

ON IDEAL CONVERGENCE IN PROBABILISTIC NORMED SPACES

M. MURSALEEN* — S. A. MOHIUDDINE**

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ABSTRACT. An interesting generalization of statistical convergence is I -convergence which was introduced by P. Kostyrko et al [KOSTYRKO, P.—ŠALÁT, T. —WILCZYŃSKI, W.: I -Convergence, Real Anal. Exchange **26** (2000-2001), 669–686]. In this paper, we define and study the concept of I -convergence, I^* -convergence, I -limit points and I -cluster points in probabilistic normed space. We discuss the relationship between I -convergence and I^* -convergence, i.e. we show that I^* -convergence implies the I -convergence in probabilistic normed space. Furthermore, we have also demonstrated through an example that, in general, I -convergence does not imply I^* -convergence in probabilistic normed space.

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1. Introduction and preliminaries

The concept of statistical convergence for sequences of real numbers was introduced by Fast [4] and Steinhaus [22] independently in the same year 1951 and since then several generalizations and applications of this notion have been investigated by various authors, namely [3], [7], [15], [16], [17], [18], [19]. One of its interesting generalization is I -convergence which was given by Kostyrko et al [12]. Recently I -convergence for sequences of functions has been studied by Balcerzak et al. [2] and by Komisarski [13].

The theory of probabilistic normed spaces [5] originated from the concept of statistical metric spaces which was introduced by Menger [14] and further studied by Schweizer and Sklar [20, 21]. It provides an important method of

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generalizing the deterministic results of normed linear spaces. It has also very useful applications in various fields, e.g. continuity properties [1], topological spaces [5], linear operators [8], study of boundedness [9], convergence of random variables [10] etc.

In this paper we study the concept of I -convergence and I^* -convergence in a more general setting, i.e. in the probabilistic normed space. We also define I -limit points and I -cluster points in probabilistic normed space and prove some interesting results.

We recall some notations and basic definitions used in this paper.

DEFINITION 1.1. ([21]) A *triangular norm* (t-norm) is a continuous mapping $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ such that $([0, 1], *)$ is an abelian monoid with unit one and $c * d \geq a * b$ if $c \geq a$ and $d \geq b$ for all $a, b, c, d \in [0, 1]$.

DEFINITION 1.2. ([5]) A function $f: \mathbb{R} \rightarrow \mathbb{R}_0^+$ is called a *distribution function* if it is non-decreasing and left-continuous with $\inf_{t \in \mathbb{R}} f(t) = 0$ and $\sup_{t \in \mathbb{R}} f(t) = 1$.

By D , we denote the set of all distribution functions.

DEFINITION 1.3. ([5]) Let X be a real linear space and $\nu: X \rightarrow D$. Then the *probabilistic norm* or ν -norm is a t-norm satisfying the following conditions:

- (i) $\nu_x(0) = 0$,
- (ii) $\nu_x(t) = 1$ for all $t > 0$ iff $x = 0$,
- (iii) $\nu_{\alpha x}(t) = \nu_x\left(\frac{t}{|\alpha|}\right)$ for all $\alpha \in \mathbb{R} \setminus \{0\}$ and for all $t > 0$,
- (iv) $\nu_{x+y}(s+t) \geq \nu_x(s) * \nu_y(t)$ for all $x, y \in X$ and $s, t \in \mathbb{R}_0^+$;

where ν_x means $\nu(x)$ and $\nu_x(t)$ is the value of ν_x at $t \in \mathbb{R}$.

Triple $(X, \nu, *)$ is called a probabilistic normed space (for short, PNS).

DEFINITION 1.4. ([11]) Let $(X, \nu, *)$ be a PNS. Then, a sequence $x = (x_k)$ is said to be *convergent* to $\xi \in X$ with respect to the probabilistic norm ν , that is, $x_k \xrightarrow{\nu} \xi$ if for every $t > 0$ and $\varepsilon \in (0, 1)$, there is a positive integer k_0 such that $\nu_{x_k - \xi}(t) > 1 - \varepsilon$ whenever $k \geq k_0$. In this case we write $\nu\text{-}\lim x = \xi$.

Remark 1.1. Let $(X, \|\cdot\|)$ be a real normed linear space, and

$$\nu(x, t) := \frac{t}{t + \|x\|}$$

for all $x \in X$ and $t > 0$. Then $x_n \xrightarrow{\|\cdot\|} x$ if and only if $x_n \xrightarrow{\nu} x$.

