

A CLASS OF LINEAR POSITIVE OPERATORS IN WEIGHTED SPACES

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(Communicated by Michal Zająac)

ABSTRACT. In this paper, we introduce a class of linear positive operators based on q -integers. For these operators we give some convergence properties in weighted spaces of continuous functions and present an application to differential equation related to q -derivatives. Furthermore, we give a Stancu-type remainder.

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1. Introduction

The classical Meyer-Knig and Zeller (MKZ) [18] operators are defined by

$$M_n(f; x) = (1-x)^{n+1} \sum_{k=0}^{\infty} f\left(\frac{k}{n+k+1}\right) \binom{n+k}{k} x^k, \quad x \in [0, 1), \quad n \in \mathbb{N}.$$

In order to give the monotonicity properties, Cheney and Sharma [7] introduced a slight modification of these operators as follows:

$$M_n^*(f; x) = (1-x)^{n+1} \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) \binom{n+k}{k} x^k, \quad x \in [0, 1), \quad n \in \mathbb{N}.$$

There are many works about these operators (see, for instance [1, 3, 6, 7, 8, 10, 16, 18]).

The study of approximation theory based on q -integers was firstly started by Phillips in [19]. The author proposed a generalization of Bernstein polynomials called q -Bernstein polynomials. After that many researchers studied in this area.

2010 Mathematics Subject Classification: Primary 41A25, 41A36.

Keywords: linear positive operators, weighted spaces, modulus of continuity, q -derivative.

Trif [21], introduced MKZ operators based on q -integers and studied the approximation and monotonicity properties of such operators. Since it was impossible to give an explicit expression for the second moment of these operators, in [9] Doğru and Duman presented another kind MKZ operators based on q -integers and investigated their statistical approximation properties. Later, Heping [14] gave an explicit formula for the second moment of the q -MKZ operators, defined by Trif, in terms of q -hypergeometric series and discussed further approximating properties of these operators. Recently, in [2] Özarslan and Duman have constructed a new generalization of MKZ operators based on q -integers and obtained a Korovkin type approximation theorem for them.

Before introducing the operators we recall some properties of the q -calculus. For any fixed real number $q > 0$ and nonnegative integer r , the q -integer $[r]_q$ and the q -factorial $[r]_q!$ are defined by

$$[r]_q = \begin{cases} \frac{1-q^r}{1-q}, & q \neq 1 \\ r, & q = 1 \end{cases}$$

and

$$[r]_q! = \begin{cases} [1]_q[2]_q \cdots [r]_q, & r \geq 1 \\ 1, & r = 0 \end{cases}$$

respectively. Also

$$(x, q)_n = \prod_{s=0}^{n-1} (1 - q^s x)$$

and q -binomial coefficients are defined by (see [13, p. 3, p. 6])

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[r]_q![n-r]_q!}, \quad n \geq r \geq 0.$$

Throughout this work we shall assume that $q \in (0, 1)$.

Rempulska and Skorupka in [20] defined the following operators

$$M_n(f, b_n; x) = \left(1 - \frac{x}{b_n}\right)^{n+1} \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}b_n\right) \binom{n+k}{k} \left(\frac{x}{b_n}\right)^k, \quad (1.1)$$

where $x \in [0, b_n)$, $n \in \mathbb{N}$, and introducing a generalization of these operators for differentiable functions in polynomial weighted spaces proved some convergence properties of that operators. In (1.1) b_n is a sequence of real numbers having the properties

$$1 \leq b_n < b_{n+1}, \quad \lim_{n \rightarrow \infty} b_n = \infty, \quad \lim_{n \rightarrow \infty} \frac{b_n}{n} = 0. \quad (1.2)$$

In this study, we present the q -generalization of the linear positive operators given by (1.1) as follows:

$$L_{n,q}(f; x) = P_{n,q}(x) \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{[n+k]_q} b_n\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k, \quad (1.3)$$

where $x \in [0, b_n)$, $n \in \mathbb{N}$, $P_{n,q}(x) = \prod_{s=0}^n \left(1 - q^s \frac{x}{b_n}\right)$ and b_n is an increasing and unbounded sequence of positive numbers such that

$$\lim_{n \rightarrow \infty} b_n = \infty. \quad (1.4)$$

It is clear that for $x \in [0, b_n)$ and $q \in (0, 1)$ these operators are linear and positive. Note that by the condition (1.4), the interval $[0, b_n)$ which is the domain of the operators $L_{n,q}$ expands infinity when $n \rightarrow \infty$. In this case, by means of the maximum norm we do not study the approximation properties of these operators. So we will give some convergence properties of the operators $L_{n,q}$ in weighted spaces of continuous functions on positive semi-axis with weighted norm.

