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ON GENERALIZED SZÁSZ-MIRAKYAN OPERATORS OF FUNCTIONS OF TWO VARIABLES

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ABSTRACT. We introduce certain generalized Szász-Mirakyan operators in exponential weight spaces of functions of two variables and we give approximation theorems for them.

1. Introduction

1.1. The Szász-Mirakyan operators

$$S_n(f;x) := \mathrm{e}^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \qquad x \in \mathbb{R}_0, \quad n \in \mathbb{N}, \tag{1.1}$$

($\mathbb{R}_0 = [0,\infty)$, $\mathbb{N} = \{1,2,\dots\}$) and their modifications for functions f from various spaces vere examined in many papers and monographs (e.g. [1]–[8], [10], [12]–[20]). M. Becker and co-authors investigated in the paper [3] the above operators S_n in exponential weight space C_p with $p = \text{const.} > 0$ and the weight function

$$v_p(x) := \mathrm{e}^{-px}, \qquad x \in \mathbb{R}_0. \tag{1.2}$$

The space C_p ([3]) is the set of all functions $f\colon \mathbb{R}_0 \to \mathbb{R}$ for which $v_p f$ is uniformly continuous and bounded on $\mathbb{R}_0$ and the norm $\|f\|_p := \sup_{x\in\mathbb{R}_0} v_p(x)|f(x)|$.

In the paper [3] was proved that S_n defined by (1.1) is an operator from the space C_p to C_q provided that $q > p > 0$ and $n > n_0 > p/\ln(q/p)$. Moreover, in

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[3] were proved fundamental approximation theorems for these operators S_n in the space C_p.

1.2. In the paper [13] we introduced the generalized Szász-Mirakyan operators for $f \in C_p$:

$$S_{n;p}^{[r]}(f;x) := \frac{1}{A_r(nx)} \sum_{k=0}^{\infty} \frac{(nx)^{rk}}{(rk)!} f\left(\frac{rk}{n+p}\right), \qquad x \in \mathbb{R}_0, \quad n \in \mathbb{N}, \tag{1.3}$$

with a fixed $r \in \mathbb{N}$ and

$$A_r(t) := \sum_{k=0}^{\infty} \frac{t^{rk}}{(rk)!} \qquad \text{for} \quad t \in \mathbb{R}_0. \tag{1.4}$$

If $r = 1$, then $A_1(t) = \mathrm{e}^t$ and

$$S_{n;p}^{[1]}(f;x) = \mathrm{e}^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^{k}}{k!} f\left(\frac{k}{n+p}\right), \qquad x \in \mathbb{R}_0, \quad n \in \mathbb{N}, \tag{1.5}$$

The approxination properties of $S_{n;p}^{[1]}$ were examined in the paper [15].

In [13] we showed that, for a fixed $p > 0$ and every $n, r \in \mathbb{N}$, $S_{n;p}^{[r]}$ is a positive linear operator from the space C_p to the same C_p. Moreover, we proved some approximation theorems on $S_{n;p}^{[r]}$.

1.3. We mention that the formula (1.3) for $S_{n;p}^{[r]}$ is connected with the Borel method B_r of summability of sequences. As is known ([9], [11]) a sequence $(a_n)_0^{\infty}$ of real numbers is summable to g by the Borel method B_r, $r \in \mathbb{N}$, if the series $\sum\limits_{k=0}^{\infty} \frac{t^{rk}}{(rk)!} a_k$ is convergent on $\mathbb{R}_0$ and if

$$\lim_{t \to \infty} r\mathrm{e}^{-t} \sum_{k=0}^{\infty} \frac{t^{rk}}{(rk)!} a_k = g.$$

In [13] we proved that the function $u = A_r(t)$, $t \in \mathbb{R}_0$, $2 \leq r \in \mathbb{N}$, is the solution of the differential equation

$$u^{(r-1)} + u^{(r-2)} + \cdots + u' + u = \mathrm{e}^t,$$

satisfying the conditions

$$u(0) = 1, \qquad u^{(k)}(0) = 0 \quad \text{for} \;\; k = 1, 2, \ldots, r-2.$$

From (1.4) is obvious that $A_1(t) = \mathrm{e}^t$ and $A_2(t) = \cosh t = \frac{1}{2}(\mathrm{e}^t + \mathrm{e}^{-t})$.

