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$\bar{\lambda}$-STATISTICALLY CONVERGENT DOUBLE SEQUENCES IN PROBABILISTIC NORMED SPACES

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ABSTRACT. The purpose of this paper is to introduce and study the concepts of double $\bar{\lambda}$-statistically convergent and double $\bar{\lambda}$-statistically Cauchy sequences in probabilistic normed space.

1. Introduction and preliminaries

The idea of statistical convergence for sequences of real number was first introduced by Fast [5] and Steinhauss [18] independently in the same year 1951 and since then several generalizations and applications of this concept have been investigated by various authors, namely Šalát [14], Fridy [6], and many others. The concept of statistical convergence for double sequences was studied by Mursaleen and Edely [11] and further studied by Mursaleen and Mohiuddine [10]. The idea of λ-statistical convergence of single sequences $x = (x_k)$ of real numbers has been studied by Mursaleen [12]; and for double sequences of fuzzy numbers by Savas [15]. Quite recently, Mohiuddine and Lohani [9] introduced the concept of λ-statistical convergence of single sequences $x = (x_k)$ in intuitionistic fuzzy normed spaces.

An interesting and important generalization of the notion of metric space was introduced by Menger [8] under the name of statistical metric space, which is now called probabilistic metric space. The idea of Menger was to use distribution functions instead of nonnegative real numbers.

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In fact the probabilistic theory has become an area of active research for the last forty years. It has a wide range of applications in functional analysis [4]. An important family of probabilistic metric spaces are probabilistic normed spaces (briefly, PN-spaces). The notion of probabilistic normed spaces was introduced by Sherstnev [17] in 1963 and later on studied by various authors, see [2, 3, 7].

In [1], Alotaibi studied the notion of λ-statistical convergence for single sequences in probabilistic normed spaces. In this paper, we study the concepts of double $\bar{\lambda}$-statistically convergence and double $\bar{\lambda}$-statistically Cauchy for sequences in probabilistic normed spaces.

Let K be a subset of $\mathbb{N}$, the set of natural numbers. Then the *asymptotic density of* K denoted by $\delta(K)$ (see [5, 18]), is defined as

$$\delta(K) = \lim_n \frac{1}{n} |\{k \in K : \ k \leq n\}|,$$

where the vertical bars denote the cardinality of the enclosed set.

Definition 1.1. A number sequence $x = (x_k)$ is said to be *statistically convergent* to the number ℓ if for each $\varepsilon > 0$, the set

$$K(\varepsilon) = \big\{k \in \mathbb{N} : \ |x_k - \ell| > \varepsilon\big\}$$

has asymptotic density zero, i.e.

$$\lim_n \frac{1}{n} \big|\big\{k \in \mathbb{N} : \ k \leq n, \ \ |x_k - \ell| > \varepsilon\big\}\big| = 0.$$

In this case we write st-$\lim x = \ell$.

Notice that every convergent sequence is statistically convergent to the same limit, but its converse need not be true.

The following definitions were giving by Mursaleen [12].

Definition 1.2. Let $\lambda = (\lambda_n)$ be a non-decreasing sequences of positive real numbers tending to ∞ and such that

$$\lambda_{n+1} \leq \lambda_n + 1, \qquad \lambda_1 = 0.$$

Let K be a subset of $\mathbb{N}$, the set of natural numbers. The number

$$\delta_\lambda(K) = \lim_n \frac{1}{\lambda_n} |\{k \in K : \ \ n - \lambda_n + 1 \leq k \leq n\}|,$$

is said to be the λ-density of K.

If $\lambda_n = n$ for every n then every λ-density is reduced to asymptotic density.

Definition 1.3. A sequence $x = (x_k)$ is said to be λ-statistically convergent to l if for every $\varepsilon > 0$, the set $K(\varepsilon)$ has λ-density zero, i.e.,

$$\lim_n \frac{1}{\lambda_n} |K_n(\varepsilon)| = 0,$$

where

$$K_n(\varepsilon) = \{k \in I_n : |x_k - l| > \varepsilon\} \quad \text{and} \quad I_n = [n - \lambda_n + 1, n].$$

In this case we write $\mathrm{st}_\lambda\text{-}\lim x = l$.