2. Convergence properties

In order to give some convergence properties of the operators $L_{n,q}$ we state the following lemma.

LEMMA 1. *The operators $L_{n,q}$ defined by (1.3) verify*

$$\begin{aligned} L_{n,q}(1; x) &= 1 \\ L_{n,q}(t; x) &= x \\ x^2 &\leq L_{n,q}(t^2; x) \leq qx^2 + \frac{b_n}{[n]_q} x \end{aligned}$$

for all $x \in [0, b_n)$.

Proof. By means of the following equation (see [4, p. 490])

$$\frac{1}{(x, q)_N} = \sum_{k=0}^{\infty} \begin{bmatrix} N+k-1 \\ k \end{bmatrix}_q x^k, \quad (2.1)$$

it is easy to verify that $L_{n,q}(1; q, x) = 1$ and $L_{n,q}(t; x) = x$.

A direct computation yields that

$$L_{n,q}(t^2; x) = P_{n,q}(x) \sum_{k=1}^{\infty} \frac{[k]_q}{[n+k]_q} b_n^2 \frac{[n+k-1]_q!}{[n]_q! [k-1]_q!} \left(\frac{x}{b_n} \right)^k. \quad (2.2)$$

Since $[k]_q = q[k-1]_q + 1$ for $k \geq 0$, we get

$$\begin{aligned} L_{n,q}(t^2; x) &= qx^2 P_{n,q}(x) \sum_{k=2}^{\infty} \frac{1}{[n+k]_q} \frac{[n+k-1]_q!}{[n]_q! [k-2]_q!} \left(\frac{x}{b_n} \right)^{k-2} \\ &\quad + b_n x P_{n,q}(x) \sum_{k=1}^{\infty} \frac{1}{[n+k]_q} \frac{[n+k-1]_q!}{[n]_q! [k-1]_q!} \left(\frac{x}{b_n} \right)^{k-1} \\ &= qx^2 P_{n,q}(x) \sum_{k=0}^{\infty} \frac{[n+k+1]_q}{[n+k+2]_q} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n} \right)^k \\ &\quad + b_n x P_{n,q}(x) \sum_{k=0}^{\infty} \frac{1}{[n+k+1]_q} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n} \right)^k. \end{aligned} \quad (2.3)$$

By using the inequalities

$$[n+k+1]_q < [n+k+2]_q, \quad [n]_q < [n+k+1]_q$$

from (2.3) one has

$$L_{n,q}(t^2; x) \leq qx^2 + \frac{b_n}{[n]_q} x. \quad (2.4)$$

In a similar way that of (2.4) by using the facts that

$$[n+k+1]_q < [n+k+2]_q, \quad [n+k+1]_q = \frac{[n+k+2]_q - 1}{q}$$

and taking into consideration $\frac{x}{b_n} \in [0, 1)$ it can be proved that

$$x^2 \leq L_{n,q}(t^2; x). \quad (2.5)$$

Hence, combining (2.4) with (2.5) we arrive at the desired result. \square

LEMMA 2. *The operators $L_{n,q}$ defined by (1.3) verify*

$$\begin{aligned} L_{n,q_n}((t-x)^4; x) &\leq (q_n^6 - 4q_n^3 + 6q_n - 3)x^4 \\ &\quad + \left[(1 + [2]_{q_n} + [3]_{q_n})q_n^3 - 4(1 + [2]_{q_n})q_n + 6 \right] \frac{b_n}{[n]_{q_n}} x^3 \\ &\quad + \left[(1 + [2]_{q_n} + [2]_{q_n}^2)q_n - 4 \right] \frac{b_n^2}{[n]_{q_n}^2} x^2 + \frac{b_n^3}{[n]_{q_n}^3} x \end{aligned}$$

for all $x \in [0, b_n)$.

