

NON-CONSTANT PERIODIC SOLUTIONS FOR SECOND ORDER HAMILTONIAN SYSTEM WITH A p -LAPLACIAN

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ABSTRACT. In this paper, some existence theorems are obtained for non-constant periodic solutions of second order Hamiltonian system with a p -Laplacian by using the Linking Theorem.

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1. Introduction and main results

Consider the ordinary p -Laplacian system

$$\frac{d}{dt}\Phi_p(\dot{u}(t)) + \nabla F(t, u(t)) = 0, \quad \text{a.e. } t \in [0, T], \quad (1.1)$$

where $p \geq 2$, $q > 1$, $1/p + 1/q = 1$, $\Phi_p(u) = |u|^{p-2}u = \left(\sqrt{\sum_{i=1}^N u_i^2}\right)^{p-2} \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$,

and $F: \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ is T -periodic ($T > 0$) in its first variable and satisfies the following assumption:

- (A) $F(t, x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b \in L^1([0, T], \mathbb{R}^+)$ such that, for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$,

$$|F(t, x)| \leq a(|x|)b(t), \quad |\nabla F(t, x)| \leq a(|x|)b(t).$$

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As $p = 2$, there have been lots of existence results about periodic solutions of system (1.1) (see [4, 5, 6, 7, 8, 10] and references therein). However, in these references, all authors only concerned about the existence of solutions. By using the local linking theorem, in [11] and [14], the authors considered the existence of nontrivial solutions. In [12], Tao and Tang considered the existence of non-constant solutions. In 2006, Schechter [9] also studied the problem under much weaker assumptions. As $p > 1$, in [13] and [15], the authors used the dual least action principle and saddle point theorem to consider system (1.1), respectively, and they obtained the existence results of solutions for system (1.1). In [16], our first author and Zhou obtained the existence of nonconstant periodic solution for system (1.1) by using the generalized mountain pass theorem. In some sense, their results generalized [12, Theorem 1].

In this paper, inspired by [9, 13, 14, 15, 16], we shall consider the existence of non-constant solutions for system (1.1) with $p \geq 2$. Our main results are the following theorems.

THEOREM 1.1. *Suppose F satisfies assumption (A) and the following conditions:*

$$(I_1) \quad \int_0^T F(t, x) dt \geq 0, \quad \text{for all } x \in \mathbb{R}^N;$$

$$(I_2) \quad \text{there are constants } l > 0, \alpha \leq 2^p(q+1)^{p-1}l^p/(pT^p) \text{ such that}$$

$$F(t, x) \leq \alpha$$

$$\text{for all } |x| \leq l \text{ in } \mathbb{R}^N \text{ and a.e. } t \in [0, T];$$

$$(I_3) \quad \text{if } N \geq 2, \text{ there are constants } \beta > (2\pi)^p/(pT^p) \text{ and } L > 0 \text{ such that}$$

$$F(t, x) \geq \beta|x|^p \tag{1.2}$$

$$\text{for all } |x| \geq L \text{ in } \mathbb{R}^N \text{ and a.e. } t \in [0, T]; \text{ If } N = 1, \text{ there are constants } \beta > 2^{p/2-1}(2\pi)^p/(pT^p) \text{ and } L > 0 \text{ such that (1.2) holds;}$$

$$(I_4) \quad \text{there is a constant } \mu > p, \text{ such that}$$

$$\limsup_{|x| \rightarrow \infty} \frac{\mu F(t, x) - (\nabla F(t, x), x)}{|x|^p} < (\mu - p)\beta \quad \text{uniformly for a.e. } t \in [0, T].$$

Then, system (1.1) has a non-constant solution.

THEOREM 1.2. *Suppose F satisfies assumption (A), (I_1) , (I_3) , (I_4) and the following condition:*

$$(I_2)' \quad \text{there are constants } l > 0, \alpha \leq 2^p\pi^2(q+1)^{(p-2)/q}/(pT^p) \text{ such that}$$

$$F(t, x) \leq \alpha|x|^p$$

$$\text{for all } |x| \leq l \text{ in } \mathbb{R}^N \text{ and a.e. } t \in [0, T].$$

Then, system (1.1) has a non-constant solution.

Remark 1.1. As $p = 2$, since both our condition (I_4) and [9, Hypothesis 4] can not imply each other, Theorem 1.1 and Theorem 1.2 are different from [9, Theorem 1.1, Theorem 1.2], respectively.