Firstly, we recall the following concepts for which we refer the readers to [7, 8, 16] for more details.

Definition 1.4. A *triangular norm* (*t-norm*) is a continuous mapping $T\colon [0,1] \times [0,1] \to [0,1]$ such that $([0,1], T)$ is an abelian monoid with unit one and for all $a, b, c \in [0,1]$:

(i) $T(c,d) \geq T(a,b)$ if $c \geq a$ and $d \geq b$;

Definition 1.5. A function $f\colon \mathbb{R} \to \mathbb{R}_0^+$ is called a *distribution function* if it is non-decreasing and left-continuous with $\inf_{t\in\mathbb{R}} f(t) = 0$ and $\sup_{t\in\mathbb{R}} f(t) = 1$.

By D, we denote the set of all distribution functions.

Definition 1.6. Let X be a real linear space and $\nu\colon X \to D$. Then the *probabilistic norm* or *ν-norm* is a t-norm satisfying the following conditions:

(i) $\nu_x(0) = 0$;

(ii) $\nu_x(t) = 1$ for all $t > 0$ iff $x = 0$;

(iii) $\nu_{\alpha x}(t) = \nu_x\left(\frac{t}{|\alpha|}\right)$ for all $\alpha \in \mathbb{R} \setminus \{0\}$;

(iv) $\nu_{x+y}(s+t) \geq T(\nu_x(s), \nu_y(t))$ for all $x, y \in X$ and $s, t \in \mathbb{R}_0^+$.

where ν_x means $\nu(x)$ and $\nu_x(t)$ is the value of ν_x at $t \in \mathbb{R}$.

Remark 1.1. We can say that t-norm is a binary operation $*$ given by

$$T(a,b) = a * b.$$

Space $(X, \nu, *)$ is called a *probabilistic normed space* (*PN-space*), and by a PN-space X we mean the triplet $(X, \nu, *)$.

Definition 1.7. Let $(X, \nu, *)$ be an PN-space. Then, a sequence $x = (x_k)$ is said to be *convergent* to ℓ in probabilistic norm space X, that is, $x_k \xrightarrow{\nu} \ell$ if for every $\varepsilon > 0$ and $\theta \in (0,1)$, there is a positive integer k_0 such that $\nu_{x_k - \ell}(\varepsilon) > 1 - \theta$ whenever $k \geq k_0$. In this case we write $\nu\text{-}\lim x = \ell$.

Definition 1.8. Let $(X, \nu, *)$ be an PN-space. Then, $x = (x_k)$ is called *Cauchy sequence* in probabilistic norm space X, if for every $\varepsilon > 0$ and $\theta \in (0,1)$, there is a positive integer k_0 such that $\nu_{x_k - x_j}(\varepsilon) > 1 - \theta$ for all $j, k \geq k_0$.

Definition 1.9. Let $(X, \nu, *)$ be an PN-space. Then, $x = (x_k)$ is said to be *bounded* in probabilistic norm space X, if there is $r \in \mathbb{R}^+$ such that $\nu_x(r) > 1 - \theta$, $0 < \theta < 1$.

We denote by ℓ_∞^ν the space of bounded sequences in PN-space.

2. Double $\bar{\lambda}$-statistical convergence in PN-space

In this section we study the concept of $\bar{\lambda}$-statistically convergent sequences in probabilistic normed spaces. Before continuing with this paper we present the definition of density and related concepts which form the background of the present work.

By the convergence of a double sequence we mean the convergence on the Pringsheim sense that is, a double sequence $x = (x_{k,l})$ has *Pringsheim limit* L (denoted by P-$\lim x = L$) provided that given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$ ([13]).

We now recall the definition of density and related concepts which form the background of the present work.

Let $K \subseteq \mathbb{N}\times\mathbb{N}$ be a two-dimensional set of positive integers and let $K(m,n)$ be the numbers of (k,l) in K such that $k \leq m$ and $l \leq n$. Then the two-dimensional analogue of natural density can be defined as follows [11].