DEFINITION 1.5. ([6]) Let K be a subset of \mathbb{N} , the set of natural numbers. Then the *asymptotic density* of K denoted by $\delta(K)$, is defined as

$$\delta(K) = \lim_n \frac{1}{n} |\{k \leq n : k \in K\}|,$$

where the vertical bars denote the cardinality of the enclosed set.

DEFINITION 1.6. ([4, 22]) A number sequence $x = (x_k)$ is said to be *statistically convergent* to the number ℓ if for each $\varepsilon > 0$, the set $K(\varepsilon) = \{k \leq n : |x_k - \ell| > \varepsilon\}$ has asymptotic density zero, i.e.

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - \ell| > \varepsilon\}| = 0.$$

In this case we write $\text{st-lim } x = \ell$.

DEFINITION 1.7. ([11]) Let $(X, \nu, *)$ be a PNS. Then, a sequence $x = (x_k)$ is said to be *statistically convergent* to $\xi \in X$ with respect to the probabilistic norm ν provided that, for every $t > 0$ and $\varepsilon > 0$,

$$\delta(\{k \leq n : \nu_{x_k - \xi}(t) \leq 1 - \varepsilon\}) = 0,$$

or equivalently

$$\lim_n \frac{1}{n} |\{k \leq n : \nu_{x_k - \xi}(t) \leq 1 - \varepsilon\}| = 0.$$

In this case we write $\text{st}_\nu\text{-lim } x = \xi$.

DEFINITION 1.8. ([12]) If X is a non-empty set then a family $I \subset 2^{\mathbb{N}}$ of subsets of X is called an *ideal* in X if

- (a) $\emptyset \in I$,
- (b) $A, B \in I$ implies $A \cup B \in I$,
- (c) For each $A \in I$ and $B \subset A$ we have $B \in I$,

An ideal I is called nontrivial ideal if $X \notin I$.

DEFINITION 1.9. ([12]) Let X be a non-empty set. A non-empty family of sets $F \subset P(X)$ is called a *filter* on X if and only if

- (a) $\emptyset \notin F$,
- (b) $A, B \in F$ implies $A \cap B \in F$,
- (c) For each $A \in F$ and $B \supset A$ we have $B \in F$.

DEFINITION 1.10. ([12]) A non-trivial ideal I in X is called an *admissible ideal* if it is different from $P(\mathbb{N})$ and it contains all singletons i.e., $\{x\} \in I$ for each $x \in X$.

Let $I \subset P(X)$ be a non-trivial ideal. Then a class $F(I) = \{M \subset X : (\exists A \in I) (M = X \setminus A)\}$ is a filter on X , called the *filter associated with the ideal I* .

DEFINITION 1.11. ([12]) An admissible ideal $I \subset P(\mathbb{N})$ is said to satisfy the condition

(AP) if for every sequence $(A_n)_{n \in \mathbb{N}}$ of pairwise disjoint sets from I there are sets $B_n \subset \mathbb{N}$, $n \in \mathbb{N}$, such that the symmetric difference $A_n \triangle B_n$ is a finite set for every n and $\bigcup_{n \in \mathbb{N}} B_n \in I$.

DEFINITION 1.12. ([12]) Let $I \subset 2^{\mathbb{N}}$ be a nontrivial ideal in \mathbb{N} . Then, a sequence $x = (x_k)$ is said to be *I -convergent* to L if for every $\varepsilon > 0$, the set

$$\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in I.$$

In this case we write $I\text{-}\lim x = L$.

2. I -Convergence in PNS

In this section, we study the concept of ideal convergence of sequences in probabilistic normed space. Throughout the paper we take I as a nontrivial admissible ideal in \mathbb{N} .

We define:

DEFINITION 2.1. Let I be a non trivial ideal of \mathbb{N} and $(X, \nu, *)$ be a probabilistic normed space. A sequence $x = (x_k)$ of elements of X is said to be *I -convergent* to $\xi \in X$ with respect to the probabilistic norm ν (or I_ν -convergent to ξ) if for each $\varepsilon > 0$ and $t > 0$,

$$\{k \in \mathbb{N} : \nu_{x_k - \xi}(t) \leq 1 - \varepsilon\} \in I.$$

In this case we write $I_\nu\text{-}\lim x = \xi$.