In [13] was proved that if $r = 2m$, $2 \leq m \in \mathbb{N}$, then

$$A_{2m}(t) = \frac{1}{m}\left[\cosh t + \sum_{k=1}^{m-1} \exp\left(t \cos \frac{k\pi}{m}\right) \cos\left(t \sin \frac{k\pi}{m}\right)\right],$$

and if $m \in \mathbb{N}$, then

$$A_{2m+1}(t) = \frac{1}{2m+1}\left[\mathrm{e}^t + 2\sum_{k=1}^{m} \exp\left(t\cos\frac{2k\pi}{2m+1}\right)\cos\left(t\sin\frac{2k\pi}{2m+1}\right)\right],$$

for $t \in \mathbb{R}_0$.

From the above formulas we get

$$\lim_{t\to\infty} \mathrm{e}^{-t} A_r(t) = \frac{1}{r} \qquad \text{for every} \quad r \in \mathbb{N},$$

which implies that for every fixed $r \in \mathbb{N}$ there exists the positive constant $M_0(r)$ depending only on r such that

$$1 \le \frac{\mathrm{e}^t}{A_r(t)} \le M_0(r) \qquad \text{for} \quad t \in \mathbb{R}_0. \tag{1.6}$$

1.4. The Szász-Mirakyan operators

$$S_{m,n}(f;x) := \mathrm{e}^{-mx-ny} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(mx)^j (ny)^k}{j!k!} f\left(\frac{j}{m}, \frac{k}{n}\right), \tag{1.7}$$

$(x,y) \in \mathbb{R}_0^2 = \mathbb{R}_0 \times \mathbb{R}_0$ and $m,n \in \mathbb{N}$, for the functions $f\colon \mathbb{R}_0^2 \to \mathbb{R}$ (and certain modifications of $S_{m,n}$) were examined also in many papers (e.g. [14], [18]).

1.5. In this paper we shall investigate the generalized Szász-Mirakyan operators $S_{m,n;p,q}^{[r,s]}$ on exponential weight spaces of functions of two variables. The definition of these operators and some auxiliary results will be given in Section 2. The approximation theorems will be proved in Section 3.

We shall denote by $M_i(a,b)$, $i \in \mathbb{N}$, suitable positive constants depending only on indicated parameters a, b.

2. Definitions and auxiliary results

2.1. The exponential weight space of functions of two variables will be denoted by $C_{p,q}$.

Let $p,q > 0$ be fixed numbers, v_p be defined by (1.2) and let

$$v_{p,q}(x,y) := v_p(x)v_q(y) = \mathrm{e}^{-px-qy} \qquad \text{for} \quad (x,y) \in \mathbb{R}_0^2. \tag{2.1}$$

The space $C_{p,q}$ is the set of all functions $f\colon \mathbb{R}_0^2 \to \mathbb{R}$ such that $v_{p,q}f$ is uniformly continuous and bounded on $\mathbb{R}_0^2$ and the norm

$$\|f\|_{p,q} \equiv \|f(\cdot,\cdot)\|_{p,q} := \sup_{(x,y)\in\mathbb{R}_0^2} v_{p,q}(x,y)|f(x,y)|. \tag{2.2}$$

Moreover, let $C^m_{p,q}$, with $p,q>0$ and $m\in\mathbb{N}$, be the class of all m-times differentiable functions $f\in C_{p,q}$ which the partial derivatives $f^{(k)}_{x^{k-i}y^i}$, $1\le k\le m$, $0\le i\le k$, belong to $C_{p,q}$ also.

Let as usual $\omega(f;C_{p,q})$ be modulus of continuity of $f\in C_{p,q}$, i.e.

$$\omega(f;C_{p,q};t,s):=\sup_{0\le h\le t,0\le\delta\le s}\|\Delta_{h,\delta}f(\cdot,\cdot)\|_{p,q} \tag{2.3}$$

for $t,s\ge 0$, where $\Delta_{h,\delta}f(x,y)=f(x+h,y+\delta)-f(x,y)$ ([5], [6], [17]).