2. Preliminaries

Let

$$W_T^{1,p} = \left\{ u: \mathbb{R} \rightarrow \mathbb{R}^N \mid u(t) \text{ is absolutely continuous on } \mathbb{R}, \right. \\ \left. u(t+T) = u(t) \text{ and } \dot{u} \in L^p(0, T; \mathbb{R}^N) \right\}.$$

On $W_T^{1,p}$, we define the norm as follows:

$$\|u\|_{W_T^{1,p}} = \left[\int_0^T |u(t)|^p dt + \int_0^T |\dot{u}(t)|^p dt \right]^{1/p}, \quad u \in W_T^{1,p}.$$

Then, $(W_T^{1,p}, \|\cdot\|_{W_T^{1,p}})$ is a reflexive and uniformly convex Banach space (see e.g. [1, Theorem 3.3, Theorem 3.6]). It follows from [9] that a locally uniformly convex Banach space has the Kadec-Klee property, that is for any sequence $\{u_n\}$ satisfying $u_n \rightharpoonup u$ weakly in Banach space $(X, \|\cdot\|)$ and $\|u_n\| \rightarrow \|u\|$, one has $u_n \rightarrow u$ strongly in X .

Let

$$\tilde{W}_T^{1,p} = \left\{ u \in W_T^{1,p} : \int_0^T u(t) dt = 0 \right\}.$$

It is easy to know that $\tilde{W}_T^{1,p}$ is a closed subspace of $W_T^{1,p}$ and $W_T^{1,p} = \mathbb{R}^N \oplus \tilde{W}_T^{1,p}$.

For $u \in W_T^{1,p}$, let $\bar{u} = \frac{1}{T} \int_0^T u(t) dt$ and $\tilde{u}(t) = u(t) - \bar{u}$. For $u \in W_T^{1,p}$, let

$$\|u\| = \|\dot{u}\|_{L^p} + \|\bar{u}\|_{L^p}.$$

Then, the norm $\|\cdot\|$ is equivalent to the norm $\|\cdot\|_{W_T^{1,p}}$. The proof can be seen in [15, Lemma 1]. Hence, there are constants $d_1 > 0$, $d_2 > 0$ such that

$$d_1 \|\cdot\|_{W_T^{1,p}} \leq \|\cdot\| \leq d_2 \|\cdot\|_{W_T^{1,p}}. \quad (2.1)$$

When $p = 2$, there have been the following inequalities.

LEMMA 2.1. (see [5, Proposition 1.3]) *Let $u \in \tilde{H}_T^1$. Then*

$$\|u\|_\infty^2 \leq \frac{T}{12} \int_0^T |\dot{u}(t)|^2 dt, \quad (\text{Sobolev's inequality}),$$

$$\int_0^T |u(t)|^2 dt \leq \frac{T^2}{4\pi^2} \int_0^T |\dot{u}(t)|^2 dt, \quad (\text{Wirtinger's inequality}).$$

For the general case $p \geq 2$, we can obtain the following inequalities.

LEMMA 2.2. *Let $u \in \tilde{W}_T^{1,p}$. Then*

$$\|u\|_\infty \leq \frac{T^{1/q}}{2(q+1)^{1/q}} \left(\int_0^T |\dot{u}(s)|^p \, ds \right)^{1/p}, \quad (2.2)$$

and

$$\int_0^T |u(s)|^p \, ds \leq \frac{T^p}{2^p \pi^2 (q+1)^{(p-2)/q}} \int_0^T |\dot{u}(s)|^p \, ds. \quad (2.3)$$

Proof. Fix $t \in [0, T]$. For every $\tau \in [t - T/2, t + T/2]$, we have

$$u(t) = u(\tau) + \int_\tau^t \dot{u}(s) \, ds. \quad (2.4)$$

Set

$$\phi(s) = \begin{cases} s - t + \frac{T}{2}, & t - T/2 \leq s \leq t, \\ t + \frac{T}{2} - s, & t \leq s \leq t + T/2. \end{cases}$$

Integrating (2.4) over $[t - T/2, t + T/2]$ and using the Hölder inequality, we obtain

$$\begin{aligned} T|u(t)| &= \left| \int_{t-T/2}^{t+T/2} u(\tau) \, d\tau + \int_{t-T/2}^{t+T/2} \int_\tau^t \dot{u}(s) \, ds \, d\tau \right| \\ &\leq \int_{t-T/2}^t \int_\tau^t |\dot{u}(s)| \, ds \, d\tau + \int_t^{t+T/2} \int_t^\tau |\dot{u}(s)| \, ds \, d\tau \\ &= \int_{t-T/2}^t \left(s - t + \frac{T}{2} \right) |\dot{u}(s)| \, ds + \int_t^{t+T/2} \left(t + \frac{T}{2} - s \right) |\dot{u}(s)| \, ds \\ &= \int_{t-T/2}^{t+T/2} \phi(s) |\dot{u}(s)| \, ds \leq \left(\int_{t-T/2}^{t+T/2} [\phi(s)]^q \, ds \right)^{1/q} \left(\int_{t-T/2}^{t+T/2} |\dot{u}(s)|^p \, ds \right)^{1/p} \\ &= \frac{T^{(q+1)/q}}{2(q+1)^{1/q}} \left(\int_{t-T/2}^{t+T/2} |\dot{u}(s)|^p \, ds \right)^{1/p} = \frac{T^{(q+1)/q}}{2(q+1)^{1/q}} \left(\int_0^T |\dot{u}(s)|^p \, ds \right)^{1/p}. \end{aligned} \quad (2.5)$$