THEOREM 2.1. Let $(X, \nu, *)$ be a PNS. Then, the following statements are equivalent:

- (i) $I_\nu\text{-}\lim x = \xi$.
- (ii) $\{k \in \mathbb{N} : \nu_{x_k - \xi}(t) \leq 1 - \varepsilon\} \in I_\nu$ for every $\varepsilon > 0$ and $t > 0$.
- (iii) $\{k \in \mathbb{N} : \nu_{x_k - \xi}(t) > 1 - \varepsilon\} \in F(I_\nu)$ for every $\varepsilon > 0$ and $t > 0$.
- (iv) $I\text{-}\lim \nu_{x_k - \xi}(t) = 1$.

Proof is standard.

THEOREM 2.2. *Let $(X, \nu, *)$ be a PNS. If a sequence $x = (x_k)$ is I_ν -convergent then I_ν -limit is unique.*

Proof. Suppose that $I_\nu\text{-}\lim x = \xi_1$ and $I_\nu\text{-}\lim x = \xi_2$. Given $\varepsilon > 0$ and $t > 0$. Choose $r > 0$ such that $(1 - r) * (1 - r) \geq 1 - \varepsilon$. Then, we define the following sets as:

$$\begin{aligned} K_{\nu,1}(r, t) &= \{k \in \mathbb{N} : \nu_{x_k - \xi_1}(t) \leq 1 - r\}, \\ K_{\nu,2}(r, t) &= \{k \in \mathbb{N} : \nu_{x_k - \xi_2}(t) \leq 1 - r\}. \end{aligned}$$

Since $I_\nu\text{-}\lim x = \xi_1$, we have

$$K_{\nu,1}(r, t) \in I.$$

Furthermore, using $I_\nu\text{-}\lim x = \xi_2$, we get

$$K_{\nu,2}(r, t) \in I.$$

Now let $K_\nu(r, t) = K_{\nu,1}(r, t) \cup K_{\nu,2}(r, t) \in I$. Then we see that $K_\nu(r, t) \in I$. This implies that its complement $K_\nu^C(r, t)$ is non empty set in $F(I)$. If $k \in K_\nu^C(r, t)$, then we have $k \in K_{\nu,1}^C(r, t) \cap K_{\nu,2}^C(r, t)$, and so

$$\nu_{\xi_1 - \xi_2}(t) \geq \nu_{x_k - \xi_1}(t/2) * \nu_{x_k - \xi_2}(t/2) > (1 - r) * (1 - r).$$

Since $(1 - r) * (1 - r) \geq 1 - \varepsilon$, it follows that

$$\nu_{\xi_1 - \xi_2}(t) > 1 - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we get $\nu_{\xi_1 - \xi_2}(t) = 1$ for all $t > 0$, which yields $\xi_1 = \xi_2$.

This completes the proof of the theorem. \square

THEOREM 2.3. *Let $(X, \nu, *)$ be a PNS.*

- (i) *If $\nu\text{-}\lim x_k = \xi$, then $I_\nu\text{-}\lim x_k = \xi$.*
- (ii) *If $I_\nu\text{-}\lim x_k = \xi_1$ and $I_\nu\text{-}\lim y_k = \xi_2$, then $I_\nu\text{-}\lim(x_k + y_k) = (\xi_1 + \xi_2)$.*
- (iii) *If $I_\nu\text{-}\lim x_k = \xi$, then $I_\nu\text{-}\lim \alpha x_k = \alpha \xi$.*

Proof.

(i) Suppose that $\nu\text{-}\lim x_k = \xi$. Then for each $\varepsilon > 0$ and $t > 0$ there exists a positive integer N such that

$$\nu_{x_k - \xi}(t) > 1 - \varepsilon$$

for each $k > N$. Since the set

$$A(t) = \{k \in \mathbb{N} : \nu_{x_k - \xi}(t) \leq 1 - \varepsilon\}$$

is contained in $\{1, 2, 3, \dots, N-1\}$ and the ideal I is admissible so $A(t) \in I$. Hence $I_\nu\text{-}\lim x_k = \xi$.

(ii) Let $I_\nu\text{-}\lim x_k = \xi_1$ and $I_\nu\text{-}\lim y_k = \xi_2$. For a given $\varepsilon > 0$ and $t > 0$ choose $r > 0$ such that $(1-r) * (1-r) > 1 - \varepsilon$. Define the following sets:

