition. Assume that $\{G_i : i \in I\}$ is the system of all ℓ-subgroups of $G^{\wedge}$ satisfying the following conditions:*

(i) *G is an ℓ-subgroup of G_i for each $i \in I$.*

(ii) *For each $i \in I$, G_i is a wru-complete ℓ-subgroup of $G^{\wedge}$ with respect to convergence $\alpha(G_i, M)$.*

Then $H = \bigcap G_i (i \in I)$ is a wru-completion of G with respect to the convergence $\alpha(H, M)$.

P r o o f. With respect to [3], $G^{\wedge}$ is *wru*-complete with respect to convergence $\alpha(G^{\wedge}, M)$. We have to prove that H has the properties (a)–(d) from Definition 2.2.

(a) Evidently, G is an ℓ-subgroup of H;

(b) The inclusions $M \subseteq G \subseteq G_i$ for each $i \in I$ impliy $M \subseteq H$. Since H is an ℓ-subgroup of $G^{\wedge}$, $M \subseteq \mathcal{A}(H)$.

(c) Let (x_n) be a sequence in H which is Cauchy with respect to convergence $\alpha(H, M)$. Hence, for each $i \in I$, (x_n) is a Cauchy sequence in G_i with respect to the convergence $\alpha(G_i, M)$. By the assumption, for each $i \in I$, there exists $x_i \in G_i$ such that $x_n \to_{\alpha(G_i,M)} x_i$. Then $x_n \to_{\alpha(G^{\wedge},M)} x_i$. Limits in $G^{\wedge}$ are uniquely determined. Thus there is $x \in G^{\wedge}$ such that $x_i = x$ for each $i \in I$, so $x \in H$. Then $x_n \to_{\alpha(G^{\wedge},M)} x$. Finally, $x_n \to_{\alpha(H,M)} x$.

(d) Let H' be an ℓ-subgroup of H, G an ℓ-subgroup of H' and H' *wru*-complete with respect to convergence $\alpha(H', M)$. Then H' being an ℓ-subgroup of $G^{\wedge}$, we get $H \subseteq H'$. Hence $H' = H$. □

Lemma 4.13. *Let G and M be as above. Then G_{ω_1} is an ℓ-subgroup of $G^{\wedge}$.*

P r o o f. It suffices to prove that G_λ is an ℓ-subgroup of $G^{\wedge}$ for every ordinal $\lambda < \omega_1$. Assume that $\lambda < \omega_1$.

Let $\lambda = 0$. This case is clear, since $G_0 = G$.

Let λ be a non-limit ordinal $\lambda > 0$. Assume that $G_{\lambda-1}$ is an ℓ-subgroup of $G^{\wedge}$. By Lemma 4.5 and Proposition 4.6, the Cantor extension G^* of G is an ℓ-subgroup of $G^{\wedge}$. This yields that $G_\lambda = (G_{\lambda-1})^*$ is an ℓ-subgroup of $(G_{\lambda-1})^{\wedge}$. Applying the assumption, we get $(G_{\lambda-1})^{\wedge} \subset G^{\wedge}$ and the proof for this case is finished.

Let λ be a limit ordinal. Suppose that G_τ is an ℓ-subgroup of $G^{\wedge}$ for each $\tau < \lambda$. Consequently, $\bigcup\limits_{\tau<\lambda} G_\tau^*$ is an ℓ-subgroup of $G^{\wedge}$. By using the same argument as above, we get that $G_\lambda = (\bigcup\limits_{\tau<\lambda} G_\tau)^*$ is an ℓ-subgroup of $(\bigcup\limits_{\tau<\lambda} G_\tau)^{\wedge}$. The inclusion $(\bigcup\limits_{\tau<\lambda} G_\tau)^{\wedge} \subseteq G^{\wedge}$ completes the proof. □

Let H be as in 4.12. By Proposition 4.10, G_{ω_1} is a wru-completion of G with respect to convergence $\alpha(G_{\omega_1}, M)$. On account of Lemma 4.13, $H \subseteq G_{\omega_1}$. Since H is wru-complete with respect to convergence $\alpha(H, M)$ and G is an ℓ-subgroup of H, we have

Corollary 4.14. *Let H be as in* 4.12. *Then $H = G_{\omega_1}$*

5. wru-completion of convex ℓ-subgroups of G

Let G be a lattice ordered group and K a convex ℓ-subgroup of G. Then

$$c(K) = \{x \in G^* : \ a_1 \le x \le a_2 \ \text{for some} \ a_1, a_2 \in K\}$$

is a convex ℓ-subgroup of G^*.

Lemma 5.1. *Let K be a convex ℓ-subgroup of G and M be as in* Section 2. *Assume that $M \subseteq K$. Then $c(K) \simeq K^*$.*

P r o o f. We are going to verify the validity of conditions (i)–(iv) in Definition 4.4 (G and H_1 are replaced by K and $c(K)$, respectively).

(i) From definition of $c(K)$, it follows that K is an ℓ-subgroup of $c(K)$.

(ii) Since $c(K)$ is an ℓ-subgroup of G^*, the inclusion $M \subseteq \mathcal{A}(G^*)$ implies $M \subseteq \mathcal{A}(c(K))$.

