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OSCILLATION OF FIRST-ORDER DIFFERENTIAL EQUATIONS WITH RETARDED ARGUMENTS

Başak Karpuz

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ABSTRACT. In this paper, we provide a test under which every solution of a first-order delay differential equation oscillates. An example is given to illustrate the significance of the result.

1. Introduction

In this paper, we shall consider the delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0 \qquad \text{for} \quad t \geq t_0, \tag{1}$$

where $p \in \mathrm{C}([t_0, \infty), \mathbb{R}_0^+)$, $\tau \in \mathrm{C}([t_0, \infty), \mathbb{R})$ satisfies $\lim\limits_{t\to\infty} \tau(t) = \infty$ and $\tau(t) \leq t$ for all sufficiently large t.

Let $t_{-1} := \min\{\tau(t) : t \geq t_0\}$. By a *solution* of (1), we mean a function $x \in \mathrm{C}([t_{-1}, \infty), \mathbb{R})$ such that $x \in \mathrm{C}'([t_0, \infty), \mathbb{R})$ and satisfies (1) identically on $[t_0, \infty)$. Throughout the paper, we restrict our attention to those solutions of (1), which is not identically zero on any interval of the form $[t, \infty)$ for all $t \geq t_0$. It is a well-known fact that for a prescribed *initial function* $\varphi \in \mathrm{C}([t_{-1}, t_0], \mathbb{R})$, (1) admits a unique solution x satisfying $x = \varphi$ on the *initial interval* $[t_{-1}, t_0]$. As is customary, a solution of (1) is called *oscillatory* if it has arbitrarily large zeros; otherwise, a solution is called *nonoscillatory*.

Now, let us introduce a short brief concerning some basic results for the oscillation of (1). The first systematic approach to oscillation of solutions to (1)

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was given by Myskis in [8]. He showed that every solution of (1) oscillates if

$$\limsup_{t\to\infty}\big[t-\tau(t)\big]<\infty \qquad \text{and} \qquad \liminf_{t\to\infty}\big[t-\tau(t)\big]\liminf_{t\to\infty}p(t)>\frac{1}{\mathrm{e}}. \tag{2}$$

In [5], Ladas et. al. proved the same conclusion for (1) under the condition

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t}p(u)\,\mathrm{d}u>1, \tag{3}$$

but here the delay function τ is assumed to be nondecreasing, however, this condition is not as sharp as (2). In [6], Ladas, and in [4], Koplatadze and Chanturiya replaced (2) with the following one

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t}p(u)\,\mathrm{d}u>\frac{1}{\mathrm{e}}. \tag{4}$$

One can easily see that (4) is weaker than (2). We would like to mention here that (4) is almost sharp and it is not possible to replace the constant $1/\mathrm{e}$ with any smaller one or $\liminf$ with $\lim$. This fact can be seen easily from the following autonomous delay differential equation

$$x'(t)+\frac{1}{\mathrm{e}}x(t-1)=0 \qquad \text{for} \quad t\geq 0 \tag{5}$$

from which we get $p(t)\equiv 1/\mathrm{e}$ and $\tau(t)=t-1$ for $t\geq 0$ when compared with (1). For (5), it is not hard to see that

$$\lim_{t\to\infty}\int_{\tau(t)}^{t}p(u)\,\mathrm{d}u=\frac{1}{\mathrm{e}}$$

and thus (4) does not hold, and that $x(t)=1/\mathrm{e}^{t}$ for $t\geq 0$ is a nonoscillatory solution of (5) (see [2, Theorem 2.3.1]).