$$K_{\nu,1}(r, t) = \{k \in \mathbb{N} : \nu_{x_k - \xi_1}(t) \leq 1 - r\},$$

$$K_{\nu,2}(r, t) = \{k \in \mathbb{N} : \nu_{y_k - \xi_2}(t) \leq 1 - r\}.$$

Since $I_\nu\text{-}\lim x_k = \xi_1$, we have

$$K_{\nu,1}(r, t) \in I.$$

Furthermore, using $I_\nu\text{-}\lim x = \xi_2$, we get

$$K_{\nu,2}(r, t) \in I.$$

Now let $K_\nu(r, t) = K_{\nu,1}(r, t) \cup K_{\nu,2}(r, t)$. Then $K_\nu(r, t) \in I$ which implies that $K_\nu^C(r, t)$ is non empty set in $F(I)$. Now we have to show that $K_\nu^C(r, t) \subset \{k \in \mathbb{N} : \nu_{(x_k + y_k) - (\xi_1 + \xi_2)}(t) > 1 - \varepsilon\}$. If $k \in K_\nu^C(r, t)$, then we have $\nu_{x_k - \xi_1}(\frac{t}{2}) > 1 - r$ and $\nu_{y_k - \xi_2}(\frac{t}{2}) > 1 - r$. Therefore

$$\nu_{(x_k + y_k) - (\xi_1 + \xi_2)}(t) \geq \nu_{x_k - \xi_1}\left(\frac{t}{2}\right) * \nu_{y_k - \xi_2}\left(\frac{t}{2}\right) > (1 - r) * (1 - r) > 1 - \varepsilon.$$

This shows that

$$K_\nu^C(r, t) \subset \{k \in \mathbb{N} : \nu_{(x_k + y_k) - (\xi_1 + \xi_2)}(t) > 1 - \varepsilon\}.$$

Since $K_\nu^C(r, t) \in F(I)$ and hence $I_\nu\text{-}\lim(x_k + y_k) = (\xi_1 + \xi_2)$.

(iii) It is trivial for $\alpha = 0$. Now let $\alpha \neq 0$. Then for a given $\varepsilon > 0$ and $t > 0$,

$$B(t) = \{k \in \mathbb{N} : \nu_{x_k - \xi}(t) > 1 - \varepsilon\} \in F(I). \quad (2.1)$$

It is sufficient to prove that for each $\varepsilon > 0$ and $t > 0$

$$B(t) \subset \{k \in \mathbb{N} : \nu_{\alpha x_k - \alpha \xi}(t) > 1 - \varepsilon\}.$$

Let $k \in B(t)$. Then we have

$$\nu_{x_k - \xi}(t) > 1 - \varepsilon.$$

Now

$$\begin{aligned} \nu_{\alpha x_k - \alpha \xi}(t) &= \nu_{x_k - \xi}\left(\frac{t}{|\alpha|}\right) \geq \nu_{x_k - \xi}(t) * \nu_0\left(\frac{t}{|\alpha|} - t\right) \\ &= \nu_{x_k - \xi}(t) * 1 = \nu_{x_k - \xi}(t) > 1 - \varepsilon. \end{aligned}$$

Hence we have

$$B(t) \subset \{k \in \mathbb{N} : \nu_{\alpha x_k - \alpha \xi}(t) > 1 - \varepsilon\}$$

and from (2.1), we conclude that $I_\nu\text{-}\lim \alpha x_k = \alpha \xi$.

This completes the proof of the theorem. \square

3. I^* -Convergence in PNS

In this section, we introduce the concept of I^* -convergence of sequences in probabilistic normed space and show that I^* -convergence implies I -convergence but not conversely.

DEFINITION 3.1. Let $(X, \nu, *)$ be a probabilistic normed space. We say that a sequence $x = (x_k)$ of elements in X is I^* -convergent to $\xi \in X$ with respect to the probabilistic norm ν if there exists a subset $K = \{k_m : k_1 < k_2 < \dots\}$ of \mathbb{N} such that $K \in F(I)$ (i.e. $\mathbb{N} \setminus K \in I$) and $\nu\text{-}\lim_{m \in K} x_{k_m} = \xi$.

In this case we write $I_\nu^*\text{-}\lim x = \xi$ and ξ is called the I_ν^* -limit of the sequence $x = (x_k)$.

THEOREM 3.1. Let $(X, \nu, *)$ be a PNS and I be an admissible ideal. If $I_\nu^*\text{-}\lim x = \xi$ then $I_\nu\text{-}\lim x = \xi$.