It follows from (2.5) that (2.2) holds. Since $p \geq 2$, it follows from the Hölder inequality that

$$\int_0^T |\dot{u}(t)|^2 dt \leq T^{1-2/p} \left(\int_0^T |\dot{u}(t)|^p dt \right)^{2/p}. \quad (2.6)$$

By (2.2) and Wirtinger's inequality, we have

$$\begin{aligned} \int_0^T |u(t)|^p dt &\leq \|u\|_\infty^{p-2} \int_0^T |u(t)|^2 dt \\ &\leq \frac{T^2}{4\pi^2} \|u\|_\infty^{p-2} \int_0^T |\dot{u}(t)|^2 dt \\ &\leq \frac{T^p}{2^p \pi^2 (q+1)^{(p-2)/q}} \int_0^T |\dot{u}(t)|^p dt. \end{aligned}$$

It follows that (2.3) holds. The proof is complete. \square

Remark 2.1. For all $p > 1$, (2.2) always holds. Especially, (2.2) and (2.3) reproduce Sobolev's inequality and Wirtinger's inequality when $p = 2$, respectively.

LEMMA 2.3. (see [5, Proposition 1.1]) *There exists $C > 0$ such that, if $u \in W_T^{1,p}$, then $\|u\|_\infty \leq C\|u\|_{W_T^{1,p}}$.*

Let $\varphi: W_T^{1,p} \rightarrow \mathbb{R}$ be defined by

$$\varphi(u) = \frac{1}{p} \int_0^T |\dot{u}(t)|^p dt - \int_0^T F(t, u(t)) dt. \quad (2.7)$$

It follows from assumption (A) that φ is continuously differentiable on $W_T^{1,p}$ and

$$\langle \varphi'(u), v \rangle = \int_0^T (|\dot{u}(t)|^{p-2} \dot{u}(t), \dot{v}(t)) dt - \int_0^T (\nabla F(t, u(t)), v(t)) dt \quad (2.8)$$

for $u, v \in W_T^{1,p}$ (see [5, Theorem 1.4]). Similar to [5, Corollary 1.1], it is easy to know that the solutions of problem (1.1) correspond to the critical points of φ .

We say that a (nonempty) subset A of X is contractible in X if there exists a continuous $\psi: A \times [0, 1] \rightarrow X$ such that $\psi(u, 0) = u$ for every $u \in A$, and $\psi(u, 1) = u_0$ for some $u_0 \in X$ and every $u \in A$. Such a continuous map ψ will be called a contraction of A in X (see [2]).

DEFINITION 2.1. (see [2, Definition 3.1]) Let X be a metric space, and A, B two subsets of X . We say A links B if A is contractible in X , $A \cap B = \emptyset$, and $\psi(A \times [0, 1]) \cap B \neq \emptyset$ for every contraction ψ of A in X .

We shall use the following lemma to obtain the critical points of φ .

LEMMA 2.4. (see [2, Theorem 3.3]) Let X be a Banach space, $\varphi \in C^1(X, \mathbb{R})$ and A, B two subsets of X such that A links B . If

$$\inf_B \varphi \geq \sup_A \varphi,$$

then there exists a sequence $(u_n) \subset X$ with

$$\varphi(u_n) \rightarrow c \quad \text{and} \quad \varphi'(u_n) \rightarrow 0,$$

where

$$c = \inf_{\psi \in \Phi_A} \sup_{A \times [0, 1]} (\varphi \circ \psi)$$

and Φ_A is the set of contractions of A in X . Moreover, if $c = \inf_B \varphi$, we can require to satisfy also $d(u_n, B) \rightarrow 0$.

Remark 2.2. By [2], since A links B , it is easy to know that $c \geq \inf_B \varphi$ (see [2, Definition 3.1 and what follows]).

3. Proofs of Theorems

First, we prove our Theorem 1.1.

Proof of Theorem 1.1. We divide our proof into three steps.

Step 1: we will construct A and B which satisfy assumptions in Lemma 2.4. For convenience, we shall use the following norm

$$\|u\| = \|\dot{u}\|_{L^p} + \|\bar{u}\|_{L^p}.$$

If $u \in \tilde{W}_T^{1,p}$ and

$$\|u\| = \|\dot{u}\|_{L^p} = \rho = \frac{2(q+1)^{1/q}}{T^{1/q}} l,$$

then by Lemma 2.2, we have $\|u\|_\infty \leq l$, and by (I₂), we get $F(t, u(t)) \leq \alpha$ for a.e. $t \in [0, T]$. Hence, for $u \in \partial B_\rho \cap \tilde{W}_T^{1,p}$, we have

$$\begin{aligned} \varphi(u) &= \frac{1}{p} \|\dot{u}\|_{L^p}^p - \int_0^T F(t, u(t)) dt \\ &\geq \frac{1}{p} \rho^p - \alpha T \geq 0. \end{aligned}$$

By (I₃) and assumption (A), there exists constant $C_1 > 0$ such that

$$F(t, x) \geq \beta|x|^p - \beta L^p - C_1 b(t) \quad (3.1)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$.