In the next section, we shall extend the result due to Li introduced in [7]. His result concludes that every solution of

$$x'(t)+p(t)x(t-\tau_0)=0 \qquad \text{for} \quad t\geq 0, \tag{6}$$

where p is a continuous function which is not identically zero on $[t,t+\tau_0)$ for all sufficiently large t, and τ_0 is a positive constant, is oscillatory if the divergent improper integral condition

$$\int^{\infty}p(u)\ln\Bigg\{\mathrm{e}\int_{u}^{u+\tau_0}p(v)\,\mathrm{d}v\Bigg\}\,\mathrm{d}u=\infty \tag{7}$$

holds. It should be mentioned here that (4) for (6) takes the form

$$\liminf_{t\to\infty} \int_{t-\tau_0}^{t} p(u)\,\mathrm{d}u > \frac{1}{\mathrm{e}}$$

which exactly implies (7) (see [2, Theorem 2.3.2]). Thus, the result due to Li, substantially improves the ones proved previously for (6). Later, in [1], Guan extended the result due to Li to the so called Euler-type equations of the form

$$x'(t) + p(t)x(t/\tau_0) = 0 \qquad \text{for} \quad t \geq 1, \tag{8}$$

where p is a continuous function which is not identically zero on $[t, \tau_0 t)$ for all sufficiently large t, and τ_0 is a constant greater than 1, and showed that if

$$\int^{\infty} p(u) \ln\biggl\{\mathrm{e} \int_{u}^{\tau_0 u} p(v)\,\mathrm{d}v\biggr\}\,\mathrm{d}u = \infty, \tag{9}$$

then all solutions to (8) are oscillatory.

In Section 2, we give some lemmas required in the sequel; in Section 3, we give our main result together with an illustrative example. Finally, in Section 4, we make a discussion to finalize the paper. The method in the proof of our main result makes use of the so-called generalized characteristic equation introduced in [2, Section 3].

2. Some lemmas

At the beginning of this section, we define the function $\overline{\tau} \in \mathrm{C}([t_0,\infty),[t_{-1},\infty))$ by

$$\overline{\tau}(t) := \max\bigl\{\tau(s) : \ t \geq s \geq t_0\bigr\} \qquad \text{for} \quad t \geq t_0,$$

which is a nondecreasing function and satisfies $\overline{\tau}(t) \geq \tau(t)$ for all $t \geq t_0$ and the function $\overline{\tau}_{-1} \in \mathrm{C}([t_{-1},\infty),[t_0,\infty))$ by

$$\overline{\tau}_{-1}(t) := \max\bigl\{s \geq t_0 : \ \overline{\tau}(s) = t\bigr\} \qquad \text{for} \quad t \geq t_{-1},$$

which satisfies $\overline{\tau}(\overline{\tau}_{-1}(t)) = t$ for all $t \geq t_0$. It is clear that if τ is increasing, then we have $\overline{\tau} = \tau$ on $[t_0,\infty)$ and $\overline{\tau}_{-1} = \tau^{-1}$ on $[t_{-1},\infty)$. We are now ready to prove two lemmas which will be required in the proof of our main result.

Lemma 1. *Let x be a nonoscillatory solution of* (1)*. If*

$$\limsup_{t\to\infty} \int_{t}^{\overline{\tau}_{-1}(t)} p(u)\,\mathrm{d}u > 0, \tag{10}$$

then

$$\liminf_{t\to\infty} \frac{x(\overline{\tau}(t))}{x(t)} < \infty. \tag{11}$$

P r o o f. Let x be a nonoscillatory solution of (1). Because of the linearity of (1), we may only consider the case that x is an eventually positive solution. Then, there exists $t_1 \geq t_0$ such that $x(t), x(\tau(t)) > 0$ for all $t \geq t_1$, from which together with (1), we learn that x is nonincreasing on the interval $[t_1, \infty)$. Therefore, we have

$$x'(t) + p(t)x(\overline{\tau}(t)) \leq 0 \qquad \text{for all} \quad t \geq t_1. \tag{12}$$

In view of (10), we may pick an increasing divergent sequence $\{\xi_k\}_{k\in\mathbb{N}} \subset [t_1, \infty)$ and a positive constant ε such that

$$\int_{\xi_k}^{\overline{\tau}_{-1}(\xi_k)} p(u)\,\mathrm{d}u \geq \varepsilon \qquad \text{for all} \quad k \in \mathbb{N}.$$