For every $f\in C_{p,q}$ we have

$$\omega(f;C_{p,q};t_1,s_1)\le\omega(f;C_{p,q};t_2,s_1)\le\omega(f;C_{p,q};t_2,s_2) \tag{2.4}$$

for $0\le t_1<t_2$ and $0\le s_1<s_2$, and

$$\lim_{t,s\to 0+}\omega(f;C_{p,q};t,s)=0. \tag{2.5}$$

2.2. Let now $r,s\in\mathbb{N}$ be fixed and let A_r be given by (1.4). For functions $f\in C_{p,q}$, $p,q>0$, we define the following generalized Szász-Mirakyan operators

$$S^{[r,s]}_{m,n;p,q}(f;x,y):=\frac{1}{A_r(mx)A_s(ny)}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(mx)^{rj}(ny)^{sk}}{(rj)!(sk)!}f\left(\frac{rj}{m+p},\frac{sk}{n+q}\right), \tag{2.6}$$

$(x,y)\in\mathbb{R}^2_0$ and $m,n\in\mathbb{N}$. In particular, for $f_0(x,y)\equiv 1$ we have

$$S^{[r,s]}_{m,n;p,q}(f_0;x,y)=1 \qquad \text{for} \quad (x,y)\in\mathbb{R}^2_0 \ \text{ and } \ m,n\in\mathbb{N}. \tag{2.7}$$

If $r=s=1$, then by (2.6) and (1.4),

$$S^{[1,1]}_{m,n;p,q}(f;x,y):=\mathrm{e}^{-mx-ny}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(mx)^{j}(ny)^{k}}{j!k!}f\left(\frac{j}{m+p},\frac{k}{n+q}\right), \tag{2.8}$$

and

$$S^{[r,s]}_{m,n;p,q}(f;0,0)=f(0,0), \tag{2.9}$$

for every $f\in C_{p,q}$ and $m,n\in\mathbb{N}$. Moreover, if $f(x,y)=f_1(x)f_2(y)$ and $f_1\in C_p$ and $f_2\in C_q$, then by (2.6) and (1.3) follows

$$S^{[r,s]}_{m,n;p,q}(f;x,y)=S^{[r]}_{m;p}(f_1;x)S^{[s]}_{n;q}(f_2;y) \qquad \text{for} \quad (x,y)\in\mathbb{R}^2_0, \quad m,n\in\mathbb{N}. \tag{2.10}$$

We shall show that for fixed $p,q>0$, $r,s\in\mathbb{N}$ and $m,n\in\mathbb{N}$ the above $S^{[r,s]}_{m,n;p,q}$ is a positive linear operator from the space $C_{p,q}$ to $C_{p,q}$.

We shall omit r and s in the symbols $S^{[r]}_{m;p}$, $S^{[s]}_{n;q}$ and $S^{[r,s]}_{m,n;p,q}$ if $r=1$ and $s=1$.

2.3. Here we shall give some results obtained in the papers [13] and [15] for the operators $S^{[r]}_{n;p}$. For a fixed $x\in\mathbb{R}_0$ we denote

$$\varphi_x(t):=t-x \qquad \text{for} \quad t\in\mathbb{R}_0. \tag{2.11}$$

Lemma 1. ([15]) *Let $p > 0$ be a fixed. Then*

$$S_{m;p}\left(\varphi_x^2(t);x\right) = (p^2x^2 + mx)/(m+p)^2 \le (p+1)(x^2+x)/(m+p), \tag{2.12}$$

$$\begin{aligned} S_{m;p}\left(\varphi_x^4(t);x\right) &= (p^4x^4 + 6p^2mx^3 + (3m-4p)mx^2 + mx)/(m+p)^4 \\ &\le (p^2x^4 + 6px^3 + 3x^2 + x)/(m+p)^2, \end{aligned} \tag{2.13}$$

and

$$S_{m;p}(1/v_p(t);x) = \exp\left[mx\left(\mathrm{e}^{p/(m+p)} - 1\right)\right] \le \mathrm{e}^{px},$$

for $x \in \mathbb{R}_0$ and $m \in \mathbb{N}$, which implies that

$$\|S_{m;p}(1/v_p)\|_p \le 1 \qquad \textit{for} \quad m \in \mathbb{N}. \tag{2.14}$$

We observe that for every fixed $p > 0$ there holds also

$$\|S_{m;p}(1/v_{2p})\|_{2p} \le 1 \qquad \text{for} \quad m \in \mathbb{N}. \tag{2.15}$$