(iii) Let (c_n) be a sequence in K that is Cauchy with respect to convergence $\alpha(K, M)$. Hence (x_n) is bounded in K, so $a_1 \le x_n \le a_2$ for some $a_1, a_2 \in K$ and for each $n \in \mathbb{N}$. The sequence (x_n) is in G and it is Cauchy with respect to convergence $\alpha(G, M)$. Then by Definition 4.4 there exists $x \in G^*$ such that $x_n \to_{\alpha(G^*, M)} x$. From $a_1 \le x \le a_2$ we infer that $x \in c(K)$. The sequence (x_n) is in $c(K)$, so $x_n \to_{\alpha(c(K), M)} x$.

(iv) Let $x \in c(K)$, $x \ge 0$. From $x \in G^*$, it follows that there exists a sequence (x_n) in G such that $x_n \to_{\alpha(G^*, M)} x$. Without loss of generality, we can suppose that $x_n \ge 0$ for each $n \in \mathbb{N}$. There exists $a \in K$ with $x \le a$. We get $x_n \wedge a \to_{\alpha(G^*, M)} x \wedge a = x$. The convexity of K in G yields that $(x_n \wedge a)$ is a sequence in K. We conclude $x_n \wedge a \to_{\alpha(c(K), M)} x$. □

Now, let us form the set

$$c_1(K) = \{x \in G_{\omega_1} : \ a_1 \le x \le a_2 \ \text{for some} \ a_1, a_2 \in K\}.$$

Then, $c_1(K)$ is a convex ℓ-subgroup of G_{ω_1}.

Proposition 5.2. *Let K be a convex ℓ-subgroup of G and let M be as in* 5.1. *Then $c_1(K) \simeq K_{\omega_1}$.*

P r o o f. We have to verify that the conditions in Definition 2.2 are fulfilled (with K and $c_1(K)$ instead of G and K_1, respectively).

(a) This is a consequence of definition of $c_1(K)$.

(b) The lattice ordered group $c_1(K)$ is an ℓ-subgroup of G_{ω_1} and according to Lemma 4.13, G_{ω_1} is an ℓ-subgroup of $G^\wedge$. Then $M \subseteq \mathcal{A}(G^\wedge)$ and $M \subseteq K \subseteq c_1(K)$ imply $M \subseteq \mathcal{A}(c_1(K))$.

(c) Assume that (x_n) is a sequence in $c_1(K)$ such that (x_n) is Cauchy with respect to convergence $\alpha(c_1(K), M)$; (x_n) is a sequence in G_{ω_1} and it is Cauchy with respect to convergence $\alpha(G_{\omega_1}, M)$. G_{ω_1} being wru-complete, there is $x \in G_{\omega_1}$ such that $x_n \to_{\alpha(G_{\omega_1},M)} x$. The sequence (x_n) is bounded in $c_1(K)$. Thus there are $v, w \in c_1(K)$ with $v \leq x_n \leq w$ for each $n \in \mathbb{N}$. Let $a_1, a_2 \in K$ with $a_1 \leq v, w \leq a_2$. Then $a_1 \leq x_n \leq a_2$ for each $n \in \mathbb{N}$. Hence $a_1 \leq x \leq a_2$, so $x \in c_1(K)$. This yields $x_n \to_{\alpha(c_1(K),M)} x$ and the proof of this part is finished.

(d) Let H be an ℓ-subgroup of $c_1(K)$ and suppose that H is wru-complete with respect to convergence $\alpha(H, M)$. Further, assume that K is an ℓ-subgroup of H. For each $\lambda < \omega_1$ we have $B_\lambda = G_\lambda \cap c_1(K) \supseteq K$ and $\bigcup\limits_{\lambda<\omega_1} B_\lambda = c_1(K)$. In order to show that $c_1(K) \subseteq H$, it suffices to prove that $B_\lambda \subseteq H$ for each $\lambda < \omega_1$. We will prove it by induction.

(i) Let $\lambda = 0$. We get $B_0 = G_0 \cap c_1(K) = G \cap c_1(K) = K \subseteq H$. Now, let $\lambda > 0$ and suppose that $B_\tau \subseteq H$ for each $\tau < \lambda$. Take $x \in B_\lambda$, $x \geq 0$.

(ii) First suppose that λ is a limit ordinal. Then from $G_\lambda = (\bigcup\limits_{\tau<\lambda} G_\tau)^*$ and $x \in G_\lambda$, we infer that there exists a sequence (x_n) in $\bigcup\limits_{\tau<\lambda} G_\tau$ with $0 \leq x_n$ for each $n \in \mathbb{N}$, $x_n \to_{\alpha(G_\lambda,M)} x$. Since $x \in c_1(K)$, there is $a \in K$, $x \leq a$. The sequence (x_n) is in G_{ω_1}, consequently $(x'_n) = (x_n \wedge a)$ is a sequence in $c_1(K)$ and $x'_n \to_{\alpha(c_1(K),M)} x \wedge a = a$. We have $0 \leq x'_n \leq a$ for each $n \in \mathbb{N}$. Since $a \in B_\tau$ for each $\tau < \lambda$, $x'_n \in B_{\tau_n}$ for some $\tau_n < \lambda$. The assumption implies $B_{\tau_n} \subseteq H$ for each $n \in \mathbb{N}$, so (x'_n) is a sequence in H. As (x'_n) is Cauchy in H with respect to convergence $\alpha(H, M)$ and H is wru-complete with respect to this convergence, $x \in H$.