Proof. Suppose that $I_\nu^*\text{-}\lim x = \xi$. Then $K = \{k_m : k_1 < k_2 < \dots\} \in F(I)$ (i.e. $\mathbb{N} \setminus K = H$ (say) $\in I$) such that $\nu\text{-}\lim_{m \in K} x_{k_m} = \xi$. But then for each $\varepsilon > 0$ and $t > 0$ there exists a positive integer N such that $\nu_{x_{k_m} - \xi}(t) > 1 - \varepsilon$ for all $m > N$. Since $\{k_m \in K : \nu_{x_{k_m} - \xi}(t) \leq 1 - \varepsilon\}$ is contained in $\{k_1 < k_2 < \dots < k_{N-1}\}$ and the ideal I is admissible, we have

$$\{k_m \in K : \nu_{x_{k_m} - \xi}(t) \leq 1 - \varepsilon\} \in I.$$

Hence

$$\{k \in \mathbb{N} : \nu_{x_k - \xi}(t) \leq 1 - \varepsilon\} \subseteq H \cup \{k_1 < k_2 < \dots < k_{N-1}\} \in I$$

for all $\varepsilon > 0$ and $t > 0$. Therefore, we conclude that $I_\nu\text{-}\lim x = \xi$. \square

Remark 3.1. The following example shows that the converse of Theorem 3.1 need not be true.

Example 3.1. Let $(\mathbb{R}, |\cdot|)$ denote the space of all real numbers with the usual norm, and let $a * b = ab$ for all $a, b \in [0, 1]$. For all $x \in \mathbb{R}$ and every $t > 0$, consider

$$\nu_x(t) := \frac{t}{t + |x|}.$$

Then $(\mathbb{R}, \nu, *)$ is a PNS.

Let $\mathbb{N} = \bigcup_j \Delta_j$ be a decomposition of \mathbb{N} such that for any $n \in \mathbb{N}$ each Δ_j contains infinitely many j 's where $j \geq n$ and $\Delta_j \cap \Delta_n = \emptyset$ for $j \neq n$. Let I be the class of all subsets of \mathbb{N} which intersect at most a finite number of Δ_j 's. Then I is an admissible ideal. Now we define a sequence $x_n = \frac{1}{j}$ if $n \in \Delta_j$. Then

$$\nu_{x_n}(t) = \frac{t}{t + |x_n|} \longrightarrow 1 \quad \text{as } n \longrightarrow \infty.$$

Hence $I_\nu\text{-}\lim_n x_n = 0$.

Now suppose that $I_\nu^*\text{-}\lim_n x_n = 0$. Then there exists a subset $K = \{n_1 < n_2 < \dots\}$ of \mathbb{N} such that $K \in F(I)$ and $\nu\text{-}\lim_j x_{n_j} = 0$. Since $K \in F(I)$, there is a set $H \in I$ such that $K = \mathbb{N} \setminus H$. Now, from the definition of I , there exists, say $p \in \mathbb{N}$ such that

$$H \subset \left(\bigcup_{n=1}^p \Delta_n \right).$$

But then $\Delta_{p+1} \subset K$, and therefore

$$x_{n_j} = \frac{1}{(p+1)} > 0$$

for infinitely many n_j 's from K which contradicts $\nu\text{-}\lim_j x_{n_j} = 0$. Therefore the assumption $I_\nu^*\text{-}\lim_n x_n = 0$ leads to the contradiction.

Hence the converse of the theorem need not be true.

Remark 3.2. From the above result we have seen that I^* -convergence implies I -convergence but not conversely. Now the question arises under what condition the converse may hold. The following theorem shows that the converse holds if the ideal I satisfies condition (AP).

Theorem 3.2. *Let $(X, \nu, *)$ be a PNS and the ideal I satisfy the condition (AP). If $x = (x_k)$ is a sequence in X such that $I_\nu\text{-}\lim x = \xi$, then $I_\nu^*\text{-}\lim x = \xi$.*

Proof. Suppose I satisfies condition (AP) and $I_\nu\text{-}\lim x = \xi$. Then for each $\varepsilon > 0$ and $t > 0$,

$$\{k \in \mathbb{N} : \mu_{x_k - \xi}(t) \leq 1 - \varepsilon\} \in I.$$

We define the set A_p for $p \in \mathbb{N}$ and $t > 0$ as

$$A_p = \left\{ k \in \mathbb{N} : 1 - \frac{1}{p} \leq \nu_{x_k - \xi}(t) < 1 - \frac{1}{p+1} \right\}.$$

