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SOME PROPERTIES OF RETRACT LATTICES OF MONOUNARY ALGEBRAS

Danica Jakubíková-Studenovská* — Jozef Pócs**

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ABSTRACT. Necessary and sufficient conditions for a connected monounary algebra (A, f), under which the lattice $\mathbf{R}^{\emptyset}(A, f)$ of all retracts of (A, f) (together with $\emptyset$) is algebraic, are proved. Simultaneously, all connected monounary algebras in which each retract is a union of completely join-irreducible elements of $\mathbf{R}^{\emptyset}(A, f)$ are characterized. Further, there are described all connected monounary algebras (A, f) such that the lattice $\mathbf{R}^{\emptyset}(A, f)$ is complemented. In this case $\mathbf{R}^{\emptyset}(A, f)$ forms a boolean lattice.

1. Introduction

Monounary algebras, i.e., algebras with one unary operation, play a significant role in the study of algebraic structures (cf., e.g., Jónsson [8], Skornjakov [13], Chvalina [2]). The foundations of the theory of monounary and of partial monounary algebras were laid by M. Novotný [10]; cf. also his expository paper [9]. Further, there exists a close connection between (partial) monounary algebras and some types of automata [1], [12].

The notion of retract was first introduced and studied for topological spaces. The importance of this notion is now well known and commonly appreciated: the notion has been investigated for many types of algebraic structures, groups, partially ordered sets, lattices, etc. From these results let us mention the paper of Novotný [11], where it is proved that constructions of homomorphisms and retracts of monounary algebras enable to make constructions of homomorphisms and retracts of arbitrary algebras.

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Let us recall that a substructure $\mathcal{M}$ of a structure $\mathcal{A}$ is said to be a retract of $\mathcal{A}$ if there exists and endomorphism (called a retraction) φ of $\mathcal{A}$ onto $\mathcal{M}$ such that $\varphi(z) = z$ for each element z of $\mathcal{M}$. We will deal with retracts of monounary algebras which were studied first in [4]–[6].

This paper is a continuation of [7], where some properties of lattices $\mathbf{R}^{\emptyset}(A, f)$ of all retracts (together with the empty set) of a monounary algebra (A, f) were dealt with. Namely, there were studied semimodularity and concepts related to semimodularity (M–symmetry and Mac Lane's condition) of retract lattices of monounary algebras. Next, there was given a description of all connected monounary algebras such that their retract lattice is modular.

Clearly, a retract is a subalgebra of a given algebra, though the lattice of retracts need not be a sublattice of the subalgebra lattice in general. However, though the retract lattice of a monounary algebra is always complete, it is not algebraic in general. We prove necessary and sufficient conditions for a connected monounary algebra (A, f), under which the lattice $\mathbf{R}^{\emptyset}(A, f)$ is algebraic. Simultaneously, all connected monounary algebras in which each retract is a union of completely join-irreducible elements of $\mathbf{R}^{\emptyset}(A, f)$ are characterized.

Further, there are described all connected monounary algebras (A, f) such that the lattice $\mathbf{R}^{\emptyset}(A, f)$ is complemented. In this case $\mathbf{R}^{\emptyset}(A, f)$ forms a boolean lattice.

2. Preliminaries

In the paper we will use notations of universal algebra due to [8] and of lattice theory due to [3].

By a monounary algebra we understand a pair (A, f) where A is a nonempty set and $f : A \to A$ is a mapping.

Let (A, f) be a monounary algebra. A nonempty subset M of A is said to be a *retract* of (A, f) if there is a mapping φ of A onto M such that φ is an endomorphism of (A, f) and $\varphi(x) = x$ for each $x \in M$. The mapping φ is then called a *retraction endomorphism* corresponding to the retract M.

If $\emptyset \neq B \subseteq A$, then we denote $f \restriction B$ the partial operation on B such that $\mathrm{dom}(f \restriction B) = \{b \in B \cap \mathrm{dom}(f) : \ f(b) \in B\}$ and if $b \in \mathrm{dom}(f \restriction B)$, then $(f \restriction B)(b) = f(b)$. Then $(B, f \restriction B)$ is called a *relative subalgebra* of (A, f). Instead of $(B, f \restriction B)$ we will write also (B, f). If $\mathrm{dom}(f \restriction B) = B$ then (B, f) is said to be a subalgebra of (A, f).

A monounary algebra (A, f) is called *connected* if for arbitrary elements $x, y \in A$ there are non-negative integers n, m such that $f^n(x) = f^m(y)$. A maximal connected subalgebra of a monounary algebra is called a *(connected) component.*

An element $x \in A$ is referred to as cyclic, if there exists a positive integer n such that $f^n(x) = x$. In this case the set $\{x, f^1(x), f^2(x), \dots, f^{n-1}(x)\}$ is said to be a cycle.

The notion of *degree* $s_f(x)$ of an element $x \in A$ was introduced in [10] (cf. also [9]) as follows. Let us denote by $A^{(\infty)}$ the set of all elements $x \in A$ such that there exists a sequence $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$ of elements belonging to A with the property $x_0 = x$ and $f(x_n) = x_{n-1}$ for each $n \in \mathbb{N}$. Further, we put $A^{(0)} = \{x \in A : \ f^{-1}(x) = \emptyset\}$. Now we define a set $A^{(\lambda)} \subseteq A$ for each ordinal λ by induction. Assume that we have defined A^α for each ordinal $\alpha < \lambda$. Then we put

$$A^{(\lambda)} = \Big\{x \in A \smallsetminus \bigcup_{\alpha<\lambda} A^{(\alpha)} : \ f^{-1}(x) \subseteq \bigcup_{\alpha<\lambda} A^{(\alpha)}\Big\}.$$

The sets $A^{(\lambda)}$ are pairwise disjoint. For each $x \in A$, either $x \in A^{(\infty)}$ or there is an ordinal λ with $x \in A^{(\lambda)}$. In the former case we put $s_f(x) = \infty$, in the latter we set $s_f(x) = \lambda$. We put $\lambda < \infty$ for each ordinal λ.

Let (A, f) be a connected monounary algebra. We say that (A, f) is *unbounded*, if

(i) $s_f(x) \neq \infty$ for each $x \in A$,

(ii) if $x \in A$, $n \in \mathbb{N}$, then there is $m \in \mathbb{N}$ such that $f^{-(m+n)}(f^m(x)) \neq \emptyset$.

The connected monounary algebra (A, f) is said to be *bounded* if (A, f) satisfies (i) and does not fulfill (ii).

Further we will use the following notation for some algebras.

As usual, the symbols $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of all integers or of positive integers, respectively. Next, for $n \in \mathbb{N}$ we denote by $\mathbb{Z}_n = \{0_n, 1_n, \dots, (n-1)_n\}$ the set of all integers modulo n. We denote by $(\mathbb{Z}, \mathrm{suc})$ and $(\mathbb{N}, \mathrm{suc})$ the algebras such that suc is the operation of the successor.