Indeed, by (1.2) and (1.5),

$$\begin{aligned} S_{m;p}(1/v_{2p};x) &= \mathrm{e}^{-mx}\sum_{k=0}^{\infty}\frac{(mx)^k}{k!}\mathrm{e}^{2pk/(m+p)} \\ &= \exp\left[mx\left(\mathrm{e}^{2p/(m+p)} - 1\right)\right], \qquad x \in \mathbb{R}_0, \quad m \in \mathbb{N}, \end{aligned}$$

and, by $k! \ge 2^{k-1}$ for $k \in \mathbb{N}$,

$$\begin{aligned} \mathrm{e}^{2p/(m+p)} - 1 &= \sum_{k=1}^{\infty}\frac{[2p/(m+p)]^k}{k!} \le 2\sum_{k=1}^{\infty}\left(\frac{p}{m+p}\right)^k \\ &\le \frac{2p/(m+p)}{1 - p/(m+p)} \\ &= \frac{2p}{m} \qquad \text{for} \quad m \in \mathbb{N}. \end{aligned}$$

Consequently,

$$\|S_{m;p}(1/v_{2p})\|_{2p} = \sup_{x\in\mathbb{R}_0} \mathrm{e}^{-2px}|S_{m;p}(1/v_{2p};x) \le 1 \qquad \text{for} \quad m \in \mathbb{N}.$$

Lemma 2. ([13]) *Let $e_k(x) = x^k$ for $x \in \mathbb{R}_0$ and $k = 0,1,2$, and let $p > 0$ and $r \in \mathbb{N}$ be fixed. Then*

$$S_{m;p}^{[r]}(e_0;x) = 1, \tag{2.16}$$

$$S_{m;p}(e_1;x) = \frac{mx}{m+p}\frac{A_r'(mx)}{A_r(mx)},$$

and

$$S_{m;p}^{[r]}(e_2;x) = \frac{m^2x^2}{(m+p)^2}\frac{A_r''(mx)}{A_r(mx)} + \frac{x}{(m+p)^2}\frac{A_r'(mx)}{A_r(mx)} \tag{2.17}$$

for $x \in \mathbb{R}_0$ and $m \in \mathbb{N}$.

Lemma 3. ([13]) *For fixed $p > 0$ and $r \in \mathbb{N}$ there holds*

$$\lim_{m\to\infty} mS^{[r]}_{m;p}(\varphi_x(t);x) = -px$$

and

$$\lim_{m\to\infty} mS^{[r]}_{m;p}(\varphi^2_x(t);x) = x,$$

at every $x \in \mathbb{R}_0$.

From the formulas (1.3)–(1.5) and the inequalities (1.6), (2.14) and (2.15) we immediately derive the following

Lemma 4. *Let $p > 0$ and $r \in \mathbb{N}$ be fixed. Then, for every non-negative function $f \in C_p$, there holds*

$$S^{[r]}_{m;p}(f;x) \le M_0(r)S_{m;p}(f;x) \qquad \text{for} \quad x \in \mathbb{R}_0, \quad m \in \mathbb{N}_0,$$

where $M_0(r) = \mathrm{const.} > 0$ is given by (1.6).

In particular we have

$$S^{[r]}_{m;p}\left(\varphi^{2i}_x(t);x\right) \le M_0(r)S_{m;p}\left(\varphi^{2i}_x(t);x\right), \qquad i \in \mathbb{N}, \tag{2.18}$$

$$v_p(x)S^{[r]}_{m;p}(1/v_p(t);x) \le M_0(r) \tag{2.19}$$

and

$$v_{2p}(x)S^{[r]}_{m;p}(1/v_{2p}(t):x) \le M_0(r),$$

for $x \in \mathbb{R}_0$ and $m \in \mathbb{N}$.

Applying (2.1), (2.2), (2.6), (2.10) and (2.19), we can easily obtain the basic lemma.

Lemma 5. *For fixed $p, q > 0$ and $r, s \in \mathbb{N}$ there is $M_1(r,s) = M_0(r)M_0(s)$ ($M_0(\cdot)$ is given by (1.6)) such that for every $f \in C_{p,q}$,*

$$\|S^{[r,s]}_{m,n;p,q}(f)\|_{p,q} \le M_1(r,s)\|f\|_{p,q} \qquad \text{for} \quad m,n \in \mathbb{N}. \tag{2.20}$$

The formulas (2.6) and (1.4) and the inequality (2.20) show that $S^{[r,s]}_{m,n;p,q}$ is a positive linear operator acting from the space $C_{p,q}$ to the same $C_{p,q}$.