The *lower asymptotic density* of the set $K \subseteq \mathbb{N}\times\mathbb{N}$ is defined as

$$\underline{\delta_2}(K) = \liminf_{m,n} \frac{K(m,n)}{mn}.$$

In case the sequence $(K(m,n)/mn)$ has a limit in Pringsheim's sense then we say that K has a *double natural density* and is defined as

$$\lim_{m,n} \frac{K(m,n)}{mn} = \delta_2(K).$$

For example, let $K = \{(i^2, j^2) : \ i, j \in \mathbb{N}\}$. Then

$$\delta_2(K) = \lim_{m,n} \frac{K(m,n)}{mn} \leq \lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0,$$

i.e. the set K has double natural density zero, while the set $\{(i, 2j) : \ i, j \in \mathbb{N}\}$ has double natural density $1/2$.

Definition 2.1. A real double sequence $x = (x_{kl})$ is said to be double statistically convergent ([11]) to the number ℓ if for each $\varepsilon > 0$, the set

$$\{(k,l) : \ k,l \in \mathbb{N}, \ \ k \leq m, \ \ l \leq n, \ \ |x_{kl} - \ell| \geq \varepsilon\}$$

has double natural density zero.

We denote the set of all double statistically convergent sequences by st_2. In this case we write $\mathrm{st}_2\text{-}\lim_{k,l} x_{kl} = \ell$.

Definition 2.2. Let $(X, \nu, *)$ be a PN-space. We say that a double sequence $x = (x_{kl})$ is said to be double statistically convergent to ℓ in probabilistic norm space X (for short, $S^{(PN)}$-*convergent*), if for every $\varepsilon > 0$ and $t \in (0,1)$,

$$\delta\big(\{(k,l) : \ k,l \in \mathbb{N}, \ \ k \leq m, \ \ l \leq n, \ \ \nu_{x_{kl}-\ell}(\varepsilon) \leq 1 - t\}\big) = 0$$

or equivalently

$$\delta\big(\big\{(k,l):\ k,l\in\mathbb{N},\ \ k\le m\ \ l\le n,\ \ \nu_{x_{kl}-\ell}(\varepsilon)>1-t\big\}\big)=1.$$

In this case we write $x_{kl}\xrightarrow{\nu}\ell$ or $S^{(PN)}\text{-}\lim x=\ell$, and denote the set of all S-convergent double sequences in probabilistic normed spaces by $(S)_\nu$.

In this paper, we introduce the concept of double $\bar{\lambda}$-statistical convergence of sequences in PN-Spaces.

The idea of λ-statistical convergence of single sequences in PN-spaces was studied by Alotaibi [1].

First we define the concept of $\bar{\lambda}$-density:

Let $\lambda=(\lambda_n)$ and $\mu=(\mu_m)$ be two non-decreasing sequences of positive real numbers each tending to ∞ and such that

$$\lambda_{n+1}\le\lambda_n+1,\qquad \lambda_1=0$$

and

$$\mu_{m+1}\le\mu_m+1,\qquad \mu_1=0.$$

Let $K\subseteq\mathbb{N}\times\mathbb{N}$. The number

$$\delta_{\bar{\lambda}}(K)=P\text{-}\lim_{n,m}\frac{1}{\bar{\lambda}_{nm}}\big|\big\{k\in I_n,\ \ l\in J_m:\ (k,l)\in K\big\}\big|$$

is said to be the $\bar{\lambda}$-density of K, provided the limit exists, where $\bar{\lambda}_{nm}=\lambda_n\mu_m$

We now ready to define the double $\bar{\lambda}$-statistical convergence.

Definition 2.3. A double sequence $x=(x_{kl})$ is said to be double $\bar{\lambda}$- statistically convergent or $S_{\bar{\lambda}}$-convergent to ℓ if for every $\varepsilon>0$,

$$P\text{-}\lim_{n,m}\frac{1}{\bar{\lambda}_{nm}}\big|\big\{k\in I_n,\ l\in J_m:\ |x_{kl}-\ell|\ge\varepsilon\big\}\big|=0.$$

In this case we write $S_{\bar{\lambda}}\text{-}\lim x=\ell$ or $x_{jk}\xrightarrow{P}\ell(S_{\bar{\lambda}})$ and we denote the set of all double $S_{\bar{\lambda}}$-statistically convergent sequences by $(S_{\bar{\lambda}})$.