Next we denote $Z_{n,\infty} = \mathbb{Z}_n \cup \mathbb{N}$ and for $x \in Z_{n,\infty}$ we set

$$\mathrm{suc}(x) = \begin{cases} x + 1_n & \text{if } x \in \mathbb{Z}_n \\ x - 1 & \text{if } x \in \mathbb{N} \smallsetminus \{1\} \\ 0_n & \text{if } x = 1. \end{cases}$$

Let M be a subset of $\mathbb{N}_0$. We denote $Z_M = \mathbb{Z} \cup \bigcup_{j \in M} \big(\mathbb{N} \times \{j\}\big)$ and for $x \in Z_M$ we set

$$\mathrm{suc}(x) = \begin{cases} x + 1 & \text{if } x \in \mathbb{Z}, \\ (i-1, j) & \text{if } x = (i, j), \ i \in \mathbb{N} \smallsetminus \{1\}, \ j \in M, \\ -j & \text{if } x = (1, j), \ j \in M. \end{cases}$$

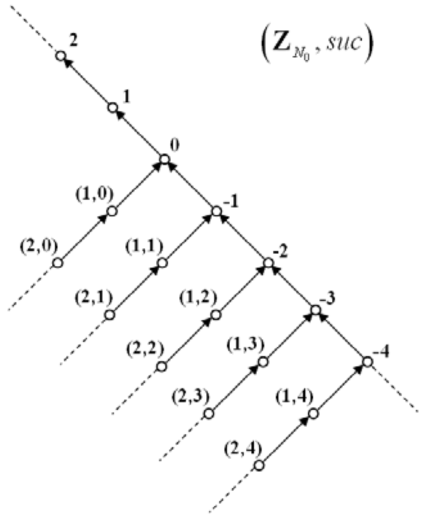

FIGURE 1. An example of (Z_M, suc) for $M = \mathbb{N}_0$

Let M be a subset of $\mathbb{N}$. We denote $Z_M^{(k)} = \mathbb{Z}_k \cup \mathbb{N} \cup \bigcup_{j \in M} \big(\mathbb{N} \times \{j\}\big)$ and for $x \in Z_M^{(k)}$ we set

$$\mathrm{suc}(x) = \begin{cases} x + 1_k & \text{if } x \in \mathbb{Z}_k, \\ x - 1 & \text{if } x \in \mathbb{N} - \{1\}, \\ 0_k & \text{if } x = 1, \\ (i-1, j) & \text{if } x = (i,j),\ i \in \mathbb{N} \setminus \{1\},\ j \in M, \\ j & \text{if } x = (1,j),\ j \in M. \end{cases}$$

In [4] the following theorem characterizing retracts of connected monounary algebras was proved.

Theorem 2.1. *Let (A, f) be a connected monounary algebra and let (M, f) be a subalgebra of (A, f). Then M is a retract of (A, f) if and only if the following condition is satisfied:*

If $y \in f^{-1}(M)$, then there is $z \in M$ with $f(y) = f(z)$ and $s_f(y) \le s_f(z)$.

The second one characterizes retracts of monounary algebras in a general (non-connected) case.

Theorem 2.2. ([4]) *Let (A, f) be a monounary algebra and let (M, f) be a subalgebra of (A, f). Then M is a retract of (A, f) if and only if the following conditions are satisfied:*

(a) *If $y \in f^{-1}(M)$, then there is $z \in M$ such that $f(y) = f(z)$ and $s_f(y) \leq s_f(z)$.*

(b) *For any connected component K of (A, f) with $K \cap M = \emptyset$, the following conditions are satisfied:*

(b1) *If K contains a cycle with d elements, then there is a connected component K' of (A, f) with $K' \cap M \neq \emptyset$ and there is $n \in \mathbb{N}$ such that n/d and K' has a cycle with n elements.*

(b2) *If K contains no cycle and x_0 is a fixed element of K, then there is $y_0 \in M$ such that $s_f(f^k(x_0)) \leq s_f(f^k(y_0))$ for each $k \in \mathbb{N} \cup \{0\}$.*

Through this paper we will use the following notation: Let (A, f) be a connected monounary algebra. For a subset $B \subseteq A$ denote by $B^0 = \{x \in B : f^{-1}(x) = \emptyset\}$. If (A, f) contains no cycle, we denote by $B^{\mathbb{Z}} = \{X \subseteq B : (X, f) \cong (\mathbb{Z}, \mathrm{suc})\}$.

Let (A, f) be a connected monounary algebra. Without quotation we will use the following facts which can be derived from Theorem 2.1.

(i) If (A, f) contains a cycle C, then C is a retract of (A, f).

(ii) If (A, f) contains a subalgebra (M, f), $(M, f) \cong (\mathbb{Z}, \mathrm{suc})$, then M is a retract of (A, f).

(iii) Suppose that (A, f) contains no cycle. Then every retract M of (A, f) as subalgebra is equal to a subalgebra generated by the set $\bigcup M^{\mathbb{Z}} \cup M^0$. We note that for any system X of sets, $\bigcup X = \{x : (\exists X' \in X)(x \in X')\}$.

(iv) If M is a retract of (A, f) and $A^{\mathbb{Z}} \neq \emptyset$ then $M^{\mathbb{Z}} \neq \emptyset$.

(v) Let M be a retract, and $x \in M$ be an element with $s_f(x) = \alpha$, $\alpha \in \mathrm{Ord}$, α limit. Then the set $\{s_f(y) : y \in f^{-1}(x) \cap M\}$ is cofinal in α.

We note that a subset B of a partially ordered set A is *cofinal* if for every a in A there is b in B such that $a \leq b$.

Next, we remind some results proved in [7]. The system of all retracts of a given monounary algebra (A, f) is denoted $R(A, f)$. This system together with empty set ordered by inclusion is denoted $\mathbf{R}^{\emptyset}(A, f)$.

Retracts of monounary algebra (A, f) are closed under arbitrary unions, hence $\mathbf{R}^{\emptyset}(A, f)$ forms a complete lattice with the greatest element A and with the least element $\emptyset$. For $B \in R(A, f) \cup \{\emptyset\}$ denote by $\langle \mathbf{B})$ the sublattice consisting of all retracts of (A, f) containing B. Similarly $(\mathbf{B}\rangle$ denote the sublattice consisting of all retracts of (A, f) contained in B.

We also denote by $\mathbf{R}(A, f)$ the system of all nonempty retracts of (A, f) ordered by inclusion. If there exists the least retract of (A, f), then this system forms a complete lattice, too. We note, that the conditions under which $\mathbf{R}(A, f)$ contains the least element were described in [7].

Let (A, f) be a monounary algebra and $A = \bigcup_{i \in I} A_i$ where (A_i, f), $i \in I$ are connected components of (A, f). Put $I_0 = \{i \in I; B \cap A_i \neq \emptyset\}$.

Lemma 2.3. ([7, Lemma 3.2]) *Let (A, f) be a monounary algebra with the system (A_i, f), $i \in I$ of connected components and let $B \in R(A, f)$ be a proper retract. Then*

$$\langle \mathbf{B} \rangle \cong \prod_{i \in I_0} \langle \mathbf{B_i} \rangle \times \prod_{i \in I \setminus I_0} \mathbf{R}^{\emptyset}(A_i, f) \ .$$

Let (A, f) be a connected monounary algebra, $x \in A$ be an arbitrary noncyclic element. Denote $A^x = \bigcup_{n \in \mathbb{N}_0} f^{-n}(x) \cup \mathbb{N}$ (we may assume that the sets $\mathbb{N}$ and $\bigcup_{n \in \mathbb{N}_0} f^{-n}(x)$ are disjoint) and for $y \in A^x$ we set $\tilde{f}(y) = f(y)$ if $y \in \bigcup_{n \in \mathbb{N}} f^{-n}(x)$, $\tilde{f}(x) = 1$ and $\tilde{f}(y) = y + 1$ for $y \in \mathbb{N}$.

Note that $s_f(x) = s_{\tilde{f}}(x)$ for all $x \in \bigcup_{n \in \mathbb{N}_0} f^{-n}(x)$.