In this case, we may find a sequence $\{\zeta_k\}_{k\in\mathbb{N}} \subset [t_1, \infty)$ such that

$$\int_{\xi_k}^{\zeta_k} p(u)\,\mathrm{d}u \geq \frac{\varepsilon}{2} \quad \text{and} \quad \int_{\zeta_k}^{\overline{\tau}_{-1}(\xi_k)} p(u)\,\mathrm{d}u \geq \frac{\varepsilon}{2} \qquad \text{for all} \quad k \in \mathbb{N}. \tag{13}$$

Clearly, $\{\zeta_k\}_{k\in\mathbb{N}}$ is divergent. For any $k \in \mathbb{N}$, integrating (12) over the intervals $[\xi_k, \zeta_k)$ and $[\zeta_k, \overline{\tau}_{-1}(\xi_k))$, we find

$$x(\zeta_k) - x(\xi_k) + \int_{\xi_k}^{\zeta_k} p(u)x(\overline{\tau}(u))\,\mathrm{d}u \leq 0 \tag{14}$$

and

$$x(\overline{\tau}_{-1}(\xi_k)) - x(\zeta_k) + \int_{\zeta_k}^{\overline{\tau}_{-1}(\xi_k)} p(u)x(\overline{\tau}(u))\,\mathrm{d}u \leq 0. \tag{15}$$

Dropping the first (positive) terms in (14) and (15), using the nonincreasing nature of $x \circ \overline{\tau}$ (since both x and $\overline{\tau}$ are monotonic) and (13), we get

$$\frac{\varepsilon}{2}x(\overline{\tau}(\zeta_k)) < x(\xi_k) \quad \text{and} \quad \frac{\varepsilon}{2}x(\xi_k) < x(\zeta_k) \qquad \text{for all} \quad k \in \mathbb{N},$$

which implies

$$\frac{x(\overline{\tau}(\zeta_k))}{x(\zeta_k)} < \left(\frac{2}{\varepsilon}\right)^2 \qquad \text{for all} \quad k \in \mathbb{N}.$$

This shows that (11) is valid, and completes the proof. □

Lemma 2. *Assume that* (1) *admits a nonoscillatory solution. Then*

$$\int_{t}^{\overline{\tau}_{-1}(t)} p(u)\,\mathrm{d}u < 1 \quad \textit{for all sufficiently large } t. \tag{16}$$

Proof. Let x be an eventually positive solution of (1). Then, there exists $t_1 \geq t_0$ such that $x(t), x(\tau(t)) > 0$ for all $t \geq t_1$. Then x is nonincreasing on the interval $[t_1, \infty)$. Integrating (12) over the interval $[t, \overline{\tau}_{-1}(t))$, where $t \geq t_1$, and using the nonincreasing nature of $x \circ \overline{\tau}$, we are led to

$$x(\overline{\tau}_{-1}(t)) + \left(\int_t^{\overline{\tau}_{-1}(t)} p(u)\,\mathrm{d}u - 1 \right) x(t) \leq 0 \qquad \text{for all} \quad t \geq t_1,$$

which implies (16). This completes the proof. □

As an immediate consequence of the lemma above, we may give the following remark for the oscillation of (1).

Remark 1. Assume existence of an increasing divergent sequence $\{\xi_k\}_{k\in\mathbb{N}} \subset [t_0, \infty)$ satisfying

$$\int_{\xi_k}^{\overline{\tau}_{-1}(\xi_k)} p(u)\,\mathrm{d}u \geq 1 \qquad \text{for all} \quad k \in \mathbb{N}.$$

Then, every solution of (1) is oscillatory.

Obviously, (3) implies the condition in Remark 1.