Now, we distinguish two cases.

Case (i): assume that $N \geq 2$. Let $u(t) = x + sw_0(t)$, where $x \in \mathbb{R}^N$, $s \geq 0$ and

$$w_0(t) = \left(\sin \frac{2\pi t}{T}, \cos \frac{2\pi t}{T}, 0, \dots, 0 \right) \in \tilde{W}_T^{1,p}.$$

Then

$$\int_0^T |w_0(t)|^2 dt = T, \quad \|\dot{u}\|_{L^p}^p = \|s\dot{w}_0\|_{L^p}^p = \int_0^T |s\dot{w}_0(t)|^p dt = \frac{(2\pi)^p}{T^{p-1}} s^p$$

and

$$\begin{aligned} \int_0^T |u(t)|^p dt &= \int_0^T |x + sw_0(t)|^p dt \\ &\geq T^{1-p/2} \left(\int_0^T |x + sw_0(t)|^2 dt \right)^{p/2} \\ &= T^{1-p/2} \left(\int_0^T |x|^2 dt + \int_0^T s^2 |w_0(t)|^2 dt \right)^{p/2} \\ &= T^{1-p/2} (|x|^2 T + s^2 T)^{p/2} \\ &\geq T(|x|^p + s^p). \end{aligned}$$

Hence, by (I₃) with $\beta > (2\pi)^p/(pT^p)$, (3.1) and the above inequality, we obtain

$$\begin{aligned} \varphi(u) &\leq \frac{1}{p} \|s\dot{w}_0\|_{L^p}^p - \beta \int_0^T |u(t)|^p dt + T\beta L^p + C_1 \int_0^T b(t) dt \\ &= \frac{(2\pi)^p}{pT^{p-1}} s^p - \beta \int_0^T |u(t)|^p dt + T\beta L^p + C_1 \int_0^T b(t) dt \\ &\leq \left(\frac{(2\pi)^p}{pT^{p-1}} - \beta T \right) s^p - \beta T |x|^p + T\beta L^p + C_1 \int_0^T b(t) dt \rightarrow -\infty \\ &\quad \text{as } s^p + |x|^p \rightarrow \infty. \end{aligned}$$

Case (ii): assume that $N = 1$. Let $u(t) = x + sw_0(t)$, where $x \in \mathbb{R}$, $s \geq 0$ and

$$w_0(t) = \sin \frac{2\pi t}{T} \in \tilde{W}_T^{1,p}.$$

Then

$$\begin{aligned} \int_0^T |w_0(t)|^2 dt &= \frac{T}{2}, \\ \|\dot{u}\|_{L^p}^p &= \int_0^T |s\dot{w}_0(t)|^p dt \leq \left(\frac{2\pi}{T}\right)^p s^p \int_0^T \left|\cos \frac{2\pi}{T}t\right|^2 dt = \frac{2^{p-1}\pi^p}{T^{p-1}} s^p \end{aligned}$$

and

$$\begin{aligned} \int_0^T |u(t)|^p dt &\geq T^{1-p/2}(|x|^2T + s^2T/2)^{p/2} \\ &\geq T\left(|x|^p + \frac{s^p}{2^{p/2}}\right). \end{aligned}$$

Hence, by (I₃) with $\beta > 2^{p/2-1}(2\pi)^p/(pT^p)$, (3.1) and the above inequality, we obtain

$$\begin{aligned} \varphi(u) &\leq \frac{2^{p-1}\pi^p}{pT^{p-1}} s^p - \beta \int_0^T |u(t)|^p dt + T\beta L^p + C_1 \int_0^T b(t) dt \\ &\leq \left(\frac{2^{p-1}\pi^p}{pT^{p-1}} - \frac{\beta T}{2^{p/2}}\right) s^p - \beta T|x|^p + T\beta L^p + C_1 \int_0^T b(t) dt \rightarrow -\infty \\ &\quad \text{as } s^p + |x|^p \rightarrow \infty. \end{aligned}$$

By (I₁), it is easy to get that

$$\varphi(x) \leq 0, \quad \text{for all } x \in \mathbb{R}^N.$$

Let

$$\begin{aligned} A &= \{x \in \mathbb{R}^N : \|x\| \leq G\} \cup \{sw_0 + x : x \in \mathbb{R}^N, s \geq 0, \|sw_0 + x\| = G\}, \\ B &= \partial B_\rho \cap \tilde{W}_T^{1,p}, \quad 0 < \rho < G, \end{aligned}$$

where

$$B_\rho = \{x \in W_T^{1,p} : \|x\| < \rho\}.$$

By [2, Example 5.2], we know that A links B and if G is sufficiently large, then

$$\sup_A \varphi \leq 0 \leq \inf_B \varphi.$$

Then, by Lemma 2.4 and Remark 2.2, we know there exists a sequence $\{u_n\} \subset W_T^{1,p}$ such that

$$\varphi(u_n) = \frac{1}{p} \|\dot{u}_n\|_{L^p}^p - \int_0^T F(t, u_n(t)) dt \rightarrow c \geq \inf_B \varphi \geq 0, \quad \text{as } n \rightarrow \infty, \quad (3.2)$$

$$\|\varphi'(u_n)\|_{(W_T^{1,p})^*} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (3.3)$$