Lemma 2.4. ([7, Lemma 3.3]) *If (A, f) is a connected monounary algebra, $B \in R(A, f)$, then*

$$\langle \mathbf{B} \rangle \cong \prod_{x \in f^{-1}(B) \setminus B} \mathbf{R}^{\emptyset}(A^x, \tilde{f}) \ .$$

The following lemma was proved in [7], too:

Lemma 2.5. ([7, Lemma 4.2]) *Let (A, f) be a connected monounary algebra. The lattice $\mathbf{R}^{\emptyset}(A, f)$ is atomic if and only if (A, f) is not unbounded.*

Lattice L is *atomic* if L has the least element 0, and for every $a \in L$, $a \neq 0$, there is an atom $p \leq a$.

3. Completely join-irreducible elements

Let L be a lattice. An element $a \in L$ is *join-irreducible*, if $a = b \vee c$ implies that $a = b$ or $a = c$, it is *completely join-irreducible* if $a = \bigvee_{i \in I} a_i$ implies that $a = a_{i_0}$ for some $i_0 \in I$.

Theorem 3.1. *Let (A, f) be a connected monounary algebra possessing no cycle. A retract $R \in R(A, f)$ is completely join-irreducible in $\mathbf{R}^{\emptyset}(A, f)$ if and only if the following conditions are satisfied.*

(1) *For all $x, y \in R$, $x \neq y$, $f(x) = f(y)$ implies that $s_f(x) \neq s_f(y)$.*

(2) *For all $x \in R$, $\left| f^{-1}(x) \cap R \right| \leq 2$.*

(3) *For all $x \in R^0 \cup \bigcup R^{\mathbb{Z}}$ there exist at most one $n \in \mathbb{N}$ and $z \in R$ with $f(z) = f^n(x)$ and $s_f(z) < s_f(f^{n-1}(x))$.*

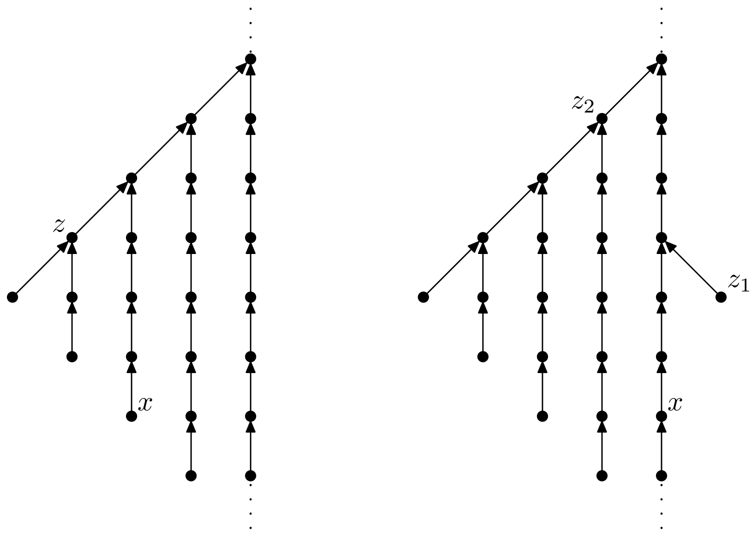

FIGURE 2. An example of an algebra satisfying the condition (3) and not satisfying (3)

P r o o f. Assume that a retract R does not fulfill the conditions of theorem. First suppose that the condition (2) is not valid. Let $x \in R$ be such an element, that $\left|f^{-1}(x) \cap R\right| > 2$. We are able to choose a subset S of $f^{-1}(x) \cap R$ with the property that the set $\{s_f(y) : \ y \in S\}$ is cofinal in the set of ordinals $\{s_f(y) : \ y \in f^{-1}(x) \cap R\}$ and $\left|(f^{-1}(x) \cap R) \smallsetminus S\right| \geq 2$. Denote by z an arbitrary element of $(f^{-1}(x) \cap R) \smallsetminus S$ and denote $Z = (f^{-1}(x) \cap R) \smallsetminus (S \cup \{z\})$. Since $\left|(f^{-1}(x) \cap R) \smallsetminus S\right| \geq 2$ we obtain that $Z \neq \emptyset$. Now put $R_1 = \big(R \smallsetminus \bigcup_{n \in \mathbb{N}} f^{-n}(x)\big) \cup \big(R \cap \bigcup_{n \in \mathbb{N}_0} f^{-n}(S \cup \{z\})\big)$ and $R_2 = \big(R \smallsetminus \bigcup_{n \in \mathbb{N}} f^{-n}(x)\big) \cup \big(R \cap \bigcup_{n \in \mathbb{N}_0} f^{-n}(S \cup Z)\big)$. Using theorem 2.1 a the fact that the set $\{s_f(y) : \ y \in S\}$ is cofinal in the set of ordinals $\{s_f(y) : \ y \in f^{-1}(x) \cap R\}$ and hence is also cofinal in $\{s_f(y) : \ y \in f^{-1}(x)\}$, we obtain that R_1, R_2 are retracts, $R_1, R_2 \subset R$ and $R = R_1 \cup R_2$.

Now assume that the condition (1) is not valid, i.e., there exist $x, y, z \in R$ with $z = f(x) = f(y)$ and $s_f(x) = s_f(y)$. We may assume that $\left|f^{-1}(z) \cap R\right| = 2$ (in other case the condition (2) is not satisfied, too).

Denote by $R_x = \big(R \smallsetminus \bigcup_{n \in \mathbb{N}} f^{-n}(z)\big) \cup \big(R \cap \bigcup_{n \in \mathbb{N}_0} f^{-n}(x)\big)$ and similarly denote by $R_y = \big(R \smallsetminus \bigcup_{n \in \mathbb{N}} f^{-n}(z)\big) \cup \big(R \cap \bigcup_{n \in \mathbb{N}_0} f^{-n}(y)\big)$. Now $R_x, R_y \in R(A, f)$ by 2.1 and $R_x, R_y \subset R$, $R = R_x \cup R_y$.

Finally suppose that the condition (3) is not valid. Let $x \in R^0 \cup \bigcup R^{\mathbb{Z}}$ be such an element that there exist $m, n \in \mathbb{N}$, $m < n$ and $z, y \in R$ with $f(z) = f^m(x)$, $f(y) = f^n(x)$ and $s_f(z) < s_f(f^{m-1}(x))$, $s_f(y) < s_f(f^{n-1}(x))$. Since $s_f(z) < s_f(f^{m-1}(x))$, according to 2.1 we obtain that $R_1 = R \smallsetminus \bigcup_{j \in \mathbb{N}_0} f^{-j}(z)$ is a retract and $R_1 \subset R$. Likewise the same holds for $R_2 = R \smallsetminus \bigcup_{j \in \mathbb{N}_0} f^{-j}(y)$. Hence $R = R_1 \cup R_2$ and R is not completely join–irreducible.

Conversely, suppose that a retract R satisfies the conditions of the theorem and $R = \bigcup_{i \in I} R_i$. Due to the condition (1), R contains at most one copy of $(\mathbb{Z}, \mathrm{suc})$ as a subalgebra, thus we may assume that $R^0 \neq \emptyset$, otherwise R is an atom. Also, we may assume that (R, f) is not isomorphic to $(\mathbb{N}, \mathrm{suc})$. In this case for $x \in R^0$ define

$$p(x) = \min\bigl\{s_f(f^n(x)) : \ (\exists n \in \mathbb{N}_0)(\exists z \in R)\bigl(z \neq f^n(x) \ \& \ f(z) = f^{n+1}(x)\bigr)\bigr\}.$$

Let $\tilde{x} \in R^0$ be an element with $p(\tilde{x}) = \min\bigl\{p(x) : \ x \in R^0\bigr\}$. Since $\tilde{x} \in R = \bigcup_{i \in I} R_i$, there exist $i_0 \in I$ with $\tilde{x} \in R_{i_0}$. We show that $R^0 \subseteq R^0_{i_0}$, what implies $R = R_{i_0}$, since $R^{\mathbb{Z}} = R^{\mathbb{Z}}_{i_0}$.