3. The main result

The main objective of this section is to establish the following theorem. In the proof of this result, the inequality

$$r\mathrm{e}^x \geq x + \ln(\mathrm{e}r) \qquad \text{for all} \quad x \geq 0 \ \text{ and } \ r > 0, \tag{17}$$

which can be proved by using elementary calculus, plays an important role.

Theorem 1. *Assume that p is a continuous function which is not identically zero on $[t, \overline{\tau}_{-1}(t))$ for all sufficiently large t, and that*

$$\int^{\infty} p(u) \ln\left\{ \mathrm{e} \int_u^{\overline{\tau}_{-1}(u)} p(v)\,\mathrm{d}v \right\} \mathrm{d}u = \infty. \tag{18}$$

Then, every solution of (1) is oscillatory.

P r o o f. For the sake of contradiction assume that (1) admits an eventually positive solution x. Pick $t_1 \geq t_0$ such that $x(t), x(\tau(t)) > 0$ for all $t \geq t_1$. Then, x is nonincreasing on the interval $[t_1, \infty)$. Set

$$\lambda(t) := -\frac{x'(t)}{x(t)} \geq 0 \qquad \text{for} \quad t \geq t_1. \tag{19}$$

Then, integrating (19) over the interval $[t_1, t)$, where $t \geq t_1$, we get

$$x(t) = x(t_1) \exp\Biggl\{-\int_{t_1}^{t} \lambda(u)\,\mathrm{d}u\Biggr\} \qquad \text{for all} \quad t \geq t_1. \tag{20}$$

Substituting (20) into (12), we obtain

$$\lambda(t) \geq p(t) \exp\Biggl\{\int_{\overline{\tau}(t)}^{t} \lambda(u)\,\mathrm{d}u\Biggr\} \qquad \text{for all} \quad t \geq t_1. \tag{21}$$

Using (17) after multiplying (21) with $\int_{t}^{\overline{\tau}_{-1}(t)} p(u)\,\mathrm{d}u > 0$, we get

$$\lambda(t) \int_{t}^{\overline{\tau}_{-1}(t)} p(u)\,\mathrm{d}u \geq p(t)\Biggl(\int_{\overline{\tau}(t)}^{t} \lambda(u)\,\mathrm{d}u + \ln\Biggl\{\mathrm{e} \int_{t}^{\overline{\tau}_{-1}(t)} p(u)\,\mathrm{d}u\Biggr\}\Biggr) \qquad \text{for all} \quad t \geq t_1,$$

and by collecting the terms involving λ on the right-hand side of the equation, we have

$$p(t) \ln\Biggl\{\mathrm{e} \int_{t}^{\overline{\tau}_{-1}(t)} p(u)\,\mathrm{d}u\Biggr\} \leq \lambda(t) \int_{t}^{\overline{\tau}_{-1}(t)} p(u)\,\mathrm{d}u - p(t) \int_{\overline{\tau}(t)}^{t} \lambda(u)\,\mathrm{d}u \qquad \text{for all} \quad t \geq t_1. \tag{22}$$

For the last term on the right-hand side of (22), we see by changing the order of the integration that

$$\int_{t_2}^{t} p(u) \int_{\overline{\tau}(u)}^{u} \lambda(v)\,\mathrm{d}v\,\mathrm{d}u \geq \int_{t_2}^{\overline{\tau}(t)} \lambda(v) \int_{v}^{\overline{\tau}_{-1}(v)} p(u)\,\mathrm{d}u\,\mathrm{d}v \qquad \text{for all} \quad t \geq t_2, \tag{23}$$

where $t_2 \geq t_1$ with $\overline{\tau}(t_2) \geq t_1$. Using (23) after integrating (22) over the interval $[t_2, t)$, where $t \geq t_2$, we obtain