Since the proof is similar the proof of Lemma 1, we can omit it.

For a fixed value of q with $0 < q < 1$ we have $\lim_{n \rightarrow \infty} [n]_q = \frac{1}{1-q}$. So, to guarantee convergence properties of the operators $L_{n,q}$, we will replace q by a sequence q_n such that $\lim_{n \rightarrow \infty} q_n = 1$.

Now we give approximation properties of the operators L_{n,q_n} in weighted spaces of continuous functions with the help of weighted Korovkin type theorem given by Gadjiev in [11, 12]. For this purpose, let us recall the notations and results of [11, 12].

$B_\rho[0, \infty)$: The space of all functions satisfying the condition

$$|f(x)| \leq M_f \rho(x)$$

where $x \in [0, \infty)$, M_f is a positive constant depending only on f and $\rho(x) = 1 + x^2$.

$C_\rho[0, \infty)$: The space of all continuous functions in the space $B_\rho[0, \infty)$.

$C_\rho^0[0, \infty)$: The subspace of all functions $f \in C_\rho[0, \infty)$ for which

$$\lim_{x \rightarrow \infty} \frac{|f(x)|}{\rho(x)} < \infty.$$

The space $B_\rho[0, \infty)$ is a linear normed space with the following norm:

$$\|f\|_\rho = \sup_{x \in [0, \infty)} \frac{|f(x)|}{\rho(x)}.$$

THEOREM A. ([11, 12]) *Let A_n be a sequence of positive linear operators acting from $C_\rho[0, \infty)$ to $B_\rho[0, \infty)$ satisfying the conditions*

$$\lim_{n \rightarrow \infty} \|A_n(t^\nu; x) - x^\nu\|_\rho = 0, \quad \nu = 0, 1, 2.$$

Then for any function $f \in C_\rho^0[0, \infty)$,

$$\lim_{n \rightarrow \infty} \|A_n(f; x) - f(x)\|_\rho = 0$$

where $\rho(x) = 1 + x^2$.

We observe that a sequence of positive linear operators A_n acts from $C_\rho[0, \infty)$ to $B_\rho[0, \infty)$ if and only if

$$\|A_n(\rho; x)\|_\rho \leq M_\rho$$

where ρ defined as in Theorem A and M_ρ is a positive constant. This fact is a simple corollary of the necessary and sufficient condition that $A_n(\rho; x) \leq M_\rho(x)$ given in [11, 12].

LEMMA 3. *Let q_n be a sequence such that $\lim_{n \rightarrow \infty} q_n = 1$ for $0 < q_n < 1$. If $\lim_{n \rightarrow \infty} \frac{b_n}{[n]_{q_n}} = 0$, then for all n the sequence of positive linear operators L_{n,q_n} acts from $C_\rho[0, \infty)$ to $B_\rho[0, \infty)$ where $\rho(x) = 1 + x^2$.*

Proof. By Lemma 1, we get

$$\sup_{x \in [0, b_n)} \frac{L_{n,q_n}(\rho; x)}{1 + x^2} \leq 1 + q_n + \frac{b_n}{[n]_{q_n}}.$$

Since, $\lim_{n \rightarrow \infty} q_n = 1$ and $\lim_{n \rightarrow \infty} \frac{b_n}{[n]_{q_n}} = 0$ there is a positive constant M such that $q_n + \frac{b_n}{[n]_{q_n}} < M$ for each n . Thus we have

$$\sup_{x \in [0, b_n)} \frac{L_{n,q_n}(\rho; x)}{1 + x^2} \leq 1 + M$$

which completes the proof. \square

We observe that for example, for $q_n = \frac{n}{n+1}$ and $b_n = \sqrt{n}$ the conditions of Lemma 3 are satisfied.