3. Theorems

3.1. First we shall prove two theorems on the degree of approximation of functions $f \in C_{p,q}$ by $S^{[r,s]}_{m,n;p,q}(f)$.

Denote by

$$R_{m,n}(f;x,y) := S^{[r,s]}_{m,n;p,q}(f;x,y) - f(x,y). \tag{3.1}$$

Theorem 1. *For fixed $p,q>0$ and $r,s\in\mathbb{N}$ there exists $M_2=M_2(p,q,r,s)$ $=\text{const.}>0$ such that if $f\in C^1_{p,q}$, then*

$$v_{p,q}(x,y)\,|R_{m;n}(f;x,y)|\le M_2\left\{\|f'_x\|_{p,q}\sqrt{\frac{x^2+x}{m+p}}+\|f'_y\|_{p,q}\sqrt{\frac{y^2+y}{n+q}}\right\},\quad(3.2)$$

for $(x,y)\in\mathbb{R}^2_0$ and $m,n\in\mathbb{N}$.

P r o o f. Choose $f\in C^1_{p,q}$ and $(x,y)\in\mathbb{R}^2_0$. Then by (3.1), (2.6)–(2.8) and (1.6) we have

$$\begin{aligned}|R_{m;n}(f;x,y)|&=\Big|S^{[r,s]}_{m,n;p,q}\Big(f(t,z)-f(x,y);x,y\Big)\Big|\\&\le S^{[r,s]}_{m,n;p,q}\Big(|f(t,z)-f(x,y)|\,;x,y\Big)\\&\le M_0(r)M_0(s)S_{m,n;p,q}\Big(|f(t,z)-f(x,y)|\,;x,y\Big)\qquad\text{for}\quad m,n\in\mathbb{N}.\end{aligned}$$

But, by (2.1) and (2.2),

$$\begin{aligned}|f(t,z)-f(x,y)|&=\Big|\int_x^t f'_u(u,z)\,\mathrm{d}u+\int_y^z f'_w(x,w)\,\mathrm{d}w\Big|\\&\le\|f'_x\|_{p,q}\Big|\int_x^t\frac{\mathrm{d}u}{v_{p,q}(u,z)}\Big|+\|f'_y\|_{p,q}\Big|\int_y^z\frac{\mathrm{d}w}{v_{p,q}(x,w)}\Big|\\&\le\|f'_x\|_{p,q}\left(\frac{1}{v_{p,q}(t,z)}+\frac{1}{v_{p,q}(x,z)}\right)|t-x|\\&\quad+\|f'_y\|_{p,q}\left(\frac{1}{v_{p,q}(x,z)}+\frac{1}{v_{p,q}(x,y)}\right)|z-y|,\end{aligned}$$

for every $(t,z)\in\mathbb{R}^2_0$. From the above and by (2.8), (2.10), (2.11) and (2.1) we get

$$\begin{aligned}&v_{p,q}(x,y)S_{m,n;p,q}\left(|f(t,z)-f(x,y)|;x,y\right)\\&\quad\le\|f'_x\|_{p,q}\big\{v_{p,q}(x,y)S_{m,n;p,q}\left(|\varphi_x(t)/v_{p,q}(t,z):x,y\right)\\&\qquad+v_q(y)S_{m,p}(|\varphi_x(t)|;x)S_{n;q}(1/v_q(z);y)\big\}\\&\qquad+\|f'_y\|_{p,q}\big\{v_q(y)S_{n;q}\left(|\varphi_y(z)|/v_q(z);y\right)+S_{n;q}\left(|\varphi_y(z)|;y\right)\big\}\\&\quad:=\|f'_x\|_{p,q}\big(T_1(x,y)+T_2(x,y)\big)+\|f'_y\|_{p,q}\big(T_3(x,y)+T_4(x,y)\big).\end{aligned}$$