$$\begin{aligned}
&\int_{t_2}^{t} p(v) \ln\left\{\mathrm{e} \int_{v}^{\overline{\tau}_{-1}(v)} p(u)\,\mathrm{d}u\right\} \mathrm{d}v \\
&\qquad \leq \int_{t_2}^{t} \lambda(v) \int_{v}^{\overline{\tau}_{-1}(v)} p(u)\,\mathrm{d}u\,\mathrm{d}v - \int_{t_2}^{t} p(u) \int_{\overline{\tau}(u)}^{u} \lambda(v)\,\mathrm{d}v\,\mathrm{d}u \\
&\qquad \leq \int_{t_2}^{t} \lambda(v) \int_{v}^{\overline{\tau}_{-1}(v)} p(u)\,\mathrm{d}u\,\mathrm{d}v - \int_{t_2}^{\overline{\tau}(t)} \lambda(v) \int_{v}^{\overline{\tau}_{-1}(v)} p(u)\,\mathrm{d}u\,\mathrm{d}v \\
&\qquad = \int_{\overline{\tau}(t)}^{t} \lambda(v) \int_{v}^{\overline{\tau}_{-1}(v)} p(u)\,\mathrm{d}u\,\mathrm{d}v < \int_{\overline{\tau}(t)}^{t} \lambda(v)\,\mathrm{d}v, \qquad (24)
\end{aligned}$$

where we have used Lemma 2 in the last step. From (18), (19) and (24), we get

$$\exp\left\{\int_{t_2}^{t} p(v) \ln\left\{\mathrm{e} \int_{v}^{\overline{\tau}_{-1}(v)} p(u)\,\mathrm{d}u\right\} \mathrm{d}v\right\} < \exp\left\{\int_{\overline{\tau}(t)}^{t} \lambda(v)\,\mathrm{d}v\right\} = \frac{x(\overline{\tau}(t))}{x(t)}$$
$$\text{for all} \quad t \geq t_2,$$

which yields by letting $t \to \infty$ that

$$\lim_{t\to\infty} \frac{x(\overline{\tau}(t))}{x(t)} = \infty. \qquad (25)$$

On the other hand, (18) implies existence of an increasing divergent sequence $\{\xi_k\}_{k\in\mathbb{N}} \subset [t_0, \infty)$ such that

$$\int_{\xi_k}^{\overline{\tau}_{-1}(\xi_k)} p(u)\,\mathrm{d}u \geq \frac{1}{\mathrm{e}} \qquad \text{for all} \quad k \in \mathbb{N},$$

which shows that (11) holds by Lemma 1. This contradicts (25), and thus every solution of (1) must be oscillatory. □

Now, we have the following example.

Example 1. Let $\alpha > 0$ and consider the delay differential equation

$$x'(t) + \frac{\exp\left\{\alpha \sin\left(\ln(\ln(t))\right)\right\}}{\mathrm{e}t\ln(t)} x\left(\sqrt[\mathrm{e}]{t}\right) = 0 \qquad \text{for} \quad t \geq 1. \qquad (26)$$

It is not hard to see that $p(t) = \exp\{\alpha \sin(\ln(\ln(t)))\}/(\mathrm{e}t\ln(t))$ and $\tau(t) = \sqrt[\mathrm{e}]{t}$ for $t \geq 1$. By making change of variables, we have

$$\int_{\tau(t)}^{t} p(u)\,\mathrm{d}u = \frac{1}{\mathrm{e}}\int_{\sqrt[\mathrm{e}]{t}}^{t} \frac{\exp\{\alpha\sin(\ln(\ln(u)))\}}{\mathrm{e}u\ln(u)}\,\mathrm{d}u = \frac{1}{\mathrm{e}}\int_{\ln(\ln(t))-1}^{\ln(\ln(t))} \mathrm{e}^{\alpha\sin(v)}\,\mathrm{d}v \qquad (27)$$
$$\text{for all} \quad t \geq 1.$$