Step 2: we prove $\{u_n\}$ is bounded. The proof is motivated by [14]. By (3.2) and (3.3), we know that there exists a constant $M_0 > 0$ such that

$$|\varphi(u_n)| \leq M_0, \quad \|\varphi'(u_n)\|_{(W_T^{1,p})^*} \leq M_0$$

for all $n \in \mathbb{N}$. It follows from (I_4) that there is a constant $M_1 > L$ such that

$$\mu F(t, x) - (\nabla F(t, x), x) \leq D_0 |x|^p \quad (3.4)$$

for all $|x| \geq M_1$ and a.e. $t \in [0, T]$, where $D_0 < (\mu - p)\beta$. By assumption (A),

$$\mu F(t, x) - (\nabla F(t, x), x) \leq a_0 b(t)$$

for all $|x| \leq M_1$ and a.e. $t \in [0, T]$, where $a_0 = (\mu + M_1) \max_{s \in [0, M_1]} a(s)$. Thus we get

$$(\nabla F(t, x), x) \geq \mu F(t, x) - D_0 |x|^p - a_0 b(t) \quad (3.5)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$. Define

$$f(s) = F(t, sx), \quad \text{for all } s \geq \frac{M_1}{|x|},$$

for all $x \in \mathbb{R}^N \setminus \{0\}$ and a.e. $t \in [0, T]$. Then it follows from (3.4) that

$$\begin{aligned} f'(s) &= \frac{1}{s} (\nabla F(t, sx), sx) \\ &\geq \frac{\mu}{s} F(t, sx) - D_0 s^{p-1} |x|^p \\ &= \frac{\mu}{s} f(s) - D_0 s^{p-1} |x|^p, \end{aligned}$$

which implies that

$$g(s) = f'(s) - \frac{\mu}{s} f(s) + D_0 s^{p-1} |x|^p \geq 0.$$

By solving the above equation, we obtain for $s \geq M_1/|x|$

$$f(s) = \left(\int_{\frac{M_1}{|x|}}^s \frac{g(r) - D_0 r^{p-1} |x|^p}{r^\mu} dr + D_1 \right) s^\mu \quad (3.6)$$

and

$$f\left(\frac{M_1}{|x|}\right) = D_1\left(\frac{M_1}{|x|}\right)^\mu.$$

Then, we get

$$D_1 = \left(\frac{|x|}{M_1}\right)^\mu f\left(\frac{M_1}{|x|}\right).$$

For all $x \in \mathbb{R}^N$ with $|x| > M_1$, it follows from (3.6) and $g(s) \geq 0$ that

$$\begin{aligned} F(t, x) &= f(1) = \int_{\frac{M_1}{|x|}}^1 \frac{g(r) - D_0 r^{p-1} |x|^p}{r^\mu} dr + \left(\frac{|x|}{M_1}\right)^\mu f\left(\frac{M_1}{|x|}\right) \\ &\geq -D_0 |x|^p \int_{\frac{M_1}{|x|}}^1 r^{p-\mu-1} dr + \left(\frac{|x|}{M_1}\right)^\mu F\left(t, \frac{M_1}{|x|}x\right) \\ &= \left(\frac{|x|}{M_1}\right)^\mu F\left(t, \frac{M_1}{|x|}x\right) + \frac{D_0 |x|^p}{\mu - p} - \frac{M_1^{p-\mu}}{\mu - p} D_0 |x|^\mu. \end{aligned}$$

So, by $\mu > p$, we obtain

$$F(t, x) \geq \left[M_1^{-\mu} F\left(t, \frac{M_1}{|x|}x\right) - \frac{D_0}{(\mu - p)M_1^{\mu-p}} \right] |x|^\mu$$

for all $x \in \mathbb{R}^N$ with $|x| > M_1$ and a.e. $t \in [0, T]$. By (I_3) , the above inequality and $D_0 < (\mu - p)\beta$, we have

$$F(t, x) \geq D_2 |x|^\mu$$

for all $x \in \mathbb{R}^N$ with $|x| > M_1$ and a.e. $t \in [0, T]$, where $D_2 = \left(\beta - \frac{D_0}{\mu-p}\right) M_1^{p-\mu} > 0$. Because $F(t, x)$ satisfies assumption (A), then

$$F(t, x) \geq D_2 |x|^\mu - D_2 M_1^\mu - \max_{s \in [0, M_1]} a(s) b(t) \quad (3.7)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$. On one hand, it follows from (3.5) and (2.1) that