Let $x \in R^0$, $x \neq \tilde{x}$ be an arbitrary element. Since (A, f) is connected, there are $m, n \in \mathbb{N}$ such that $f^m(\tilde{x}) = f^n(x)$. According to the conditions (1) and (3) and the definition of the element $\tilde{x}$, we obtain $s_f(f^{m-1}(\tilde{x})) < s_f(f^{n-1}(x))$.

Also, using the conditions (3) and (1) we obtain that, $\bigl|f^{-1}(f^k(x))\bigr| = 1$ for all $k = 1, 2, \dots, n-1$. Due to the characterization of retracts in 2.1 and by (2) we obtain $f^{n-1}(x) \in R_{i_0}$ and this also implies $x \in R_{i_0}$. Hence $R^0 \subseteq R^0_{i_0}$, what complete the proof. □

We note that in the first part of the previous proof we were able to find two retracts R_1, R_2 with $R = R_1 \cup R_2$, hence we obtain the following corollary.

Corollary 3.2. *Let (A, f) be a connected monounary algebra possessing no cycle. A retract $R \in R(A, f)$ is completely join-irreducible in $\mathbf{R}^{\emptyset}(A, f)$ if and only if R is join-irreducible.*

If a connected algebra (A, f) possess a cycle C, then due to Lemma 2.4 $\mathbf{R}(A, f) = \langle \mathbf{C} \rangle \cong \prod_{x \in f^{-1}(C) \smallsetminus C} \mathbf{R}^{\emptyset}(A^x, \tilde{f})$, thus a retract $R \in R(A, f)$ is completely join-irreducible in $\mathbf{R}^{\emptyset}(A, f)$ if and only if R is equal to the cycle C or there is exactly one $x \in R \cap \bigl(f^{-1}(C) \smallsetminus C\bigr)$ and $\tilde{R}_x = \bigl(R \cap \bigcup_{i=0}^{\infty} f^{-i}(x)\bigr) \cup \bigcup_{i=1}^{\infty} \tilde{f}^i(x)$ is completely join-irreducible in the corresponding algebra $(A^x, \tilde{f})$. Again $R \in R(A, f)$ is completely join-irreducible if and only if R is join-irreducible.

Theorem 3.3. *Let (A, f) be a connected monounary algebra, $R \in R(A, f)$ be a completely join-irreducible element. Then each retract $Q \in R(A, f)$, $Q \subseteq R$ is completely join-irreducible and moreover $(\mathbf{R}\rangle$ forms a decreasing chain of order type $\leq \omega$. This order type is equal to ω if and only if (A, f) is unbounded.*

Proof. We use the characterization of completely join-irreducible elements from Theorem 3.1. If R satisfies the conditions (1),(2),(3) then also any retract $Q \subseteq R$ does. To prove that $(\mathbf{R}\rangle$ forms a chain it is sufficient to show that for any two incomparable retracts S, T there is no completely join-irreducible upper bound. Since S and T are incomparable retracts, $S \cup T$ is not completely join-irreducible, hence $S \cup T$ does not satisfy one of the conditions of Theorem 3.1 and so does any retract $R' \supseteq S \cup T$.

Finally we prove that the order type of $(\mathbf{R}\rangle$ is $\leq \omega$. First consider that (A, f) possesses no cycle. We may assume that R is not an atom. In this case, similarly as in the proof of the theorem 3.1, define for all $Q \subseteq R$, Q being not an atom and for all $x \in Q^0$, $p_Q(x) = \min\bigl\{s_f(f^n(x)) : \ (\exists n \in \mathbb{N}_0)(\exists z \in R)\bigl(z \neq f^n(x) \ \& \ f(z) = f^{n+1}(x)\bigr)\bigr\}$. Let x_Q be such an element that $p_Q(x_Q) = \min\bigl\{p_Q(x) : \ x \in Q^0\bigr\}$. For all $Q \subseteq R$ the element x_Q is unique and x_Q determines Q uniquely in the following sense: if $Q' \subseteq Q$ and $x_Q \in Q'$ then $Q = Q'$. Hence for all $Q, Q' \subseteq R$, $Q \neq Q'$ implies $x_Q \neq x_{Q'}$. Moreover, if $Q' \subset Q$ and $f^k(x_Q) = f^l(x_{Q'})$ then $k < l$.

Since for all $j \in \mathbb{N}$ there is at most one x_Q such that $x_Q \in \bigcup\limits_{i=0}^{\infty} f^{-i}(f^j(x_R)) \smallsetminus \bigcup\limits_{i=0}^{\infty} f^{-i}(f^{j-1}(x_R))$ we obtain that the order type of $(\mathbf{R}\rangle$ is $\leq \omega$. It is obvious that the order type of $(\mathbf{R}\rangle$ is equal to ω if and only if $\mathbf{R}^{\emptyset}(A, f)$ contains no atom, i.e., (A, f) is unbounded.

Suppose that (A, f) possesses a cycle C. Let R be a completely join-irreducible element of $\mathbf{R}^{\emptyset}(A, f)$. Suppose that R is not equal to the cycle. Then there is exactly one $x \in R \cap \bigl(f^{-1}(C) \smallsetminus C\bigr)$ and $\tilde{R}_x = \bigl(R \cap \bigcup\limits_{i=0}^{\infty} f^{-i}(x)\bigr) \cup \bigcup\limits_{i=1}^{\infty} \tilde{f}^i(x)$ is completely join-irreducible in $\mathbf{R}^{\emptyset}(A^x, \tilde{f})$. We may also assume that $\tilde{R}_x$ is not an atom of $\mathbf{R}^{\emptyset}(A^x, \tilde{f})$. In this case $R^0 \neq \emptyset$. Since there is $m \in \mathbb{N}$ such that $f^m(x_R) \in C$ we obtain that $(\mathbf{R}\rangle$ forms a finite chain. $\square$

4. Algebraic lattice

The aim of this section is to describe all connected monounary algebras (A, f), such that a lattice $\mathbf{R}^{\emptyset}(A, f)$ is algebraic. Simultaneously we will investigate under which conditions each element of $\mathbf{R}^{\emptyset}(A, f)$ is a union of completely join-irreducible elements.

Let L be a complete lattice and let a be an element of L. Then a is called *compact* if $a \leq \bigvee X$, for some $X \subseteq L$, implies that $a \leq \bigvee X_1$, for some finite $X_1 \subseteq X$.

A complete lattice is called *algebraic* if every element is a join of compact elements.

Lemma 4.1. *Let (A, f) be a connected monounary algebra. If there exists an element $x \in A$ with $s_f(x) = \omega$, then the lattice $\mathbf{R}^{\emptyset}(A, f)$ is not algebraic.*

Proof. To prove this statement it is sufficient to show that no retract of (A, f) containing the element x is compact.

Suppose that $R \in R(A, f)$ and $x \in R$. Since $s_f(x) = \omega$, according to 2.1 we obtain that the set $S^R_{f^{-1}(x)} = \{s_f(y) : \ y \in R \cap f^{-1}(x)\}$ is cofinal in ω. We note that the set S of ordinals is cofinal in ω if and only if S is infinite. Let $\mathcal{S} = \{S_i : \ i \in \mathbb{N}\}$ be an infinite partition of $S^R_{f^{-1}(x)}$ where S_i is infinite for all $i \in \mathbb{N}$. Each set S_i is cofinal in $S^R_{f^{-1}(x)}$ and hence in ω. Put $R_i = \big(R \smallsetminus \bigcup\limits_{n=1}^{\infty} f^{-n}(x)\big) \cup \big(R \cap \bigcup\limits_{y \in S_i^A} \bigcup\limits_{n=0}^{\infty} f^{-n}(y)\big)$, where $S_i^A = \{y \in f^{-1}(x) : \ s_f(y) \in S_i\}$. It is not difficult to verify that R_i is a retract for all $i \in \mathbb{N}$. We have $R = \bigcup\limits_{i \in \mathbb{N}} R_i$ and $R \neq \bigcup\limits_{i \in I} R_i$ for all finite subset $I \subset \mathbb{N}$, hence R is not compact. $\square$

Obviously, if a retract R contains an element x with $s_f(x) = \omega$, then according to Theorem 3.1, R is not completely join-irreducible. Hence we obtain the following lemma.