Using the Hölder inequality, (2.1), (2.10) and (2.14)-(2.16), we deduce that

$$\begin{aligned} T_1(x,y) &\leq \Big(v_{2p,2q}(x,y)S_{m,n;p,q}(1/v_{2p,2q};x,y)S_{m;p}\left(\varphi_x^2(t);x\right)\Big)^{1/2} \\ &\leq \Big(\|S_{m;p}(1/v_{2p})\|_{2p}\|S_{n;q}(1/v_{2q})\|_{2q}\, S_{m;p}\left(\varphi_x^2(t);x\right)\Big)^{1/2} \\ &\leq \Big(S_{m;p}\left(\varphi_x^2(t);x\right)\Big)^{1/2} \end{aligned}$$

and analogously

$$T_2(x,y) = \|S_{n;q}(1/v_q)\|_q\Big(S_{m;p}\left(\varphi_x^2(t);x\right)\Big)^{1/2} \leq \Big(S_{m;p}\Big(\varphi_x^2(t);x\Big)\Big)^{1/2},$$

$$T_3(x,y) \leq \Big(\|S_{n;q}(1/v_{2q})\|_{2q}S_{n;q}\left(\varphi_y^2(z);y\right)\Big)^{1/2} \leq \Big(S_{n;q}\left(\varphi_y^2(z);y\right)\Big)^{1/2}$$

and

$$T_4(x,y) \leq \Big(S_{n;q}\left(\varphi_y^2(z);y\right)\Big)^{1/2} \qquad \text{for} \quad m,n\in\mathbb{N}.$$

Combining the above and using the inequality (2.12), we obtain the desired estimation (3.2). □

Theorem 2. *Let $p,q>0$ and $r,s\in\mathbb{N}$ be fixed and let $R_{m,n}(f)$ be defined by* (3.1)*. Then there exists $M_3 = M_3(p,q,r,s) = \text{const.} > 0$ such that if $f\in C_{p,q}$, then*

$$v_{p,q}(x,y)\,|R_{m;n}(f;x,y)| \leq M_3\omega\left(f;C_{p,q};\sqrt{\frac{x^2+x}{m+p}},\sqrt{\frac{y^2+y}{n+q}}\right), \tag{3.3}$$

for $(x,y)\in\mathbb{R}_0^2$ and $m,n\in\mathbb{N}$, where $\omega(f;C_{p,q})$is the modulus of continuity of f defined by (2.3)*.*

P r o o f. The inequality (3.3) for $(x,y)=(0,0)$ is obvious by (3.1), (2.9) and (2.3). Let now $x,y>0$. For $f\in C_{p,q}$ we consider the Stieklov function

$$f_{h,\delta}(x,y) := \frac{1}{h\delta}\int_0^h \mathrm{d}u\int_0^\delta f(x+u,y+w)\,\mathrm{d}w, \qquad (x,y)\in\mathbb{R}_0^2,\ \ h,\delta>0,$$

for which the partial derivatives are given by the formulas:

$$\Big(f_{h,\delta}(x,y)\Big)'_x = \frac{1}{h\delta}\int_0^\delta\Big(\Delta_{h,w}f(x,y) - \Delta_{0,w}f(x,y)\Big)\,\mathrm{d}w,$$

$$\Big(f_{h,\delta}(x,y)\Big)'_y = \frac{1}{h\delta}\int_0^h\Big(\Delta_{u,\delta}f(x,y) - \Delta_{w,0}f(x,y)\Big)\,\mathrm{d}u.$$

From this we deduce that $f_{h,\delta} \in C^1_{p,q}$ and by (2.3) and (2.4),

$$\begin{aligned} \|f_{h,\delta} - f\|_{p,q} &\leq \omega(f; C_{p,q}; h, \delta), &(3.4)\\ \|(f_{h,\delta})'_x\|_{p,q} &\leq 2h^{-1}\omega(f; C_{p,q}; h, \delta) &(3.5)\end{aligned}$$

and

$$\|(f_{h,\delta})'_y\|_{p,q} \leq 2\delta^{-1}\omega(f; C_{p,q}; h, \delta) \qquad \text{for} \quad h, \delta > 0. \tag{3.6}$$

Now, by (2.6), (2.7) and (3.1), we can write

$$\begin{aligned} &|R_{m,n}(f; x, y)| \\ &\leq \left|S^{[r,s]}_{m,n;p,q}\big(f(t,z) - f_{h,\delta}(t,z); x, y\big)\right| + |R_{m,n}(f_{h,\delta}; x, y)| + |f_{h,\delta}(x,y) - f(x,y)| \\ &:= \sum_{k=1}^{3} W_k(x,y) \qquad \text{for} \quad m, n \in \mathbb{N}, \quad h, \delta > 0. \end{aligned} \tag{3.7}$$