$$\begin{aligned} &\mu M_0 + M_0 d_2 \|u_n\|_{L^p} + M_0 d_2 \|\dot{u}_n\|_{L^p} \\ &\geq \mu M_0 + M_0 d_2 \|u_n\|_{W_T^{1,p}} \\ &\geq \mu M_0 + M_0 \|u_n\| \\ &\geq \mu \varphi(u_n) - \langle \varphi'(u_n), u_n \rangle \end{aligned}$$

$$\begin{aligned}
 &= \frac{\mu}{p} \int_0^T |\dot{u}_n(t)|^p dt - \mu \int_0^T F(t, u_n(t)) dt - \int_0^T |\dot{u}_n(t)|^p dt \\
 &\quad + \int_0^T (\nabla F(t, u_n(t)), u_n(t)) dt \\
 &= \frac{\mu-p}{p} \int_0^T |\dot{u}_n(t)|^p dt - \int_0^T [\mu F(t, u_n(t)) - (\nabla F(t, u_n(t)), u_n(t))] dt \\
 &\geq \frac{\mu-p}{p} \int_0^T |\dot{u}_n(t)|^p dt - D_0 \int_0^T |u_n(t)|^p dt - a_0 \int_0^T b(t) dt \\
 &= \frac{\mu-p}{p} \|\dot{u}_n\|_{L^p}^p - D_0 \|u_n\|_{L^p}^p - D_3,
 \end{aligned}$$

where $D_3 = a_0 \int_0^T b(t) dt$. Thus, we have

$$\mu M_0 + M_0 d_2 \|u_n\|_{L^p} + D_0 \|u_n\|_{L^p}^p + D_3 \geq \frac{\mu-p}{2p} \|\dot{u}_n\|_{L^p}^p + D_4,$$

where

$$D_4 = \min_{s \in [0, +\infty)} \left\{ \frac{\mu-p}{2p} s^p - M_0 d_2 s \right\}.$$

The fact $\mu > p$ implies that $-\infty < D_4 < 0$. Hence,

$$\frac{\mu-p}{2p} \|\dot{u}_n\|_{L^p}^p \leq D_0 \|u_n\|_{L^p}^p + M_0 d_2 \|u_n\|_{L^p} + D_3 + \mu M_0 - D_4$$

and then

$$\|\dot{u}_n\|_{L^p} \leq D_5 \|u_n\|_{L^p} + D_6 \|u_n\|_{L^p}^{1/p} + D_7 \quad (3.8)$$

for some positive constants D_5, D_6, D_7 . On the other hand, by (2.1), (3.5) and (3.7) and Hölder's inequality, we have

$$\begin{aligned}
 &pM_0 + M_0 d_2 \|u_n\|_{L^p} + M_0 d_2 \|\dot{u}_n\|_{L^p} \\
 &\geq pM_0 + M_0 d_2 \|u_n\|_{W_T^{1,p}} \\
 &\geq p\varphi(u_n) - \langle \varphi'(u_n), u_n \rangle \\
 &= \int_0^T (\nabla F(t, u_n(t)), u_n(t)) dt - p \int_0^T F(t, u_n(t)) dt \\
 &\geq \int_0^T [\mu F(t, u_n(t)) - D_0 |u_n(t)|^p - a_0 b(t)] dt - p \int_0^T F(t, u_n(t)) dt
 \end{aligned}$$

$$\begin{aligned}
 &\geq (\mu - p) \int_0^T F(t, u_n(t)) \, dt - D_0 \int_0^T |u_n(t)|^p \, dt - D_3 \\
 &\geq (\mu - p) D_2 \int_0^T |u_n(t)|^\mu \, dt - D_0 \int_0^T |u_n(t)|^p \, dt - D_8 \\
 &\geq (\mu - p) D_3 T^{1-\mu/p} \|u_n\|_{L^p}^\mu - D_0 \|u_n\|_{L^p}^p - D_8
 \end{aligned}$$

for some constant $D_8 > 0$. Combining (3.8) with the above inequality, we get

$$\begin{aligned}
 pM_0 + M_0 d_2 D_7 + D_8 &\geq (\mu - p) D_2 T^{1-\mu/p} \|u_n\|_{L^p}^\mu - D_0 \|u_n\|_{L^p}^p \\
 &\quad - M_0 d_2 (1 + D_5) \|u_n\|_{L^p} - M_0 d_2 D_6 \|u_n\|_{L^p}^{1/p},
 \end{aligned}$$

which implies that $\|u_n\|_{L^p}$ is bounded since $\mu > p$. By (3.8), it is easy to know $\|\dot{u}_n\|_{L^p}$ is also bounded. Hence $\|u_n\|_{W_T^{1,p}}$ is bounded. Furthermore, by (2.1), $\{u_n\}$ is bounded in $W_T^{1,p}$.