THEOREM 1. *Let q_n be a sequence such that $\lim_{n \rightarrow \infty} q_n = 1$ for $0 < q_n < 1$. If $\lim_{n \rightarrow \infty} \frac{b_n}{[n]_{q_n}} = 0$, then for each $f \in C_\rho^0[0, \infty)$,*

$$\lim_{n \rightarrow \infty} \sup_{x \in [0, b_n)} \frac{|L_{n,q_n}(f; x) - f(x)|}{1 + x^2} = 0.$$

Proof. We have seen that L_{n,q_n} acts from $C_\rho[0, \infty)$ to $B_\rho[0, \infty)$ for all n . Therefore, to complete the proof applying Theorem A to the operators

$$A_n(f; x) = \begin{cases} L_{n,q_n}(f; x), & \text{if } x \in [0, b_n) \\ f(x), & \text{if } x \geq b_n \end{cases}$$

it is sufficient to verify that the conditions

$$\lim_{n \rightarrow \infty} \sup_{x \in [0, b_n)} \frac{|L_{n,q_n}(t^\nu; x) - x^\nu|}{1 + x^2} = 0, \quad \nu = 0, 1, 2,$$

are valid. From Lemma 1, we immediately see that this condition is satisfied for $\nu = 0, 1$. Again by Lemma 1, one gets

$$\sup_{x \in [0, b_n)} \frac{|L_{n,q_n}(t^2; x) - x^2|}{1 + x^2} \leq 1 - q_n + \frac{b_n}{[n]_{q_n}}.$$

Since $\lim_{n \rightarrow \infty} q_n = 1$ and $\lim_{n \rightarrow \infty} \frac{b_n}{[n]_{q_n}} = 0$, this implies that

$$\lim_{n \rightarrow \infty} \sup_{x \in [0, b_n)} \frac{|L_{n, q_n}(t^2; x) - x^2|}{1 + x^2} = 0.$$

Thus the proof is completed. \square

Now recall the definition of the weighted modulus of continuity $\Omega(f; \delta)$ which given in [15]. Let $f \in C_\rho^0[0, \infty)$. The weighted modulus of continuity of f is defined by

$$\Omega(f; \delta) = \sup_{|h| \leq \delta, x \in [0, b_n)} \frac{|f(x+h) - f(x)|}{(1+h^2)(1+x^2)}. \quad (2.6)$$

In [15], it is proved that

$$\lim_{\delta \rightarrow 0} \Omega(f; \delta) = 0$$

and

$$|f(t) - f(x)| \leq 2(1 + \delta^2)\Omega(f; \delta)(1 + x^2) \left(1 + \frac{|t-x|}{\delta}\right) (1 + (t-x)^2) \quad (2.7)$$

for each $f \in C_\rho^0[0, \infty)$ and $\delta > 0$.

THEOREM 2. Let q_n be a sequence such that $\lim_{n \rightarrow \infty} q_n = 1$ for $0 < q_n < 1$. If $\lim_{n \rightarrow \infty} \frac{b_n}{[n]_{q_n}} = 0$, then for each $f \in C_\rho^0[0, \infty)$ the inequality

$$\sup_{x \in [0, b_n)} \frac{|L_{n, q_n}(f; x) - f(x)|}{1 + x^2} \leq 2w(f; \delta_{n, q_n}),$$

where $w(f; \delta)$ is the usual modulus of continuity of the function f on the semi-axis $[0, \infty)$ and $\delta_{n, q_n} = \sqrt{1 - q_n + \frac{b_n}{[n]_{q_n}}}$, is valid for sufficiently large n .

Proof. By the linearity and monotonicity of the operators L_{n, q_n} we have

$$|L_{n, q_n}(f; x) - f(x)| \leq L_{n, q_n}(|f(t) - f(x)|; x). \quad (2.8)$$

On the other hand, by using the well known property of the modulus of continuity $w(f; \delta)$

$$|f(t) - f(x)| \leq \left(1 + \frac{|t-x|}{\delta}\right) w(f; \delta)$$

for every $\delta > 0$ and applying the Cauchy-Schwarz inequality, from (2.8) it follows that

$$|L_{n, q_n}(f; x) - f(x)| \leq w(f; \delta) \left(1 + \frac{1}{\delta} \sqrt{L_{n, q_n}((t-x)^2; x)}\right). \quad (2.9)$$

By means of Lemma 1 we have

$$\begin{aligned} L_{n,q_n}((t-x)^2; x) &= [L_{n,q_n}(t^2; x) - x^2] - 2x[L_{n,q_n}(t; x) - x] \\ &= |L_{n,q_n}(t^2; x) - x^2| \\ &\leq (1 - q_n)x^2 + \frac{b_n}{[n]_{q_n}}x \end{aligned}$$

which yields

$$\sup_{x \in [0, b_n]} \frac{L_{n,q_n}((t-x)^2; x)}{(1+x^2)^2} \leq 1 - q_n + \frac{b_n}{[n]_{q_n}}.$$