If $\bar{\lambda}_{nm}=nm$, for all n, m, then the set of $S_{\bar{\lambda}}$-convergent sequences reduces to the space st^2.

Now we define the $S_{\bar{\lambda}}$-convergence in PN-space.

Definition 2.4. Let $(X,\nu,*)$ be a PN-space. We say that a double sequence $x=(x_{kl})$ is said to be $S_{\bar{\lambda}}$-*convergent* to ℓ in probabilistic norm space X (for short, $S_{\bar{\lambda}}^{(PN)}$-*convergent*), if for every $\varepsilon>0$ and $t\in(0,1)$,

$$\delta_{\bar{\lambda}}\big(\big\{k\in I_n,\ l\in J_m:\ \nu_{x_{kl}-\ell}(\varepsilon)\le1-t\big\}\big)=0$$

or equivalently

$$\delta_{\bar{\lambda}}\big(\big\{k\in I_n,\ l\in J_m:\ \nu_{x_{kl}-\ell}(\varepsilon)>1-t\big\}\big)=1.$$

In this case we write $x_{kl} \xrightarrow{\nu} \ell(S_{\bar{\lambda}})$ or $S_{\bar{\lambda}}^{(PN)}\text{-}\lim x = \ell$, and denote the set of all $S_{\bar{\lambda}}$-convergent double sequences in probabilistic normed spaces by $(S_{\bar{\lambda}})_{\nu}$.

Theorem 2.1. *Let $(X, \nu, *)$ be a PN-space. If a sequence $x = (x_{kl})$ is a double $\bar{\lambda}$-statistically convergent in probabilistic normed space X, then $S_{\bar{\lambda}}^{(PN)}$-limit is unique.*

Proof. Suppose that $S_{\bar{\lambda}}^{(PN)}\text{-}\lim x = \ell_1$ and $S_{\bar{\lambda}}^{(PN)}\text{-}\lim x = \ell_2$. Let $\varepsilon > 0$ and $t > 0$. Choose $s \in (0,1)$ such that $(1-s) * (1-s) \geq 1-t$. Then we define the following sets as

$$K_1(s,\varepsilon) = \big\{k \in I_n,\ l \in J_m :\ \nu_{x_{kl}-\ell_1}(\varepsilon/2) \leq 1-s\big\},$$
$$K_2(s,\varepsilon) = \big\{k \in I_n,\ l \in J_m :\ \nu_{x_{kl}-\ell_2}(\varepsilon/2) \leq 1-s\big\}.$$

So that we have $\delta_{\bar{\lambda}}(K_1(s,\varepsilon)) = 0$ and $\delta_{\bar{\lambda}}(K_2(s,\varepsilon)) = 0$ for all $\varepsilon > 0$. Now let

$$K_3(s,\varepsilon) = K_1(s,\varepsilon) \cup K_2(s,\varepsilon).$$

It follows that $\delta_{\bar{\lambda}}(K_3(s,\varepsilon)) = 0$, which implies

$$\delta_{\bar{\lambda}}(\mathbb{N} \times \mathbb{N} \setminus K_3(s,\varepsilon)) = 1.$$

If $(k,l) \in \mathbb{N} \times \mathbb{N} \setminus K_3(s,\varepsilon)$, we have

$$\begin{aligned}
\nu_{\ell_1-\ell_2}(\varepsilon) &= \nu_{(\ell_1-x_{kl})+(x_{kl}-\ell_2)}(\varepsilon/2+\varepsilon/2) \\
&\geq \nu_{x_{kl}-\ell_1}(\varepsilon/2) * \nu_{x_{kl}-\ell_2}(\varepsilon/2) \\
&> (1-s) * (1-s) \geq 1-t.
\end{aligned}$$

Since $t > 0$ was arbitrary, we get $\nu_{\ell_1-\ell_2}(\varepsilon) = 1$ for all $\varepsilon > 0$, which gives $\ell_1 = \ell_2$.

Hence $S_{\bar{\lambda}}^{(PN)}$ limit is unique.