Step 3: we prove that system (1.1) has a non-constant periodic solution. By Step 2, we know that there exists $M > 0$ such that $\|u_n\| \leq M$. Since $W_T^{1,p}$ is a reflexive Banach space, then there is a renamed subsequence such that

$$u_n \rightharpoonup u \quad \text{weakly in } W_T^{1,p}. \quad (3.9)$$

Furthermore, by [5, Proposition 1.2], we have

$$u_n \rightarrow u \quad \text{strongly in } C([0, T], \mathbb{R}^N). \quad (3.10)$$

By (2.8), we have

$$\begin{aligned}
 \langle \varphi'(u_n), u_n - u \rangle &= \int_0^T (|\dot{u}_n(t)|^{p-2} \dot{u}_n(t), \dot{u}_n(t) - \dot{u}(t)) \, dt \\
 &\quad - \int_0^T (\nabla F(t, u_n(t)), u_n(t) - u(t)) \, dt.
 \end{aligned} \quad (3.11)$$

Since $\{\|u_n\|\}$ is bounded and $\varphi'(u_n) \rightarrow 0$, we have

$$|\langle \varphi'(u_n), u_n - u \rangle| \leq \|\varphi'(u_n)\|_{(W_T^{1,p})^*} \|u_n - u\| \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (3.12)$$

By assumption (A) and (3.10), we have

$$\int_0^T (\nabla F(t, u_n(t)), u_n(t) - u(t)) \, dt \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.13)$$

Hence, it follows from (3.11), (3.12) and (3.13) that

$$\int_0^T (|\dot{u}_n(t)|^{p-2} \dot{u}_n(t), \dot{u}_n(t) - \dot{u}(t)) dt \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.14)$$

On the other hand, it is easy to derive from (3.10) and the boundedness of $\{u_n\}$ that

$$\int_0^T (|u_n(t)|^{p-2} u_n(t), u_n(t) - u(t)) dt \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.15)$$

Set

$$\psi(u) = \frac{1}{p} \left(\int_0^T |u(t)|^p dt + \int_0^T |\dot{u}(t)|^p dt \right).$$

Then we have

$$\begin{aligned} \langle \psi'(u_n), u_n - u \rangle &= \int_0^T (|u_n(t)|^{p-2} u_n(t), u_n(t) - u(t)) dt \\ &\quad + \int_0^T (|\dot{u}_n(t)|^{p-2} \dot{u}_n(t), \dot{u}_n(t) - \dot{u}(t)) dt, \end{aligned} \quad (3.16)$$

and

$$\begin{aligned} \langle \psi'(u), u_n - u \rangle &= \int_0^T (|u(t)|^{p-2} u(t), u_n(t) - u(t)) dt \\ &\quad + \int_0^T (|\dot{u}(t)|^{p-2} \dot{u}(t), \dot{u}_n(t) - \dot{u}(t)) dt. \end{aligned} \quad (3.17)$$

From (3.14) and (3.15), we obtain

$$\langle \psi'(u_n), u_n - u \rangle \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.18)$$

On the other hand, it follows from (3.9) that

$$\langle \psi'(u), u_n - u \rangle \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.19)$$

By (3.16), (3.17) and by using Hölder's inequality, we get

$$\begin{aligned}
& \langle \psi'(u_n) - \psi'(u), u_n - u \rangle \\
&= \int_0^T (|u_n(t)|^{p-2} u_n(t), u_n(t) - u(t)) dt + \int_0^T (|\dot{u}_n(t)|^{p-2} \dot{u}_n(t), \dot{u}_n(t) - \dot{u}(t)) dt \\
&\quad - \int_0^T (|u(t)|^{p-2} u(t), u_n(t) - u(t)) dt - \int_0^T (|\dot{u}(t)|^{p-2} \dot{u}(t), \dot{u}_n(t) - \dot{u}(t)) dt \\
&= \|u_n\|_{W_T^{1,p}}^p + \|u\|_{W_T^{1,p}}^p - \int_0^T (|u_n(t)|^{p-2} u_n(t), u(t)) dt \\
&\quad - \int_0^T (|\dot{u}_n(t)|^{p-2} \dot{u}_n(t), \dot{u}(t)) dt - \int_0^T (|u(t)|^{p-2} u(t), u_n(t)) dt \\
&\quad - \int_0^T (|\dot{u}(t)|^{p-2} \dot{u}(t), \dot{u}_n(t)) dt \\
&\geq \|u_n\|_{W_T^{1,p}}^p + \|u\|_{W_T^{1,p}}^p - \left(\|u_n\|_{L^p}^{p-1} \|u\|_{L^p} + \|\dot{u}_n\|_{L^p}^{p-1} \|\dot{u}\|_{L^p} \right) \\
&\quad - \left(\|u\|_{L^p}^{p-1} \|u_n\|_{L^p} + \|\dot{u}\|_{L^p}^{p-1} \|\dot{u}_n\|_{L^p} \right) \\
&\geq \|u_n\|_{W_T^{1,p}}^p + \|u\|_{W_T^{1,p}}^p - \left(\|u\|_{L^p}^p + \|\dot{u}\|_{L^p}^p \right)^{1/p} \left(\|u_n\|_{L^p}^p + \|\dot{u}_n\|_{L^p}^p \right)^{1/q} \\
&\quad - \left(\|u_n\|_{L^p}^p + \|\dot{u}_n\|_{L^p}^p \right)^{1/p} \left(\|u\|_{L^p}^p + \|\dot{u}\|_{L^p}^p \right)^{1/q} \\
&= \|u_n\|_{W_T^{1,p}}^p + \|u\|_{W_T^{1,p}}^p - \|u\|_{W_T^{1,p}}^{p-1} \|u_n\|_{W_T^{1,p}}^{p-1} - \|u_n\|_{W_T^{1,p}}^{p-1} \|u\|_{W_T^{1,p}}^{p-1} \\
&= \left(\|u_n\|_{W_T^{1,p}}^{p-1} - \|u\|_{W_T^{1,p}}^{p-1} \right) \left(\|u_n\|_{W_T^{1,p}} - \|u\|_{W_T^{1,p}} \right).
\end{aligned}$$