On the other hand, the periodicity and the oscillating nature of the sin function, we learn that

$$\sin(t) \leq \sin\big((3\pi+1)/2\big) < -\frac{1}{2}$$

for all $t \in [2k\pi + (3\pi-1)/2, 2k\pi + (3\pi+1)/2)$ (which is an interval with a length of 1) and all $k \in \mathbb{N}$, which yields by the increasing nature of the exponential function that $\mathrm{e}^{\alpha\sin(t)} < 1$ for all $t \in [2k\pi + (3\pi-1)/2, 2k\pi + (3\pi+1)/2)$ and all $k \in \mathbb{N}$. Then, it follows from (27), the discussion above and making use of simple calculus that

$$\liminf_{t\to\infty} \frac{1}{\mathrm{e}}\int_{\ln(\ln(t))-1}^{\ln(\ln(t))} \mathrm{e}^{\alpha\sin(v)}\,\mathrm{d}v = \frac{1}{\mathrm{e}}\int_{(3\pi-1)/2}^{(3\pi+1)/2} \mathrm{e}^{\alpha\sin(v)}\,\mathrm{d}v < \frac{1}{\mathrm{e}}.$$

Hence, (2) does not hold for (26). Clearly, τ is increasing, hence $\overline{\tau}_{-1}(t) = t^{\mathrm{e}}$ for $t \geq 1$. By using change of variables and Jensen's famous inequality for concave functions, we get

$$\begin{aligned}
&\int_{1}^{\infty} \frac{\exp\{\alpha\sin(\ln(\ln(u)))\}}{\mathrm{e}u\ln(u)} \ln\bigg\{\mathrm{e}\int_{u}^{u^{\mathrm{e}}} \frac{\exp\{\alpha\sin(\ln(\ln(v)))\}}{\mathrm{e}v\ln(v)}\,\mathrm{d}v\bigg\}\,\mathrm{d}u \\
&\quad= \int_{1}^{\infty} \frac{\exp\{\alpha\sin(\ln(\ln(u)))\}}{\mathrm{e}u\ln(u)} \ln\bigg\{\int_{\ln(\ln(u))}^{\ln(\ln(u))+1} \mathrm{e}^{\alpha\sin(r)}\,\mathrm{d}r\bigg\}\,\mathrm{d}u \\
&\quad= \frac{1}{\mathrm{e}}\int_{0}^{\infty} \mathrm{e}^{\alpha\sin(s)} \ln\bigg\{\int_{s}^{s+1} \mathrm{e}^{\alpha\sin(r)}\,\mathrm{d}r\bigg\}\,\mathrm{d}s \qquad (28) \\
&\quad\geq \frac{\alpha}{\mathrm{e}}\int_{0}^{\infty} \mathrm{e}^{\alpha\sin(s)} \bigg(\int_{s}^{s+1} \sin(r)\,\mathrm{d}r\bigg)\,\mathrm{d}s \\
&\quad= \frac{\alpha}{\mathrm{e}}\bigg(\sin(1)\int_{0}^{\infty} \mathrm{e}^{\alpha\sin(s)}\sin(s)\,\mathrm{d}s + \big(1-\cos(1)\big)\int_{0}^{\infty} \mathrm{e}^{\alpha\sin(s)}\cos(s)\,\mathrm{d}s\bigg).
\end{aligned}$$

Set $f_\alpha(t) := \mathrm{e}^{\alpha t} t$ for $t \in \mathbb{R}$. Hence, we have

$$f_\alpha(\sin(t)) + f_\alpha(-\sin(t)) = f_\alpha(\sin(t))\big(1 - \mathrm{e}^{-2\alpha \sin(t)}\big) \geq 0 \qquad \text{for all} \quad t \in [0, \pi),$$

which holds with equality if and only if $t = 0$, and this yields

$$\begin{aligned}\int_0^{2\pi} f_\alpha(\sin(s))\,\mathrm{d}s &= \int_0^{\pi} f_\alpha(\sin(s))\,\mathrm{d}s + \int_\pi^{2\pi} f_\alpha(\sin(s))\,\mathrm{d}s \\ &= \int_0^{\pi} f_\alpha(\sin(s))\,\mathrm{d}s + \int_0^{\pi} f_\alpha(\sin(s+\pi))\,\mathrm{d}s \\ &= \int_0^{\pi} \big[f_\alpha(\sin(s)) + f_\alpha(-\sin(s))\big]\,\mathrm{d}s > 0.\end{aligned}$$