Next, by (2.20), we have

$$v_{p,q}(x,y)W_1(x,y) \leq M_1(r,s)\|f - f_{h,\delta}\|_{p,q} \tag{3.8}$$

and, by Theorem 1,

$$\begin{aligned} &v_{p,q}(x,y)W_2(x,y) \\ &\leq M_2(p,q,r,s)\left\{\|(f_{h,\delta})'_x\|_{p,q}\sqrt{\frac{x^2+x}{m+p}} + \|(f_{h,\delta})'_y\|_{p,q}\sqrt{\frac{y^2+y}{n+q}}\right\}. \end{aligned} \tag{3.9}$$

Applying (3.8), (3.9) and (3.4)-(3.6), we get from (3.7):

$$v_{p,q}(x,y)\,|R_{m,n}(f; x, y)| \leq M_4\omega(f; C_{p,q}; h, \delta)\left\{1 + h^{-1}\sqrt{\frac{x^2+x}{m+p}} + \delta^{-1}\sqrt{\frac{y^2+y}{n+q}}\right\},$$

$(M_4 = M_4(p,q,r,s) = \text{const.} > 0)$ which for given $m, n \in \mathbb{N}$ and $x, y > 0$ and choosed $h = \sqrt{(x^2+x)/(m+p)}$ and $\delta = \sqrt{(y^2+y)/(n+q)}$ implies the estimation (3.3).

The proof of (3.3) at the points $(x, 0)$ and $(0, y)$ is similar. □

Theorem 2 and the property (2.5) imply the following

Corollary 1. *For fixed $p, q > 0$ and $r, s \in \mathbb{N}$ and every $f \in C_{p,q}$ there holds*

$$\lim_{m,n\to\infty} S^{[r,s]}_{m,n;p,q}(f; x, y) = f(x,y) \qquad \text{at every} \quad (x,y) \in \mathbb{R}^2_0.$$

This convergence is uniform on every rectangle $[x_1, x_2] \times [y_1, y_2]$ with $x_1, y_1 \geq 0$.

3.2. The purpose of this section is to prove the Voronovskaya-type theorem for the operatos $S^{[r,s]}_{n,n;p,q}$.

Theorem 3. *Let p, q, r and s satisfy the assumptions of* Theorem 2. *If $f \in C^2_{p,q}$, then*

$$\begin{aligned}
&\lim_{n\to\infty} n \left\{ S^{[r,s]}_{n,n;p,q}(f;x,y) - f(x,y) \right\} \\
&= -pxf'_x(x,y) - qyf'_y(x,y) + \frac{x}{2} f''_{x^2}(x,y) + \frac{y}{2} f''_{y^2}(x,y)
\end{aligned} \tag{3.10}$$

at every $(x,y) \in \mathbb{R}^2_0$.

P r o o f. The statement (3.10) is obvious for $(x,y) = (0,0)$ by (2.9).

Choose $f \in C^2_{p,q}$ and $(x,y) \in \mathbb{R}^2_0$. By the Taylor formula we have

$$\begin{aligned}
f(t,z) = f(x,y) &+ f'_x(x,y)(t-x) + f'_y(x,y)(z-y) \\
&+ \frac{1}{2} \left\{ f''_{x^2}(x,y)(t-x)^2 + 2f''_{xy}(x,y)(t-x)(z-y) \right. \\
&+ \left. f''_{y^2}(x,y)(z-y)^2 \right\} + \psi_{x,y}(t,z) \left\{ (t-x)^4 + (z-y)^4 \right\}^{1/2}
\end{aligned} \tag{3.11}$$

for $(t,z) \in \mathbb{R}^2_0$, where $\psi_{x,y}$ is a function belonging to the space $C_{p,q}$ and $\lim\limits_{(t,z)\to(x,y)} \psi_{x,y}(t,z) = \psi_{x,y}(x,y) = 0$. Hence the formulas (2.6), (3.11), (2.7), (2.10), (2.11) and (1.3) imply that

$$\begin{aligned}
&S^{[r,s]}_{n,n;p,q}(f(t,z);x,y) \\
&= f(x,y) + f'_x(x,y) S^{[r]}_{n;p}(\varphi_x(t);x) + f'_y(x,y) S^{[s]}_{n;q}\big(\varphi_y(z);y\big) \\
&+ \frac{1}{2} \Big\{ f''_{x^2}(x,y) S^{[r]}_{n;p}\big(\varphi^2_x(t);x\big) + 2f''_{xy}(x,y) S^{[r]}_{n;p}\big(\varphi_x(t);x\big) S^{[s]}_{n;q}\big(\varphi_y(z);y\big) \\
&+ f''_{y^2}(x,y) S^{[s]}_{n;q}\big(\varphi^2_y(z);y\big) \Big\} + Z_n(x,y), \qquad n \in \mathbb{N},
\end{aligned} \tag{3.12}$$