Thus if we choose $\delta = \delta_{n,q_n}$ from (2.9) we find the desired result. \square

THEOREM 3. *Let q_n be a sequence such that $\lim_{n \rightarrow \infty} q_n = 1$ for $0 < q_n < 1$. If $\lim_{n \rightarrow \infty} \frac{b_n}{[n]_{q_n}} = 0$, then for each $f \in C_\rho^0[0, \infty)$ the inequality*

$$\sup_{x \in [0, b_n]} \frac{|L_{n,q_n}(f; x) - f(x)|}{(1+x^2)^3} \leq M\Omega(f; K_{n,q_n}^{1 \setminus 4})$$

is valid for sufficiently large n . Here M is a positive constant independent of n , $\Omega(f; \delta)$ is the weighted modulus of continuity defined by (2.6) and

$$\begin{aligned} K_{n,q_n} = \max \left\{ q_n^6 - 4q_n^3 + 6q_n - 3, \right. \\ \left[(1 + [2]_{q_n} + [3]_{q_n})q_n^3 - 4(1 + [2]_{q_n})q_n + 6 \right] \frac{b_n}{[n]_{q_n}}, \\ \left. \left[(1 + [2]_{q_n} + [2]_{q_n}^2)q_n - 4 \right] \frac{b_n^2}{[n]_{q_n}^2}, \frac{b_n^3}{[n]_{q_n}^3} \right\}. \end{aligned}$$

Proof. By the linearity and monotonicity of the operators L_{n,q_n} from (2.7) one gets

$$\begin{aligned} &|L_{n,q_n}(f; x) - f(x)| \\ &\leq 2(1 + \delta^2)\Omega(f; \delta)(1+x^2)L_{n,q_n}\left(\left(1 + \frac{|t-x|}{\delta}\right)(1+(t-x)^2); x\right). \end{aligned}$$

For all $x \in [0, b_n]$ and $t \in [0, \infty)$ using the inequality (see [15, p. 361])

$$\left(1 + \frac{|t-x|}{\delta}\right)(1+(t-x)^2) \leq 2(1 + \delta^2)\left(1 + \frac{(t-x)^4}{\delta^4}\right)$$

we have

$$|L_{n,q_n}(f; x) - f(x)| \leq 16\Omega(f; \delta)(1+x^2)\left[1 + \frac{1}{\delta^4}L_{n,q_n}((t-x)^4; x)\right]. \quad (2.10)$$

From Lemma 2 and (2.10) it follows that

$$|L_{n,q_n}(f; x) - f(x)| \leq 16\Omega(f; \delta)(1 + x^2) \left[1 + \frac{1}{\delta^4} K_{n,q_n}(x^4 + x^3 + x^2 + x) \right].$$

Choosing $\delta = K_{n,q_n}^{1 \setminus 4}$ then we have

$$|L_{n,q_n}(f; x) - f(x)| \leq 16\Omega(f; K_{n,q_n}^{1 \setminus 4})(1 + x^2)(x^4 + x^3 + x^2 + x + 1)$$

which gives the desired result. \square

Remark 1. In Theorem 2, we obtained the rate of convergence of the operators L_{n,q_n} in the space $C_\rho[0, \infty)$ by means of the usual modulus of continuity and in Theorem 3, we presented the rate of convergence for the operators L_{n,q_n} in the space $C_{\rho^3}[0, \infty)$ with the help of the weighted modulus of continuity.

3. Application to differential equations

Finally, we shall give a functional differential equation related the q -derivatives for the operators $L_{n,q}$ defined by (1.3). In [3, 8, 10, 17, 22] there exist equations without q -integers similar to the one in Theorem 4.

We now introduce the q -derivative.