Lemma 4.2. *No retract containing an element x with $s_f(x) = \omega$ is a union of completely join-irreducible elements.*

Lemma 4.3. *Let (A, f) be a connected monounary algebra. If (A, f) contains a subalgebra isomorphic to (Z_M, suc), $M \subseteq \mathbb{N}_0$, M infinite, then the lattice $\mathbf{R}^{\emptyset}(A, f)$ is not algebraic.*

Proof. Suppose that (A, f) contains a subalgebra (B, f) isomorphic to (Z_M, suc), $M \subseteq \mathbb{N}_0$. We will not distinguish (B, f) from (Z_M, suc). We show that the retract $\mathbb{Z}$ is not compact in $\mathbf{R}^{\emptyset}(A, f)$. Denote by Z_i, $i \in M$ the retract generated by the set $\{(n, i) : \ n \in \mathbb{N}\}$. We note that each (Z_i, suc) is isomorphic to $(\mathbb{Z}, \mathrm{suc})$. Now $\mathbb{Z} \subseteq \bigcup\limits_{i \in M} Z_i$ and it is evident that $\mathbb{Z} \nsubseteq \bigcup\limits_{i \in X_1} Z_i$ for any $X_1 \subseteq M$, X_1 finite. Hence $\mathbb{Z}$ is not compact. Since $\mathbb{Z}$ is an atom, it cannot be a join of compact elements. $\square$

Similarly as Lemma 4.3 we can prove the following statement.

Lemma 4.4. *Let (A, f) be a connected monounary algebra. If (A, f) contains a subalgebra isomorphic to $(Z_M^{(k)}, \mathrm{suc})$, $k \in \mathbb{N}$, $M \subseteq \mathbb{N}$, M infinite, then the lattice $\mathbf{R}^{\emptyset}(A, f)$ is not algebraic.*

4.1. Not unbounded algebras

Now we will deal with connected monounary algebras which are not unbounded. Further we will assume that (A, f) possesses no cycle. We say that a retract R of a connected monounary algebra (A, f) is *finitely generated* if R^0 and $R^{\mathbb{Z}}$ is finite. We note that the retract R as subalgebra is equal to the subalgebra generated by the set $R^0 \cup \bigcup R^{\mathbb{Z}}$.

Lemma 4.5. *Let (A, f) be a connected not unbounded monounary algebra possessing no cycle. If (A, f) contains no element x with $s_f(x) = \omega$ and (A, f) contains no subalgebra isomorphic to (Z_M, suc), $M \subseteq \mathbb{N}_0$, M infinite, then finitely generated retracts are compact elements of $\mathbf{R}^{\emptyset}(A, f)$ and $\mathbf{R}^{\emptyset}(A, f)$ is an algebraic lattice.*

Proof. First suppose that there is no element of degree equal to ∞, i.e., (A, f) is bounded. Let R be a finitely generated retract. Since retracts are subalgebras, R is a compact element of the subalgebra lattice of (A, f) and thus R is a compact element of $\mathbf{R}^{\emptyset}(A, f)$ since the operation of join in the both lattices is equal to union. Now assume that $A^{\mathbb{Z}} \neq \emptyset$. Since (A, f) contains no subalgebra isomorphic to (Z_M, suc), $M \subseteq \mathbb{N}_0$, M infinite, for all $X \in A^{\mathbb{Z}}$ there exists $\hat{x} \in X$ such that $\hat{x}$ belongs to no other $Y \in A^{\mathbb{Z}}$, i.e., $\hat{x}$ determines X uniquely. If R is a finitely generated retract, $R \subseteq \bigcup\limits_{i \in I} R_i$, then it is evident, that there exists a finite subset $I_0 \subseteq I$ with $R^0 \cup \{\hat{x} \in X : X \in R^{\mathbb{Z}}\} \subseteq \bigcup\limits_{i \in I_0} R_i$. Thus $R \subseteq \bigcup\limits_{i \in I_0} R_i$. Hence R is a compact element of $\mathbf{R}^{\emptyset}(A, f)$.

Now we show that $\mathbf{R}^{\emptyset}(A, f)$ is an algebraic lattice. For this purpose we prove that for all $R \in R(A, f)$ and $x \in R^0$ there exists a finitely generated retract K_x, with $x \in K_x \subseteq R$.

Let $R \in R(A, f)$, $y \in R, f^{-1}(y) \neq \emptyset$. If $s_f(y) = \infty$ then $y \in \bigcup R^{\mathbb{Z}}$ and there is $X \in R^{\mathbb{Z}}$ with $y \in X$. Now suppose that $s_f(y) \neq \infty$. There is no element $z \in A$ with $s_f(z) = \omega$, hence the algebra $(A^y, \tilde{f})$ is bounded. It is not difficult to verify that the set $\tilde{R} = R \cap \bigcup\limits_{n=0}^{\infty} f^{-n}(y) \cup \bigcup\limits_{n=1}^{\infty} \tilde{f}^n(y)$ is a retract of $(A^y, \tilde{f})$. Due to Lemma 2.5 the lattice $\mathbf{R}^{\emptyset}(A^y, \tilde{f})$ is atomic, hence there is a retract $Y \in R(A^y, \tilde{f})$, $Y \subseteq \tilde{R}$, $(Y, \tilde{f}) \cong (\mathbb{N}, \mathrm{suc})$. So we have proved that for each $R \in R(A, f)$, $y \in R, f^{-1}(y) \neq \emptyset$ there exists a "chain" Y with the following properties: $Y \subseteq R$, $y \in Y$ and for all $y' \in f^{-1}(Y) \cap \bigcup\limits_{n=0}^{\infty} f^{-n}(y)$, there is $z \in Y$

with $f(y') = f(z)$, $s_f(y') \leq s_f(z)$, i.e., the elements below y satisfy the condition of 2.1.

Let $x \in R^0$ be an arbitrary element. Now we construct a retract K_x. Since (A, f) is not unbounded, the lattice $\mathbf{R}^{\emptyset}(A, f)$ is atomic. Let $m \in \mathbb{N}_0$ be the least nonnegative integer, such that $f^m(x) \in B$ for some atom $B \subseteq R$. Put $K_{x,0} = \{x\}$ and for all $i = 1, \dots, m-1$, $K_{x,i} = Y_i$ where Y_i is the "chain" corresponding to the element $y_i = f^i(x)$. According to the properties of the sets $K_{x,i}$ it is not difficult to verify that the set $K_x = B \cup \bigcup\limits_{i=0}^{m-1} K_{x,i}$ satisfies the condition of Theorem 2.1, thus there is a retract of (A, f) with $K_x \subseteq R$. Since $\left|{K_x}^0\right| \leq m + 1$ and $\left|{K_0}^{\mathbb{Z}}\right| \leq 1$, the retract K_x is finitely generated.