(iii) Now, assume that λ is a non-limit ordinal. From $G_\lambda = (G_{\lambda-1})^*$ and $x \in G_\lambda$ we deduce that then there exists a sequence (x_n) in $G_{\lambda-1}$ with $x_n \to_{\alpha(G_\lambda,M)} x$. In the same way as in (ii), we construct the sequence (x'_n) and prove that (x'_n) is a sequence in $B_{\lambda-1}$. Applying the assumption, we obtain that (x'_n) is a sequence in H. Further, repeating the procedure from (ii), we finish the proof. □

When studying ru-convergence in archimedean lattice ordered groups, analogous results to 5.1 and 5.2 were obtained in [4] and [5] under the assumption that K is a direct factor of G.

Proposition 5.3. *Let K be a direct factor of G and M be as in* 5.1. *Then K_{ω_1} is a direct factor of G_{ω_1}.*

The proof is the same as in [5] for the case of ru-convergence in an archimedean lattice ordered group.

Again, let K be a convex ℓ-subgroup of a lattice ordered group G.

Let $g > a$ for each $g \in G^+ \setminus K$ and each $a \in K$. Then G is called a *lexicographic extension of* K and we write $G = \langle K \rangle$.

If $G = \langle K \rangle$ and $K \neq \{0\}$, then we obviously have $A(G) \subseteq K$.

Lemma 5.4. *Let $G = \langle K \rangle$ and Let M be as in* 5.1. *Then $G^* = \langle K^* \rangle$.*

P r o o f. According to Lemma 5.1, K^* can be viewed as a convex ℓ-subgroup of G^*.

Let $x \in (G^*)^+ \setminus K^*$. Hence $x > 0$. There exists a sequence (x_n) in G with $x_n > 0$ for each $n \in \mathbb{N}$ and $x_n \to_{\alpha(G^*,M)} x$. We claim that there exists $n_0 \in \mathbb{N}$ such that $x_n \in G^+ \setminus K$ for each $n \in \mathbb{N}$, $n \geq n_0$. If this is not the case then for each $n_1 \in \mathbb{N}$ there exists $n \in \mathbb{N}, n \geq n_1$ such that $x_n \in K$. Hence there is a subsequence (x'_n) of (x_n) such that (x'_n) is in K. We get $x'_n \to_{\alpha(G^*,M)} x$. Consequently, $x \in K^*$, a contradiction. Therefore, $x_n > a$ for each $a \in K$ and each $n \in \mathbb{N}$, $n \geq n_0$, so $x \geq a$.

Let $y \in K^*$. There exists a sequence (y_n) in K with $y_n \to_{\alpha(K^*,M)} y$. Hence $x \geq y_n$ for each $n \in \mathbb{N}$. Thus, $x \geq y$. We conclude that $x > y$, on account of $x \notin K^*$. □

Remark that a part of the above proof is similar to the proof in [10] where convergence lattice ordered groups were examined.

Proposition 5.5. *Let $G = \langle K \rangle$ and Let M be as in* 5.1. *Then, $G_{\omega_1} = \langle K_{\omega_1} \rangle$.*

P r o o f. We have $G_{\omega_1} = \bigcup\limits_{\beta<\omega_1} G_\beta$, $K_{\omega_1} = \bigcup\limits_{\beta<\omega_1} K_\beta$, K_λ is a convex ℓ-subgroup of G_λ for each $\lambda < \omega_1$. We first prove that the relation

$$G_\lambda = \langle K_\lambda \rangle \tag{r}$$

is valid for each $\lambda < \omega_1$.

Let $\lambda = 0$. We have $K_0 = K$ and $G_0 = G$. Applying the hypothesis, we get $G_0 = \langle K_0 \rangle$.

Let $\lambda > 0$ and suppose that $G_\tau = \langle K_\tau \rangle$ for all $\tau < \lambda$.

Let λ be a non-limit ordinal. By Lemma 5.4,

$$G_\lambda = (G_{\lambda-1})^* = \langle (K_{\lambda-1})^* \rangle = \langle K_\lambda \rangle.$$

Let λ be a limit ordinal. It is easy to see that $\bigcup\limits_{\tau<\lambda} G_\tau = \langle \bigcup\limits_{\tau<\lambda} K_\tau \rangle$ holds.

According to Lemma 5.4,

$$G_\lambda = \Big(\bigcup_{\tau<\lambda} G_\tau\Big)^* = \Big(\Big\langle\bigcup_{\tau<\lambda} K_\tau\Big\rangle\Big)^* = \Big\langle\Big(\bigcup_{\tau<\lambda} K_\tau\Big)^*\Big\rangle = \langle K_\lambda\rangle.$$

By using relation (r), the proof is finished. □

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