For $0 < q < 1$ the q -derivative $D_q f(x)$ of f denoted by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \quad x \neq 0, \quad D_q f(0) = \lim_{x \rightarrow 0} D_q f(x).$$

THEOREM 4. Let $g_n(t) = \frac{q^n t}{b_n - q^n t}$ for $0 < q < 1$ and also let $f \in C_\rho^0[0, \infty)$ and $x \in [0, b_n)$. Then the operators $L_{n,q}$ defined by (1.3) satisfies the functional differential equation

$$\begin{aligned} & \frac{q^n}{[n]_q} x \left(1 - \frac{x}{b_n} \right) D_q L_{n,q}(f; x) \\ & - \left(1 - q^{n+1} \frac{x}{b_n} \right) L_{n,q}(f g_n; x) + \frac{q^n [n+1]_q}{[n]_q b_n} x L_{n,q}(f; x) = 0 \end{aligned}$$

for each n .

Proof. From the definition of the q -derivative, one gets

$$\begin{aligned}
 & \frac{q^n}{[n]_q} x \left(1 - \frac{x}{b_n}\right) D_q L_{n,q}(f; x) \\
 &= \frac{q^n}{[n]_q} \left(1 - \frac{x}{b_n}\right) \left[\frac{L_{n,q}(f; x) - L_{n,q}(f; qx)}{1 - q} \right] \\
 &= \frac{q^n}{(1 - q)[n]_q} \left(1 - \frac{x}{b_n}\right) L_{n,q}(f; x) - \frac{q^n}{(1 - q)[n]_q} \left(1 - \frac{x}{b_n}\right) \prod_{s=0}^n \left(1 - q^{s+1} \frac{x}{b_n}\right) \\
 & \quad \times \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{[n+k]_q} b_n\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q q^k \left(\frac{x}{b_n}\right)^k.
 \end{aligned}$$

Since

$$\left(1 - \frac{x}{b_n}\right) \prod_{s=0}^n \left(1 - q^{s+1} \frac{x}{b_n}\right) = \left(1 - q^{n+1} \frac{x}{b_n}\right) P_{n,q}(x)$$

we can write

$$\begin{aligned}
 & \frac{q^n}{[n]_q} x \left(1 - \frac{x}{b_n}\right) D_q L_{n,q}(f; x) \\
 &= \frac{q^n}{(1 - q)[n]_q} \left(1 - \frac{x}{b_n}\right) L_{n,q}(f; x) \\
 & \quad - \frac{q^n}{(1 - q)[n]_q} \left(1 - q^{n+1} \frac{x}{b_n}\right) P_{n,q}(x) \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{[n+k]_q} b_n\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q q^k \left(\frac{x}{b_n}\right)^k.
 \end{aligned} \tag{3.1}$$

By the definition of q -integers, it is easily seen that

$$q^k = (q - 1)[k] + 1.$$

So, substitution of this into (3.1) gives

$$\begin{aligned}
 & \frac{q^n}{[n]_q} x \left(1 - \frac{x}{b_n}\right) D_q L_{n,q}(f; x) \\
 &= \frac{q^n}{(1 - q)[n]_q} \left[\left(1 - \frac{x}{b_n}\right) - \left(1 - q^{n+1} \frac{x}{b_n}\right) \right] L_{n,q}(f; x) \\
 & \quad + \left(1 - q^{n+1} \frac{x}{b_n}\right) P_{n,q}(x) \sum_{k=1}^{\infty} f\left(\frac{[k]_q}{[n+k]_q} b_n\right) \frac{q^n [k]_q}{[n]_q} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k
 \end{aligned}$$

$$\begin{aligned}
 &= -\frac{q^n[n+1]_q}{[n]_q b_n} x L_{n,q}(f; x) + \left(1 - q^{n+1} \frac{x}{b_n}\right) P_{n,q}(x) \\
 &\quad \times \sum_{k=1}^{\infty} f\left(\frac{[k]_q}{[n+k]_q} b_n\right) \frac{q^n[k]_q}{[n]_q} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k.
 \end{aligned}$$

Since

$$g_n\left(\frac{[k]_q}{[n+k]_q} b_n\right) = \frac{q^n[k]_q}{[n]_q}$$

the proof is completed. \square

4. A Stancu-type remainder of $L_{n,q}$

In this section, we give Stancu-type remainder for the operators $L_{n,q}$ by means of the divided differences.

Let x_0, x_1, \dots, x_n be distinct points in the domain of f . Then the divided differences of a function f is given by

$$f[x_0, x_1, \dots, x_n] = \sum_{r=0}^n \frac{f(x_r)}{\prod_{j \neq r} (x_r - x_j)}$$

where r remains fixed and j takes all values from 0 to n , except r (see [5]).