Obviously $\{A_1, A_2, \dots\}$ is countable and belongs to I , and $A_i \cap A_j = \emptyset$ for $i \neq j$. By condition (AP), there is a countable family of sets $\{B_1, B_2, \dots\} \in I$ such that the symmetric difference $A_i \triangle B_i$ is a finite set for each $i \in \mathbb{N}$ and $B = \bigcup_{i=1}^{\infty} B_i \in I$. From the definition of associate filter $F(I)$ there is a set $K \in F(I)$ such that $K = \mathbb{N} \setminus B$. To prove the theorem it is sufficient to show that the subsequence $(x_k)_{k \in K}$ is convergent to ξ with respect to the probabilistic norm ν . Let $\eta > 0$ and $t > 0$. Choose $q \in \mathbb{N}$ such that $\frac{1}{q} < \eta$. Then

$$\left\{ k \in \mathbb{N} : \nu_{x_k - \xi}(t) \leq 1 - \eta \right\} \subset \left\{ k \in \mathbb{N} : \nu_{x_k - \xi}(t) \leq 1 - \frac{1}{q} \right\} \subset \bigcup_{i=1}^{q+1} A_i.$$

Since $A_i \triangle B_i$, $i = 1, 2, \dots, q+1$, are finite, there exists $k_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{i=1}^{q+1} B_i \right) \cap \{k : k \geq k_0\} = \left(\bigcup_{i=1}^{q+1} A_i \right) \cap \{k : k \geq k_0\}. \quad (3.1)$$

If $k \geq k_0$ and $k \in K$ then $k \notin \bigcup_{i=1}^{q+1} B_i$. Therefore by (3.1), we have $k \notin \bigcup_{i=1}^{q+1} A_i$. Hence for every $k \geq k_0$ and $k \in K$ we have

$$\nu_{x_k - \xi}(t) > 1 - \eta.$$

Since $\eta > 0$ was arbitrary, we have $I_\nu^* - \lim x = \xi$.

This completes the proof of the theorem. \square

THEOREM 3.3. *Let $(X, \nu, *)$ be a PNS. Then the following conditions are equivalent:*

- (i) $I_\nu^* - \lim x = \xi$.
- (ii) *There exist two sequences $y = (y_k)$ and $z = (z_k)$ in X such that $x = y + z$, $\nu - \lim y = \xi$ and the set $\{k : z_k \neq \theta\} \in I$, where θ denotes the zero element of X .*

Proof. Suppose that the condition (i) holds. Then there exists a subset $K = \{k_m : k_1 < k_2 < \dots\}$ of \mathbb{N} such that

$$K \in F(I) \quad \text{and} \quad \nu - \lim_m x_{k_m} = \xi. \quad (3.2)$$

We define the sequences $y = (y_k)$ and $z = (z_k)$ as follows:

$$y_k = \begin{cases} x_k, & \text{if } k \in K \\ \xi, & \text{if } k \in K^C; \end{cases}$$

and $z_k = x_k - y_k$ for all $k \in \mathbb{N}$. For given $\varepsilon > 0$, $t > 0$ and $k \in K^C$, we have

$$\nu_{y_k - \xi}(t) = 1 > 1 - \varepsilon.$$

Using (3.2) we have $\nu\text{-}\lim y = \xi$. Since $\{k : z_k \neq \theta\} \subset K^C$, we have $\{k : z_k \neq \theta\} \in I$.

Let the condition (ii) hold. Then $K = \{k : z_k = \theta\} \in F(I)$ is an infinite set. Let $K = \{k_m : k_1 < k_2 < \dots\}$. Since $x_{k_m} = y_{k_m}$ and $\nu\text{-}\lim y = \xi$, $\nu\text{-}\lim_{m \rightarrow \infty} x_{k_m} = \xi$. Hence $I_\nu^*\text{-}\lim_k x_k = \xi$.

This completes the proof of the theorem. \square

4. I -limit points and I -cluster points in PNS

In this section we define I -limit points and I -cluster points in probabilistic normed space analogous to the statistical limit points and statistical cluster points due to Fridy [8].

DEFINITION 4.1. Let $(X, \nu, *)$ be a PNS, and $x = (x_k) \in X$. An element $\xi \in X$ is said to be a *limit point of the sequence* $x = (x_k)$ with respect to the probabilistic norm ν (or a ν -*limit point*) if there is subsequence of the sequence x which converges to ξ with respect to the probabilistic norm ν .

By $\mathcal{L}_\nu(x)$, we denote the set of all limit points of the sequence $x = (x_k)$ with respect to the probabilistic norm ν .