This completes the proof of the theorem. □

Theorem 2.2. *Let $(X, \nu, *)$ be a PN-space. If a sequence $x = (x_{kl})$ is a double statistically convergent to ℓ in probabilistic normed space X, then $S_{\bar{\lambda}}^{(PN)}\text{-}\lim x = \ell$ if*

$$P\text{-}\liminf_{nm} \frac{\bar{\lambda}_{nm}}{nm} > 0. \tag{2.1}$$

Proof. For given $\varepsilon > 0$ and $t \in (0,1)$,

$$\big\{k \leq n,\ l \leq m :\ \nu_{x_{kl}-\ell}(\varepsilon) \leq 1-t\big\} \supset \big\{k \in I_n,\ l \in J_m :\ \nu_{x_{kl}-\ell}(\varepsilon) \leq 1-t\big\}$$

Therefore,

$$\begin{aligned}
&\frac{1}{nm}\big\{k \le n,\ l \le m:\ \nu_{x_{kl}-\ell}(\varepsilon) \le 1-t\big\}\\
&\ge \frac{1}{nm}\big\{k \in I_n,\ l \in J_m:\ \nu_{x_{kl}-\ell}(\varepsilon) \le 1-t\big\}\\
&\ge \frac{\bar{\lambda}_{nm}}{nm}\frac{1}{\bar{\lambda}_{nm}}\big\{k \in I_n,\ l \in J_m:\ \nu_{x_{kl}-\ell_1}(\varepsilon) \le 1-t\big\}.
\end{aligned}$$

Taking the limit as $n, m \to \infty$ and using (2.1), we get $S_{\bar{\lambda}}^{(PN)}\text{-}\lim x = \ell$. ☐

Theorem 2.3. *Let $(X, \nu, *)$ be a PN-space. If $\nu\text{-}\lim x = \ell$ then $S_{\bar{\lambda}}^{(PN)}\text{-}\lim x = \ell$. But converse need not be true.*

Proof. Let $\nu\text{-}\lim x = \ell$. Then for every $t \in (0,1)$ and $\varepsilon > 0$, there is a couple $(k_0, l_0) \in \mathbb{N} \times \mathbb{N}$ such that $\nu_{x_{kl}-\ell}(\varepsilon) > 1-t$ for all $k \ge k_0$, $l \ge l_0$. Hence the set $\big\{k \in I_n,\ l \in J_m:\ \nu_{x_{kl}-\ell}(\varepsilon) \le 1-t\big\}$ has natural density zero and hence

$$\delta_{\bar{\lambda}}\Big(\big\{k \in I_n,\ l \in J_m:\ \nu_{x_{kl}-\ell}(\varepsilon) \le 1-t\big\}\Big) = 0,$$

that is, $S_{\bar{\lambda}}^{(PN)}\text{-}\lim x = \ell$.

For converse, we construct the following example:

Example 2.1. Let $(\mathbb{R}, |\cdot|)$ denote the space of real numbers with the usual norm. Let $a * b = ab$ and $\nu_x(\varepsilon) = \frac{\varepsilon}{\varepsilon + |x|}$, where $x \in X$ and $\varepsilon \ge 0$. In this case, we observe that $(\mathbb{R}, \nu, *)$ is a PN-space. Define a sequence $x = (x_{kl})$ by

$$x_{kl} = \begin{cases} (k,l), & \text{for } n - \big[\sqrt{\lambda_n}\,\big] + 1 \le k \le n \text{ and} \\ & \qquad m - \big[\sqrt{\mu_m}\,\big] + 1 \le l \le m \ \ (k, l \in \mathbb{N}); \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that, this sequence is $S_{\bar{\lambda}}$-convergence to zero in PN-space, i.e., $x_{kl} \xrightarrow{\nu} 0(S_{\bar{\lambda}})$, while $x_{kl} \stackrel{\nu}{\not\to} 0$.