THEOREM 5. *If $x \in [0, b_n) \setminus \left\{\frac{[k]_q}{[n+k]_q} b_n : k = 0, 1, 2, \dots\right\}$ then the following formula is valid*

$$\begin{aligned}
 L_{n,q}(f; x) - f(x) &= b_n x P_{n,q}(x) \sum_{k=0}^{\infty} \frac{q^k}{[n+k+1]_q} f\left[x, \frac{[k]_q}{[n+k]_q} b_n, \frac{[k+1]_q}{[n+k+1]_q} b_n\right] \\
 &\quad \times \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k.
 \end{aligned}$$

Proof. By the linearity of the operators $L_{n,q}$, we can write

$$L_{n,q}(f; x) - f(x) = -P_{n,q}(x) \sum_{k=0}^{\infty} \left\{ f(x) - f\left(\frac{[k]_q}{[n+k]_q} b_n\right) \right\} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k. \quad (4.1)$$

Since,

$$f(x) - f\left(\frac{[k]_q}{[n+k]_q} b_n\right) = \left(x - \frac{[k]_q}{[n+k]_q} b_n\right) f\left[x, \frac{[k]_q}{[n+k]_q} b_n\right]$$

and

$$\begin{bmatrix} n+k \\ k \end{bmatrix}_q \frac{[k]_q}{[n+k]_q} b_n = \begin{bmatrix} n+k-1 \\ k-1 \end{bmatrix}_q b_n$$

from (4.1) it follows that

$$\begin{aligned} & L_{n,q}(f; x) - f(x) \\ &= b_n P_{n,q}(x) \sum_{k=1}^{\infty} f \left[x, \frac{[k]_q}{[n+k]_q} b_n \right] \begin{bmatrix} n+k-1 \\ k-1 \end{bmatrix}_q \left(\frac{x}{b_n} \right)^k \\ &\quad - x P_{n,q}(x) \sum_{k=0}^{\infty} f \left[x, \frac{[k]_q}{[n+k]_q} b_n \right] \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n} \right)^k \\ &= x P_{n,q}(x) \sum_{k=0}^{\infty} \left\{ f \left[x, \frac{[k+1]_q}{[n+k+1]_q} b_n \right] - f \left[x, \frac{[k]_q}{[n+k]_q} b_n \right] \right\} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n} \right)^k. \end{aligned} \tag{4.2}$$

By substituting

$$\begin{aligned} & f \left[x, \frac{[k+1]_q}{[n+k+1]_q} b_n \right] - f \left[x, \frac{[k]_q}{[n+k]_q} b_n \right] \\ &= f \left[x, \frac{[k]_q}{[n+k]_q} b_n, \frac{[k+1]_q}{[n+k+1]_q} b_n \right] \left(\frac{[k+1]_q}{[n+k+1]_q} - \frac{[k]_q}{[n+k]_q} \right) b_n \end{aligned}$$

into (4.2) and by using the equalities

$$\frac{[k+1]_q}{[n+k+1]_q} - \frac{[k]_q}{[n+k]_q} = \frac{q^k [n]_q}{[n+k]_q [n+k+1]_q}$$

and

$$\begin{bmatrix} n+k \\ k \end{bmatrix}_q \frac{q^k [n]_q}{[n+k]_q [n+k+1]_q} = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \frac{q^k}{[n+k+1]_q}$$

we arrive at the desired result. \square

We recall that a function f is convex on $I \subset \mathbb{R}$ if and only if its divided differences is nonnegative. So we now state the following remark.

Remark 2. If f is convex on $[0, b_n)$, then $L_{n,q}(f; x) - f(x) \geq 0$ for all $n \in \mathbb{N}$, $x \in [0, b_n) \setminus \left\{ \frac{[k]_q}{[n+k]_q} b_n : k = 0, 1, 2, \dots \right\}$ and $0 < q < 1$.

Remark 3. When $q = 1$ all of the results of this paper is valid for the operators $M_n(f, b_n; x)$ defined by Rempulska and Skorupka [20].

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Received 1. 7. 2009

Accepted 25. 2. 2010

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