DEFINITION 4.2. Let $(X, \nu, *)$ be a PNS, and $x = (x_k) \in X$. An element $\xi \in X$ is said to be an I -*limit point of the sequence* x with respect to the probabilistic norm ν (or I_ν -*limit point*) if there is a subset $K = \{k_m : k_1 < k_2 < \dots\}$ of \mathbb{N} such that $K \notin I$ and $\nu\text{-}\lim_{m \rightarrow \infty} x_{k_m} = \xi$.

We denote by $\Lambda_\nu^I(x)$, the set of all I_ν -limit points of the sequence $x = (x_k)$.

DEFINITION 4.3. Let $(X, \nu, *)$ be a PNS, and $x = (x_k) \in X$. An element $\xi \in X$ is said to be an I -*cluster point of* x with respect to the probabilistic norm ν (or I_ν -*cluster point*) if for each $\varepsilon > 0$ and $t > 0$

$$K = \{k \in \mathbb{N} : \nu_{x_k - \xi}(t) > 1 - \varepsilon\} \notin I.$$

By $\Gamma_\nu^I(x)$, we denote the set of all I_ν -cluster points of the sequence $x = (x_k)$.

THEOREM 4.1. *Let $(X, \nu, *)$ be a PNS. Then for every sequence $x = (x_k)$ in X , we have $\Lambda_\nu^I(x) \subset \Gamma_\nu^I(x) \subset \mathcal{L}_\nu(x)$.*

Proof. Let $\xi \in \Lambda_\nu^I(x)$. Then there exists a set $K = \{k_m : k_1 < k_2 < \dots\}$ of \mathbb{N} such that $K \notin I$ and $\nu\text{-}\lim_{m \rightarrow \infty} x_{k_m} = \xi$. For each $\varepsilon > 0$ and $t > 0$ there exists $N \in \mathbb{N}$ such that for $k > N$ we have $\nu_{x_k - \xi}(t) > 1 - \varepsilon$. Hence

$$\{k \in \mathbb{N} : \nu_{x_k - \xi}(t) > 1 - \varepsilon\} \supset \{k_{N+1}, k_{N+2}, \dots\}$$

and so

$$\{k \in \mathbb{N} : \nu_{x_k - \xi}(t) > 1 - \varepsilon\} \notin I,$$

which means that $\xi \in \Gamma_\nu^I(x)$. Hence $\Lambda_\nu^I(x) \subset \Gamma_\nu^I(x)$.

Let $\xi \in \Gamma_\nu^I(x)$. Then for given $\varepsilon > 0$ and $t > 0$, we have

$$\{k \in \mathbb{N} : \nu_{x_k - \xi}(t) > 1 - \varepsilon\} \notin I.$$

Let $K = \{k_m : k_1 < k_2 < \dots\}$. Then there is a subsequence (x_{k_n}) of (x_n) that converges to ξ with respect to the probabilistic norm ν . Therefore ξ is an ordinary limit point of (x_n) , that is $\xi \in \mathcal{L}_\nu(x)$ and hence $\Gamma_\nu^I(x) \subset \mathcal{L}_\nu(x)$.

This completes the proof of the theorem. \square

THEOREM 4.2. *Let $x = (x_k)$ be a sequence in a PN-space $(X, \nu, *)$. Then $\Lambda_\nu^I(x) = \Gamma_\nu^I(x) = \{\xi\}$, provided $I_\nu\text{-}\lim_k x_k = \xi$.*

Proof. Let $\eta \in \Lambda_\nu^I(x)$, where $\xi \neq \eta$. Then there exist two subsets K and K' , that is, $K = \{k_m : k_1 < k_2 < \dots\}$ and $K' = \{q_m : q_1 < q_2 < \dots\}$ of \mathbb{N} such that

$$K \notin I \quad \text{and} \quad \nu\text{-}\lim_{m \rightarrow \infty} x_{k_m} = \xi, \quad (4.1)$$

$$K' \notin I \quad \text{and} \quad \nu\text{-}\lim_{m \rightarrow \infty} x_{q_m} = \eta. \quad (4.2)$$

By (4.2), given $\varepsilon > 0$ and $t > 0$, there exists $N \in \mathbb{N}$ such that for $m > N$ we have $\nu_{q_m - \eta}(t) > 1 - \varepsilon$. Therefore the set

$$A = \{q_m \in K' : \nu_{q_m - \eta}(t) \leq 1 - \varepsilon\} \subset \{q_m : q_1 < q_2 < \dots < q_N\}.$$