So that

$$\begin{aligned}\int_0^{\infty} f_\alpha(\sin(s))\,\mathrm{d}s &= \sum_{k=0}^{\infty} \int_{2k\pi}^{2(k+1)\pi} f_\alpha(\sin(s))\,\mathrm{d}s \\ &= \sum_{k=0}^{\infty} \int_0^{2\pi} f_\alpha(\sin(s))\,\mathrm{d}s = \infty. \end{aligned} \tag{29}$$

Also, for all $t \geq 0$, we have

$$\begin{aligned}\left|\int_0^{t} \exp\big\{\alpha \sin(s)\big\} \cos(s)\,\mathrm{d}s\right| &= \frac{1}{\alpha}\big|\mathrm{e}^{\alpha \sin(t)} - 1\big| \\ &\leq \frac{1}{\alpha}\big(\mathrm{e}^{\alpha} + 1\big).\end{aligned} \tag{30}$$

Thus, using (29) and (30) in (28), we see that (18) holds. Every solution of (26) is therefore oscillatory by Theorem 1.

4. Final comments

It is apparently obvious that (18) for (6) and (8) reduces to (7) and (9), respectively. Theorem 1 hence generalizes the main results of the papers [1] and [7]. We also would like to mention that our results in the previous section can be easily extended to the case of several delays, which has the form

$$x'(t) + \sum_{i=1}^{n} p_i(t) x(\tau_i(t)) = 0 \qquad \text{for} \quad t \geq t_0,$$

where n is a positive integer, for $i = 1, 2, \ldots, n$, $\tau_i \in \mathrm{C}([t_0, \infty), \mathbb{R})$ satisfies $\lim_{t\to\infty} \tau_i(t) = \infty$ and $\tau_i(t) \leq t$ for all sufficiently large t, and $p_i \in \mathrm{C}([t_0, \infty), \mathbb{R}_0^+)$.

REFERENCES

[1] GUAN, K.: *Oscillation of solutions for a class of unbounded delay differential equations*, Ann. Differential Equations **21** (2005), 144–152.

[2] GYŐRI, I—LADAS, G.: *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, 1991.

[3] HUNT, B. R.—YORKE, J. A.: *When all solutions of* $x'(t) = \sum_{j=1}^{n} q_j(t)x(\tau_j(t))$ *oscillate*, J. Differential Equations **53** (1984), 139–145.

[4] KOPLATADZE, R. G.—CHANTURIYA, T. A: *Oscillating and monotone solutions of first-order differential equations with deviating argument*, Differ. Uravn. **18** (1982), 1463–1465 (Russian).

[5] LADAS, G.—LAKSMIKANTHAM, V—PAPADAKIS, L. S: *Oscillation of higher-order retarded differential equations generated by the retarded arguments*. In: Delay and Functional Differential Equations and Their Applications, Academic Press, New York, 1972, pp. 219–231.

[6] LADAS, G.: *Sharp conditions for oscillation caused by delays*, Appl. Anal. **9** (1979), 93–98.

[7] LI, B. T.: *Oscillation of first-order delay differential equation*, Proc. Amer. Math. Soc. **124** (1996), 3729–3737.

[8] MYSKIS, A. R.: *Linear homogeneous differential equations of first order with deviating arguments*, Uspekhi Mat. Nauk **5** (1950), 160–1162 (Russian).

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Department of Mathematics
Faculty of Science and Literature
ANS Campus
Afyon Kocatepe University
03200 Afyonkarahisar
TURKEY
E-mail: bkarpuz@gmail.com
URL: http://www2.aku.edu.tr/~bkarpuz