This completes the proof of the theorem. ☐

Theorem 2.4. *Let $(X, \nu, *)$ be a PN-space. Then, $S_{\bar{\lambda}}^{(PN)}\text{-}\lim x = \ell$ if and only if there exists a subset $K = \big\{(k_n, l_n):\ k_1 < k_2 < \dots;\ \ l_1 < l_2 < \dots\big\} \subseteq \mathbb{N} \times \mathbb{N}$ such that $\delta_{\bar{\lambda}}(K) = 1$ and $\nu\text{-}\lim\limits_{n\to\infty} x_{k_n l_n} = \ell$.*

Proof.

Necessity. Suppose that $S_{\bar{\lambda}}^{(PN)}\text{-}\lim x = \ell$. Then, for any $\varepsilon > 0$ and $s \in \mathbb{N}$, let

$$K(s, \varepsilon) = \Big\{k \in I_n,\ l \in J_m:\ \nu_{x_{kl}-\ell}(\varepsilon) \le 1 - \frac{1}{s}\Big\},$$

and

$$M(s,\varepsilon)=\Big\{k\in I_n,\ l\in J_m:\ \nu_{x_{kl}-\ell}(\varepsilon)>1-\tfrac{1}{s}\Big\}.$$

Then $\delta_{\bar\lambda}(K(s,\varepsilon))=0$ and

$$M(1,\varepsilon)\supset M(2,\varepsilon)\supset\dots\supset M(i,\varepsilon)\supset M(i+1,\varepsilon)\supset\dots \tag{2.2}$$

and

$$\delta_{\bar\lambda}(M(s,\varepsilon))=1,\qquad s=1,2,\dots.$$

Now we have to show for $(k,l)\in M(s,\varepsilon)$, $x=(x_{kl})$ is ν-convergent to ℓ. Suppose that for some $(k,l)\in M(s,\varepsilon)$, the double sequence $x=(x_{kl})$ is not ν-convergent to ℓ. Therefore there is $t>0$ and are positive integers l_0,k_0 such that $\nu_{x_{kl}-\ell}(\varepsilon)\le 1-t$ for all $l\ge l_0$, $k\ge k_0$. Let $\nu_{x_{kl}-\ell}(\varepsilon)>1-t$ for all $k\le k_0$, $l\le l_0$. Then

$$\delta_{\bar\lambda}\big(\big\{k\in I_n,\ l\in J_m:\ \nu_{x_{kl}-\ell}(\varepsilon)>1-t\big\}\big)=0.$$

Since $t>\frac{1}{s}$, we have

$$\delta_{\bar\lambda}(M(s,\varepsilon))=0$$

which contradicts (2.2). Therefore $x=(x_{kl})$ is ν-convergent to ℓ.

Conversely, suppose that there exists a subset $K=\big\{(k_n,l_n):\ k_1<k_2<\dots;\ l_1<l_2<\dots\big\}\subseteq\mathbb{N}\times\mathbb{N}$ such that $\delta_{\bar\lambda}(K)=1$ and $\nu\text{-}\lim\limits_{(k,l)\in K}x_{kl}=\ell$, there exists $N\in\mathbb{N}$ such that for every $t\in(0,1)$ and $\varepsilon>0$

$$\nu_{x_{kl}-\ell}(\varepsilon)>1-t,\qquad\text{for all}\quad k,l\ge N.$$

Now

$$\begin{aligned}M(t,\varepsilon)&=\big\{k\in I_n,\ l\in J_m:\ \nu_{x_{kl}-\ell}(\varepsilon)\le 1-t\big\}\\&\subseteq\mathbb{N}\times\mathbb{N}-\big\{(k_{N+1},l_{N+1}),(k_{N+2},l_{N+2}),\dots\big\}.\end{aligned}$$

Therefore $\delta_{\bar\lambda}(M(t,\varepsilon))\le 1-1=0$. Hence $S_{\bar\lambda}^{(PN)}\text{-}\lim x=\ell$.

This completes the proof of the theorem. □

In the next we now define double λ-statistically Cauchy sequence in probabilistic normed space.