where

$$Z_n(x,y) := S^{[r,s]}_{n,n;p,q} \left(\psi_{x,y}(t,z) \sqrt{\varphi^4_x(t) + \varphi^4_y(z)}; x,y \right). \tag{3.13}$$

Using now Lemma 3, we get from (3.12)

$$\begin{aligned}
&\lim_{n\to\infty} n \left\{ S^{[r,s]}_{n,n;p,q}(f;x,y) - f(x,y) \right\} \\
&= -pxf'_x(x,y) - qyf'_y(x,y) + \frac{x}{2} f''_{x^2}(x,y) + \frac{y}{2} f''_{y^2}(x,y) + \lim_{n\to\infty} nZ_n(x,y).
\end{aligned} \tag{3.14}$$

Obviously the formula (3.14) yields (3.10) provided that

$$\lim_{n\to\infty} nZ_n(x,y) = 0. \tag{3.15}$$

By the Hölder inequality, (2.6), (2.10) and (2.16) we get from (3.13):

$$\begin{aligned}
&n|Z_n(x,y)| \\
&\le \left(S^{[r,s]}_{n,n;p,q}\big(\psi^2_{x,y}(t,z);x,y\big) \right)^{1/2} \left\{ n^2 S^{[r]}_{n,p}\big(\varphi^4_x(t);x\big) + n^2 S^{[s]}_{n,q}\big(\varphi^4_y(z);y\big) \right\}^{1/2}
\end{aligned}$$

and, by Corollary 1 and propreties of $\psi_{x,y}$,

$$\lim_{n\to\infty} S^{[r,s]}_{n,n;p,q}\big(\psi^2_{x,y}(t,z);x,y\big) = \psi^2_{x,y}(x,y) = 0.$$

From this and (2.18) and (2.13) immediately follows (3.15). Thus the proof of (3.10) is completed. $\square$

4. Remarks

4.1 In the paper [14] certain class of positive linear operators $L_{m,n}$ was investigated in the polynomial weight space $C^*_{p,q}$ of functions of two variables. Now $C^*_{p,q}$ is dependent on $p,q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and the weight function $w_{p,q}(x,y) = w_p(x)w_q(y)$ for $(x,y) \in \mathbb{R}^2_0$, where $w_0(x) \equiv 1$ and $w_p(x) = (1+x^p)^{-1}$ if $p \in \mathbb{N}$. $C^*_{p,q}$ is the set of all functions $f\colon \mathbb{R}^2_0 \to \mathbb{R}$ such that $w_{p,q}f$ is uniformly continuous and bounded on $\mathbb{R}^2_0$ and the norm $\|f\|^*_{p,q} = \sup_{(x,y)\in\mathbb{R}^2_0} w_{p,q}(x,y)|f(x,y)|$. In [14] was proved that $L_{m,n}\colon C^*_{p,q} \to C^*_{p,q}$ and the some approximation theorems for these operators were given. In [14] was showed also that the class of operators $L_{m,n}$ contains the Szász-Mirakyan operators $S_{m,n}$ defined by (1.7) and other well known operators of functions of two variables.

4.2. It is obvious that for $f \in C^*_{p,q}$, $p,q \in \mathbb{N}_0$, and $r,s \in \mathbb{N}$, the generalized Szász-Mirakyan operators

$$S^{[r,s]}_{m,n}(f;x,y) := \frac{1}{A_r(mx)A_s(ny)} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(mx)^{rj}(ny)^{sk}}{(rj)!(sk)!} f\left(\frac{rj}{m}, \frac{sk}{n}\right),$$

$(x,y) \in \mathbb{R}^2_0$ and $m,n \in \mathbb{N}$, can be examined (A_r is given by (1.4)).

We can verify that the above $S^{[r,s]}_{m,n}$ belong to the class of operators $L_{m,n}$ considered in [14]. Hence, the approximation properties of $S^{[r,s]}_{m,n}$ can be obtain from the paper [14].

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