Finally, $R = \bigcup\limits_{x \in R^0} K_x \cup \bigcup R^{\mathbb{Z}}$, hence each $R \in R(A, f)$ is a union of compact retracts, which completes the proof. □

We note that the retract K_x can be constructed in such a way that K_x is completely join-irreducible. We put $K_{x,i} = Y_i$ if $s_f(f^{i-1}(x)) < \max\bigl\{s_f(z) :\ z \in f^{-1}(f^i(x))\bigr\}$ and $K_{x,i} = Y_{i-1} \cup \{f^i(x)\}$ otherwise. Hence we obtain the following lemma.

Lemma 4.6. *Let (A, f) be a connected not unbounded monounary algebra containing no element of degree ω. Then every retract is a union of completely join-irreducible retracts.*

Theorem 4.7. *Let (A, f) be a connected not unbounded monounary algebra. The lattice $\mathbf{R}^{\emptyset}(A, f)$ is algebraic if and only if (A, f) satisfies the following conditions:*

(1) *There is no element $x \in A$ with $s_f(x) = \omega$.*
(2) *If (A, f) possesses no cycle, then (A, f) contains no subalgebra isomorphic to (Z_M, suc), $M \subseteq \mathbb{N}_0$, M infinite.*
(3) *If (A, f) possesses a cycle, then (A, f) contains no subalgebra isomorphic to $(Z_M^{(k)}, \mathrm{suc})$, $k \in \mathbb{N}$, $M \subseteq \mathbb{N}_0$, M infinite.*

Proof. If an algebra contains a cycle C then $\mathbf{R}(A, f) \cong \prod\limits_{x \in f^{-1}(C) \smallsetminus C} \mathbf{R}^{\emptyset}(A^x, \tilde{f})$. Since (A, f) contains no subalgebra isomorphic to $(Z_M^{(|C|)}, \mathrm{suc})$, $M \subseteq \mathbb{N}_0$, M infinite, any algebra $(A^x, \tilde{f})$, $x \in f^{-1}(C) \smallsetminus C$ contains no subalgebra isomorphic to (Z_M, suc), $M \subseteq \mathbb{N}_0$, M infinite. A direct product of algebraic lattices is again algebraic, hence $\mathbf{R}(A, f)$ is algebraic and also $\mathbf{R}^{\emptyset}(A, f)$. Now the statement follows from Lemmas 4.1 and 4.3-4.5. □

4.2. Unbounded algebras

Further we will deal with unbounded connected monounary algebras. We note that in this case no retract is finitely generated, otherwise there is a retract isomorphic to $(\mathbb{N}, \mathrm{suc})$. Also we will assume that an unbounded monounary algebra (A, f) contains no element of degree equal to ω. Then for all $x \in A$, $s_f(x) \in \omega$, i.e., $s_f(x)$ is a finite ordinal.

Let $x \in A^0$ be a fixed element. For $i \in \mathbb{N}$, denote $d_i = \max\{s_f(y) : \ y \in f^{-1}(f^i(x))\}$. Let $M = \{i \in \mathbb{N} : \ d_i > s_f(f^{i-1}(x))\}$. The algebra (A, f) is unbounded, therefore M is infinite.

By a *skeleton of* (A, f) *corresponding to* x we understand the subalgebra generated by a set $\{y_i : \ i \in M \cup \{0\}\}$, where $y_0 = x$ and y_i is an arbitrary element with $y_i \in A^0$, $f^{d_i+1}(y_i) = f^i(x)$.

We note that it is not difficult to verify that each skeleton of (A, f) corresponding to x is a retract of (A, f).

Denote by $S(x)$ the set of all skeletons corresponding to x. We say that $S(x)$ is *almost unique* if the following condition is valid: if $M' \subseteq M$ and for all $i \in M'$ there exist distinct elements $v_i, w_i \in A^0$ with $f^{d_i+1}(v_i) = f^{d_i+1}(w_i) = f^i(x)$, then M' is finite.

Lemma 4.8. *Each skeleton is completely join-irreducible in* $\mathbf{R}^{\emptyset}(A, f)$.

P r o o f. Let S be a skeleton of x with a generating set $\{y_i : \ i \in M \cup \{0\}\}$. We verify that S satisfies the conditions of Theorem 3.1. Since the only elements $z \in S$ with $|f^{-1}(z) \cap S| > 1$ are of the form $f^i(x)$, $i \in M$, we obtain that S satisfies (1) and (2) of 3.1. Further let $y \in S^0$. Then $y = x$ or $y = y_j$ for some $j \in M$. If $y = x$ then for all $i \in M$, $s_f(f^{i-1}(x)) < s_f(z)$, where $z \in S$, $f(z) = f^i(x)$. Now suppose that $y = y_j$. For $1 \le l \le d_j$ we have $f^{-1}(f^l(y_j)) \cap S = f^{l-1}(y_j)$ and for $l > d_j + 1$, $f^l(y_j) = f^{j+l-(d_j+1)}(x)$, thus $d_j + 1$ is the only positive integer with $f^{d_j+1}(y_j) = f(z)$, where $z = f^{j-1}(x)$ and $s_f(f^{d_j}(y_j)) > s_f(z)$. Hence S also satisfies the condition (3) of 3.1. □

Lemma 4.9. *Let* $R \in R(A, f)$ *be a retract and* $x \in R^0$ *be an arbitrary element. Then there is a skeleton* S_x *of* x *such that* $S_x \subseteq R$.

P r o o f. We prove that for all $i \in M$ there exists $y_i \in R^0$ with $f^{d_i+1}(y_i) = f^i(x)$. Let $i \in M$. Since all elements of (A, f) have finite degree, the same holds for an algebra $(A^{f^i(x)}, \tilde{f})$. Moreover, $(A^{f^i(x)}, \tilde{f})$ is bounded and $s_f(z) = s_{\tilde{f}}(z)$ for all $z \in \bigcup\limits_{n=0}^{\infty} f^{-n}(f^i(x))$. Since $\tilde{R} = R \cap \bigcup\limits_{n=0}^{\infty} f^{-n}(f^i(x)) \cup \bigcup\limits_{n=1}^{\infty} \tilde{f}^n(f^i(x))$ is a retract of $(A^{f^i(x)}, \tilde{f})$ and $\mathbf{R}^{\emptyset}(A^{f^i(x)}, \tilde{f})$ is atomic, there exists a retract $Y \subseteq \tilde{R}$, $(Y, \tilde{f}) \cong (\mathbb{N}, \mathrm{suc})$. Hence Y is generated by some element y_i. Obviously $y_i \in R^0$

and $f^{d_i+1}(y_i) = f^i(x)$, since $s_f(f^i(x)) = s_{\tilde{f}}(f^i(x)) = d_i + 1$. Take S_x the subalgebra generated by the set $\{y_i : \ i \in M\} \cup \{x\}$. $\square$

As a consequence of this lemma we obtain that each retract R is a union of skeletons, $R = \bigcup\limits_{x \in R^0} S_x$.

The following theorem characterizes all connected monounary algebras where any retract is a union of completely join-irreducible retracts.

Theorem 4.10. *Let (A, f) be a connected monounary algebra. In $\mathbf{R}^{\emptyset}(A, f)$, every retract is a union of completely join-irreducible retracts if and only if there is no element $x \in A$ with $s_f(x) = \omega$.*

Proof. If (A, f) possesses no cycle, then the statement follows from Lemmas 4.2, 4.6, 4.8 and 4.9.