Definition 2.5. Let $(X,\nu,*)$ be a PN-space. Then, a double sequence $x=(x_{kl})$ is said to be $S_{\bar\lambda}$-*Cauchy* in PN-space X if for every $\varepsilon>0$ and $t\in(0,1)$, there exist $N=N(\varepsilon)$ and $M=M(\varepsilon)$ such that

$$\delta_{\bar\lambda}\big(\big\{k\in I_n,\ l\in J_m:\ \nu_{x_{kl}-x_{MN}}(\varepsilon)\le 1-t\big\}\big)=0.$$

Theorem 2.5. *Let $(X,\nu,*)$ be a PN-space and θ_{rz} be any double lacunary sequence. Then, a double sequence $x=(x_{kl})$ is $S_{\bar\lambda}^{(PN)}$-convergent if and only if it is $S_{\bar\lambda}^{(PN)}$-Cauchy in probabilistic norm space X.*

Proof. Let $x = (x_{kl})$ be $S_{\bar{\lambda}}^{(PN)}$-convergent to ℓ, i.e., $S_{\bar{\lambda}}^{(PN)}\text{-}\lim x = \ell$. Then for a given $\varepsilon > 0$ and $t \in (0,1)$, choose $r > 0$ such that $(1-t) * (1-t) > 1-r$. Then, we have

$$\delta_{\bar{\lambda}}(A(t,\varepsilon)) = \delta_{\bar{\lambda}}\big(\{k \in I_n,\ l \in J_m :\ \nu_{x_{kl}-\ell}(\varepsilon/2) \le 1-t\}\big) = 0 \tag{2.3}$$

which implies that

$$\delta_{\bar{\lambda}}(A^C(t,\varepsilon)) = \delta_{\bar{\lambda}}\big(\{k \in I_n,\ l \in J_m :\ \nu_{x_{kl}-\ell}(\varepsilon/2) > 1-t\}\big) = 1.$$

Let $(p,q) \in A^C(t,\varepsilon)$. Then $\nu_{x_{pq}-\ell}(\varepsilon) > 1-t$.

Now, let

$$B(t,\varepsilon) = \{k \in I_n,\ l \in J_m :\ \nu_{x_{kl}-x_{pq}}(\varepsilon) \le 1-r\}.$$

We need to show that $B(t,\varepsilon) \subset A(t,\varepsilon)$. Let $(k,l) \in B(t,\varepsilon) \setminus A(t,\varepsilon)$. Then we have

$$\nu_{x_{kl}-x_{pq}}(\varepsilon) \le 1-r \qquad \text{and} \qquad \nu_{x_{kl}-\ell}(\varepsilon/2) > 1-t,$$

in particular $\nu_{x_{pq}-\ell}(\varepsilon/2) > 1-t$. Then

$$1-r \ge \nu_{x_{kl}-x_{pq}}(\varepsilon) \ge \nu_{x_{kl}-\ell}(\varepsilon/2) * \nu_{x_{pq}-\ell}(\varepsilon/2) > (1-t)*(1-t) > 1-r,$$

which is not possible. Hence $B(t,\varepsilon) \subset A(t,\varepsilon)$. Therefore, by (2.3), $\delta_{\bar{\lambda}}(B(t,\varepsilon)) = 0$. Hence x is $S_{\bar{\lambda}}^{(PN)}$-Cauchy.

Conversely, let $x = (x_{kl})$ be $S_{\bar{\lambda}}^{(PN)}$-Cauchy but not $S_{\bar{\lambda}}^{(PN)}$-convergent. Now

$$\nu_{x_{kl}-x_{MN}}(\varepsilon) \ge \nu_{x_{kl}-\ell}(\varepsilon/2) * \nu_{x_{MN}-\ell}(\varepsilon/2) > (1-t)*(1-t) > 1-r,$$

since x is not $S_{\bar{\lambda}}^{(PN)}$-convergent. Therefore $\delta_{\bar{\lambda}}(B^C(t,\varepsilon)) = 0$, where

$$B(t,\varepsilon) = \{(k,l) \in \mathbb{N}\times\mathbb{N} :\ \nu_{x_{kl}-x_{MN}}(\varepsilon) \le 1-r\}$$

and so $\delta_{\bar{\lambda}}(B(t,\varepsilon)) = 1$, which is contradiction, since x was $S_{\bar{\lambda}}^{(PN)}$-Cauchy. Hence x must be $S_{\bar{\lambda}}^{(PN)}$-convergent.

This completes the proof of the theorem. □

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