It follows that

$$0 \leq \left(\|u_n\|_{W_T^{1,p}}^{p-1} - \|u\|_{W_T^{1,p}}^{p-1} \right) \left(\|u_n\|_{W_T^{1,p}} - \|u\|_{W_T^{1,p}} \right) \leq \langle \psi'(u_n) - \psi'(u), u_n - u \rangle,$$

which, together with (3.18) and (3.19) yields $\|u_n\|_{W_T^{1,p}} \rightarrow \|u\|_{W_T^{1,p}}$. By the uniform convexity of $W_T^{1,p}$ and (3.9), it follows from the Kadec-Klee property that $\|u_n - u\|_{W_T^{1,p}} \rightarrow 0$. Then by (2.1), we get $\|u_n - u\| \rightarrow 0$. Furthermore, by

(3.3) and the continuity of $\varphi'(\cdot)$ in $W_T^{1,p}$, we have

$$\langle \varphi'(u), v \rangle = \int_0^T (|\dot{u}(t)|^{p-2} \dot{u}(t), \dot{v}(t)) dt - \int_0^T (\nabla F(t, u(t)), v(t)) dt = 0,$$

for all $v \in W_T^{1,p}$,

from which we conclude easily that u is a solution of (1.1). Since φ is continuous, by (3.2), we know that

$$\varphi(u) = c \geq 0.$$

If $c > 0$ and $u \in \mathbb{R}^N$, then by (I_1) , we have

$$\varphi(u) = - \int_0^T F(t, u) dt \leq 0,$$

which contradicts with $\varphi(u) = c > 0$. If $c = 0$, then by (3.2), $c = \inf_B \varphi = 0$. Furthermore, by Lemma 2.4, we know $d(u_n, B) \rightarrow 0$. Hence, there is a sequence $\{v_n\} \subset B$ such that $u_n - v_n \rightarrow 0$ in $W_T^{1,p}$. Note that

$$\|u - v_n\| \leq \|u - u_n\| + \|u_n - v_n\| \rightarrow 0.$$

We obtain that $\{v_n\} \subset B$ converges to u strongly. Since $\tilde{W}_T^{1,p}$ is closed subspace and $B = \partial B_\rho \cap \tilde{W}_T^{1,p}$ is closed subset in $W_T^{1,p}$, we have $u \in B$. Hence, u is not constant. Thus the proof is complete. \square

Remark 3.1. When $\{u_n\}$ is bounded in $W_T^{1,p}$, the fact that $\|u_n - u\|_{W_T^{1,p}} \rightarrow 0$ in the Step 3 has been proved by Xu and Tang in [15]. For readers' convenience, we present it here.

Next, we prove Theorem 1.2.

Proof of Theorem 1.2. If $u \in \tilde{W}_T^{1,p}$ and

$$\|u\| = \|\dot{u}\|_{L^p} = \rho = \frac{2(q+1)^{1/q}}{T^{1/q}} l,$$

then by Lemma 2.2, we have $\|u\|_\infty \leq l$, and by $(I_2)'$, we get $F(t, u(t)) \leq \alpha |u(t)|^p$ for a.e. $t \in [0, T]$. Then by Lemma 2.2, for $u \in \partial B_\rho \cap \tilde{W}_T^{1,p}$, we have

$$\begin{aligned} \varphi(u) &\geq \frac{1}{p} \|\dot{u}\|_{L^p}^p - \alpha \int_0^T |u(t)|^p dt \\ &\geq \frac{1}{p} \|\dot{u}\|_{L^p}^p - \frac{\alpha T^p}{2^p \pi^2 (q+1)^{(p-2)/q}} \|\dot{u}\|_{L^p}^p \end{aligned}$$

$$= \left(\frac{1}{p} - \frac{\alpha T^p}{2^p \pi^2 (q+1)^{(p-2)/q}} \right) \rho^p \geq 0.$$

The other proofs are the same as in Theorem 1.1. □

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