If (A, f) possesses a cycle C, then $\mathbf{R}(A, f) = \langle \mathbf{C} \rangle \cong \prod\limits_{x \in f^{-1}(C) \smallsetminus C} \mathbf{R}^{\emptyset}(A^x, \tilde{f})$. The statement follows from the fact that (A, f) contains no element of degree ω if and only if the same holds for all $(A^x, \tilde{f})$, $x \in f^{-1}(C) \smallsetminus C$. $\square$

Lemma 4.11. *A skeleton of x is compact if and only if $S(x)$ is almost unique.*

Proof. Let $x \in A^0$. First suppose that $S(x)$ is almost unique. Let $S \in S(x)$ be an arbitrary skeleton of x generated by a set $\{y_i : \ i \in M \cup \{0\}\}$ and $S \subseteq \bigcup\limits_{t \in T} R_t$. Let i_0 be the greatest positive number such that there exist distinct $v_{i_0}, w_{i_0} \in A^0$ with $f^{d_{i_0}+1}(v_{i_0}) = f^{d_{i_0}+1}(w_{i_0}) = f^{i_0}(x)$. Hence for all $S_1, S_2 \in S(x)$ we obtain that $S_1 \smallsetminus \bigcup\limits_{n=1}^{\infty} f^{-n}(f^{i_0}(x)) = S_2 \smallsetminus \bigcup\limits_{n=1}^{\infty} f^{-n}(f^{i_0}(x))$. Since $x \in \bigcup\limits_{t \in T} R_t$, there exists $t_0 \in T$ such that $x \in R_{t_0}$. Due to Lemma 4.9, there is a skeleton S_1 of x with $S_1 \subseteq R_{t_0}$. Since $S(x)$ is almost unique we have $\{y_i : \ i \in M, \ i > i_0\} \subseteq S_1$. Hence for each $1 \leq i \leq i_0$ there is $t_i \in T$ with $y_i \in R_{t_i}$ and we obtain that $S \subseteq \bigcup\limits_{t \in T_0} R_t$, where $|T_0| \leq i_0 + 1$.

Conversely, suppose that $S(x)$ is not almost unique. Let $M' = \{i_j : \ j \in \mathbb{N}\} \subseteq M$ be an infinite subset, such that there exist $v_i \in A^0$, $v_i \neq y_i$ with $f^{d_i+1}(v_i) = f^{d_i+1}(y_i) = f^i(x)$. Denote by R_j the skeleton generated by the set $\{y_i : \ i \in M \smallsetminus M'\} \cup \{y_{i_1}, y_{i_2}, \dots, y_{i_j}\} \cup \{v_{i_l} : \ l > j\}$. Now $S \subseteq \bigcup\limits_{j \in \mathbb{N}} R_j$ and $S \nsubseteq \bigcup\limits_{j \in J} R_j$ for any finite $J \subseteq \mathbb{N}$, hence S is not compact. $\square$

Since in any algebraic lattice, completely join-irreducible elements are compact, we obtain the following theorem.

Theorem 4.12. *Let (A, f) be an unbounded monounary algebra containing no element of degree ω. The lattice $\mathbf{R}^{\emptyset}(A, f)$ is algebraic if and only if for all $x \in A^0$ the system $S(x)$ of skeletons is almost unique.*

5. Complemented and boolean lattices

In this section we will find conditions under which $\mathbf{R}(A, f)$ or $\mathbf{R}^{\emptyset}(A, f)$ respectively, is complemented. Also we will show that in this case the system $\mathbf{R}(A, f)$ forms a boolean lattice.

Let (A, f) be a monounary algebra, $x \in A$. We say that the ordered pair $\big(x, f(x)\big)$ forms a *gap* if $s_f(x) + 1 < s_f(f(x))$.

Lemma 5.1. *Let (A, f) be a monounary algebra, $x \in A$ be an element such that $\big(x, f(x)\big)$ forms a gap. Then the lattice $\mathbf{R}^{\emptyset}(A, f)$ is not complemented.*

Proof. Suppose that there is $x \in A$ such that $\big(x, f(x)\big)$ forms a gap. Since $\big(x, f(x)\big)$ forms a gap, there exists $y \in f^{-1}(x)$ with $s_f(x) < s_f(y)$. Consider the set $R = A \smallsetminus \bigcup\limits_{n \in \mathbb{N}_0} f^{-n}(x)$. The only element $z \in f^{-1}(R)$, $z \notin R$ is equal to x and the element y satisfies the condition of 2.1 therefore R is a retract of (A, f). Let R' be a retract with $R' \cup R = A$. Then $x \in R'$ and $R' \cap \big(f^{-1}(f(x)) \smallsetminus \{x\}\big) \neq \emptyset$, since $y \in f^{-1}(x)$ and $s_f(x) < s_f(y)$. The set $R' \cap R = R' \cap \left(A \smallsetminus \bigcup\limits_{n \in \mathbb{N}_0} f^{-n}(x)\right) = R' \smallsetminus \bigcup\limits_{n \in \mathbb{N}_0} f^{-n}(x)$ is a retract and $R \wedge R' = R \cap R' \neq \emptyset$, hence $\mathbf{R}^{\emptyset}(A, f)$ is not complemented. □

The condition that a connected monounary algebra (A, f) contains no gap is not sufficient for a lattice $\mathbf{R}^{\emptyset}(A, f)$ to be complemented.

Denote by $\mathcal{C}$ the class of all infinite connected monounary algebras, consisting only of elements of degree ∞ and possessing a cycle. Next, denote by $\mathcal{Z}$ a class of all connected monounary algebras (A, f) with all elements of degree ∞ and satisfying the property: there exists a subalgebra of (A, f) isomorphic to (Z_M, suc), $M \subseteq \mathbb{N}_0$, M infinite.

Lemma 5.2. *Let (A, f) be a connected monounary algebra such that $(A, f) \in \mathcal{C} \cup \mathcal{Z}$. Then $\mathbf{R}^{\emptyset}(A, f)$ is not complemented.*

Proof. Suppose that $(A, f) \in \mathcal{C}$. Then (A, f) contains a cycle and the cycle is the only atom covering the empty set, thus $\mathbf{R}^{\emptyset}(A, f)$ is not complemented.

Further assume that $(A, f) \in \mathcal{Z}$. Let (B, f) be a subalgebra of (A, f) isomorphic to (Z_M, suc), M infinite, $M \subseteq \mathbb{N}_0$. We will not distinguish (B, f) from

(Z_M, suc). We show that the retract $\mathbb{Z}$ does not have a complement in $\mathbf{R}^{\emptyset}(A, f)$. Let R' be a retract with $R' \cup \mathbb{Z} = A$. Then all elements $(1, j)$, $j \in M$ belong to R' and since R' is closed with respect to f we obtain that $\mathbb{Z} \subseteq R'$. □

Theorem 5.3. *Let (A, f) be a connected monounary algebra, $(A, f) \notin \mathcal{C} \cup \mathcal{Z}$. The system $\mathbf{R}^{\emptyset}(A, f)$ forms a complemented lattice if and only if there is no element $x \in A$ such that $\big(x, f(x)\big)$ forms a gap. In this case $\mathbf{R}^{\emptyset}(A, f)$ is a boolean lattice.*

P r o o f. According to Lemma 5.1 we obtain that if $\mathbf{R}^{\emptyset}(A, f)$ is complemented then (A, f) contains no gap.

Conversely suppose that (A, f) contains no gap. Since (A, f) is connected, the existence of an element of degree ∞ would imply that all elements in A have degree equal to ∞.

First suppose that there is no element $x \in A$ with $s_f(x) = \infty$. Applying Theorem 2.1 we obtain that each retract of (A, f) is equal to a subalgebra generated by a set of elements of the zero degree. Let $[M]$ denote the subalgebra generated by a subset $M \subseteq A^0$. We put $[\emptyset] = \emptyset$.

We show that each subalgebra $[M]$, $M \subseteq A^0$ is a retract of (A, f). Let $y \in f^{-1}([M])$ be an element such that $y \notin [M]$. There exists $x \in M$ such that $f(y) = f^k(x)$, $k \in \mathbb{N}$. The algebra (A, f) contains no gap, thus $z = f^{k-1}(x)$ satisfies $f(z) = f(y)$ and $s_f(y) = s_f(z)$. Therefore this implies that $R \in R^{\emptyset}(A, f)$ if and only if $R = [M]$, $M \subseteq A^0$.

Hence $\mathbf{R}^{\emptyset}(A, f) \cong \big(\mathcal{P}(A^0), \subseteq\big)$ and it forms a boolean lattice.

Further suppose that $s_f(x) = \infty$ holds for all $x \in A$. Since $(A, f) \notin \mathcal{C}$ we obtain that $A^{\mathbb{Z}} \neq \emptyset$ and we note that each subset $X \in A^{\mathbb{Z}}$ determines a minimal retract isomorphic to $(\mathbb{Z}, \mathrm{suc})$. We define $\varphi\colon \mathcal{P}(A^{\mathbb{Z}}) \to R^{\emptyset}(A, f)$ by putting $\varphi(\emptyset) = \emptyset$ and $\varphi(M) = \bigcup\limits_{X \in M} X$ for all $M \subseteq A^{\mathbb{Z}}$, $M \neq \emptyset$. Every retract is a union of all subalgebras isomorphic to $(\mathbb{Z}, \mathrm{suc})$ which it contains, thus the mapping φ is surjective. Suppose that $M_1, M_2 \in \mathcal{P}(A^{\mathbb{Z}})$, $M_1 \subseteq M_2$. Then obviously $\varphi(M_1) \subseteq \varphi(M_2)$. If $M_1 \nsubseteq M_2$ then there is $X \in M_1$ such that $X \notin M_2$. Since $(A, f) \notin \mathcal{Z}$, for all $x \in X$ a set $\{y; y \in X, (\exists n \in \mathbb{N}) f^n(y) = x, \left|f^{-1}(y)\right| \geq 2\}$ i.e., the set of all $y \in X$ below x with $\left|f^{-1}(y)\right| \geq 2$ is finite. Hence there exists $x' \in X$ such that $|f^{-n}(x')| = 1$, thus x' does not belong to any other $Y \in S$, $Y \neq X$ and we obtain that $x' \in \bigcup\limits_{Z \in M_1} Z = \varphi(M_1)$ and $x' \notin \bigcup\limits_{Z \in M_2} Z = \varphi(M_2)$.

We have $M_1 \subseteq M_2$ if and only if $\varphi(M_1) \subseteq \varphi(M_2)$, hence the mapping φ is a lattice isomorphism and $\mathbf{R}^{\emptyset}(A, f) \cong \big(\mathcal{P}(A^{\mathbb{Z}}), \subseteq\big)$. Therefore $\mathbf{R}^{\emptyset}(A, f)$ forms a boolean lattice. □

Further we will deal with algebras having a least retract.

Lemma 5.4. *Let (A, f) be a connected monounary algebra with the least retract O. If the system $\mathbf{R}(A, f)$ forms a complemented lattice, then the only gaps are $\big(x, f(x)\big)$ where $f(x) \in O$.*

Proof. Suppose that there is an element $x \in A$, such that $\big(x, f(x)\big)$ forms a gap and $f(x) \notin O$. Since $(x, f(x))$ forms a gap, there exists $y \in f^{-1}(x)$ with $s_f(x) < s_f(y)$. A set $R = A \smallsetminus \bigcup\limits_{n \in \mathbb{N}_0} f^{-n}(x)$ is a retract. Let R' be a retract with $R' \cup R = A$. The set $R' \cap R = R' \cap \left(A \smallsetminus \bigcup\limits_{n \in \mathbb{N}_0} f^{-n}(x)\right) = R' \smallsetminus \bigcup\limits_{n \in \mathbb{N}_0} f^{-n}(x)$ is a retract and $R \wedge R' = R \cap R' \neq O$, since $f(x) \in R \cap R'$ and $f(x) \notin O$. Hence $\mathbf{R}(A, f)$ is not complemented. $\square$

Denote by $\mathcal{Z}_C$ the class of all connected monounary algebras (A, f) such that there exists a subalgebra of (A, f) isomorphic to $(Z_M^{(k)}, \mathrm{suc})$, $k \in \mathbb{N}$, $M \subseteq \mathbb{N}$, M being infinite.

The following lemma can be proved similarly as Lemma 5.2.

Lemma 5.5. *Let (A, f) be a connected monounary algebra such that $(A, f) \in \mathcal{Z}_C$. Then $\mathbf{R}(A, f)$ is not complemented.*

Theorem 5.6. *Let (A, f) be a connected monounary algebra with the least retract O. Suppose that $(A, f) \notin \mathcal{Z}_C$. The system $\mathbf{R}(A, f)$ forms a complemented lattice if and only if the only gaps are $\big(x, f(x)\big)$, where $f(x) \in O$. In this case $\mathbf{R}(A, f)$ is a boolean lattice.*

Proof. The necessary condition is due to Lemma 5.4.

Suppose that $x \in f^{-1}(O) \smallsetminus O$. According to the assumptions, the algebra $(A^x, \tilde{f})$ defined in Lemma 2.4 contains no gap. Since $(A, f) \notin \mathcal{Z}_C$ we also obtain that $(A^x, \tilde{f}) \notin \mathcal{Z}$. Due to Theorem 5.3, $\mathbf{R}^{\emptyset}(A^x, \tilde{f})$ forms a boolean lattice, for all $x \in f^{-1}(O) \smallsetminus O$. Since $\mathbf{R}(A, f) \cong \langle \mathbf{O} \rangle$, applying Lemma 2.4 we obtain that $\mathbf{R}(A, f)$ is a direct product of boolean lattices. $\square$

Theorem 5.7. *Let (A, f) be a monounary algebra with the least retract O. The system $\mathbf{R}(A, f)$ forms a complemented lattice if and only if*

(i) *every component B having a nonempty intersection with the least retract O satisfies $(B, f) \notin \mathcal{Z}_C$ and B contains only gaps $\big(x, f(x)\big)$, where $f(x) \in O$,*

(ii) *every component B having an empty intersection with the least retract O satisfies $(B, f) \notin \mathcal{C} \cup \mathcal{Z}$ and B contains no gap.*

Proof. Suppose that $A = \bigcup\limits_{i \in I} A_i$ where A_i, $i \in I$ are connected components of the algebra (A, f). We have $\mathbf{R}(A, f) = \langle \mathbf{O} \rangle \cong \prod\limits_{i \in I_0} \langle \mathbf{O_i} \rangle \times \prod\limits_{i \in I \smallsetminus I_0} \mathbf{R}^{\emptyset}(A_i, f)$, where

$I_0 = \{i \in I : O \cap A_i \neq \emptyset\}$ and $O_i = O \cap A_i$. Since the product of lattices is complemented if and only if every factor is complemented the assertion follows from Theorem 5.3 and Theorem 5.6. □

Corollary 5.8. *Let (A, f) be a monounary algebra. The system $\mathbf{R}(A, f)$ forms a boolean lattice if and only if $\mathbf{R}(A, f)$ is complemented.*

We remark that if in 5.8 the system $\mathbf{R}(A, f)$ is replaced by $\mathbf{R}^{\emptyset}(A, f)$, then the assertion need not be valid in general (for the non-connected case).

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* *Institute of Mathematics*
P. J. Šafárik University
Jesenná 5
SK–041 54 Košice
SLOVAKIA
E-mail: danica.studenovska@upjs.sk

** *Mathematical Institute*
Slovak Academy of Sciences
Grešákova 6
SK-040 01 Košice
SLOVAKIA
E-mail: pocs@saske.sk