As I is an admissible ideal so $A \in I$. If we take

$$B = \{q_m \in K' : \nu_{q_m - \eta}(t) > 1 - \varepsilon\} \notin I.$$

Otherwise, if $B \in I$, then $A \cup B = K' \in I$, which is contradiction to (4.2). Since $I_\nu\text{-}\lim_k x_k = \xi$, we have that for each $\varepsilon > 0$ and $t > 0$ the set

$$C = \{k \in \mathbb{N} : \nu_{x_k - \xi}(t) \leq 1 - \varepsilon\} \in I.$$

Therefore

$$C^C = \{k \in \mathbb{N} : \nu_{x_k - \xi}(t) > 1 - \varepsilon\} \in F(I).$$

Since for every $\xi \neq \eta$, we have $B \cap C^C = \emptyset$, $B \subset C$. Since $C \in I$ implies $B \in I$, is a contradiction to the fact that $B \notin I$. Hence $\Lambda_\nu^I(x) = \{\xi\}$.

On the other hand, suppose that $\eta \in \Gamma_\nu^I(x)$, where $\xi \neq \eta$. By definition, for each $\varepsilon > 0$ and $t > 0$, the sets

$$A = \{k \in \mathbb{N} : \nu_{x_k - \xi}(t) > 1 - \varepsilon\} \notin I,$$

$$B = \{k \in \mathbb{N} : \nu_{x_k - \eta}(t) > 1 - \varepsilon\} \notin I.$$

For $\xi \neq \eta$, we have $A \cap B = \emptyset$ and therefore $B \subset A^C$. Also, $I_\nu\text{-}\lim_k x_k = \xi$ implies that the set

$$A^C = \{k \in \mathbb{N} : \nu_{x_k - \xi}(t) \leq 1 - \varepsilon\} \in I.$$

Hence $B \in I$, which is a contradiction to $B \notin I$. Therefore $\Gamma_\nu^I(x) = \{\xi\}$.

This completes the proof of the theorem. \square

THEOREM 4.3. *Let $(X, \nu, *)$ be a PNS and if for any two sequences $x = (x_k)$, $y = (y_k)$ in X , the set $A = \{k \in \mathbb{N} : x_k \neq y_k\} \in I$. Then $\Lambda_\nu^I(x) = \Lambda_\nu^I(y)$ and $\Gamma_\nu^I(x) = \Gamma_\nu^I(y)$.*

Proof. Let $\xi \in \Lambda_\nu^I(x)$. Then there exists a subset $K = \{k_m : k_1 < k_2 < \dots\}$ of \mathbb{N} such that $K \notin I$ and $\nu\text{-}\lim_{m \rightarrow \infty} x_{k_m} = \xi$. Given $\varepsilon > 0$ and $t > 0$, there exists $N \in \mathbb{N}$ such that $\nu_{x_{k_m} - \xi}(t) > 1 - \varepsilon$ for $m > N$. Define $K_1 = K \cap A$ and $K_2 = K \setminus A$. Since $A \in I$, $K_1 \in I$. As $K = K_1 \cup K_2$ and $K \notin I$ so $K_2 \notin I$. It is clear that the subsequence $(y_k)_{k \in K_2}$ of the sequence $y = (y_k)$ is convergent to ξ with respect to the probabilistic norm ν . This implies that $\xi \in \Lambda_\nu^I(y)$ and hence $\Lambda_\nu^I(x) \subset \Lambda_\nu^I(y)$.

Similarly we can show that $\Lambda_\nu^I(y) \subset \Lambda_\nu^I(x)$. Hence $\Lambda_\nu^I(x) = \Lambda_\nu^I(y)$.

Now we need to show that $\Gamma_\nu^I(x) = \Gamma_\nu^I(y)$. Let $\xi \in \Gamma_\nu^I(x)$. For each $\varepsilon > 0$ and $t > 0$ the set

$$B = \{k \in \mathbb{N} : \nu_{x_k - \xi}(t) > 1 - \varepsilon\} \notin I.$$

Given $\varepsilon > 0$ and $t > 0$, define

$$C = \{k \in \mathbb{N} : \nu_{y_k - \xi}(t) > 1 - \varepsilon\}.$$

Now, we have to show that $C \notin I$. Let $C \in I$. Then

$$C^C = \{k \in \mathbb{N} : \nu_{y_k - \xi}(t) \leq 1 - \varepsilon\} \in F(I).$$

By hypothesis, the set $A^C = \{k \in \mathbb{N} : x_k = y_k\} \in F(I)$. Since $F(I)$ is a filter in \mathbb{N} , $C^C \cap A^C \in F(I)$. Since $C^C \cap A^C \subset B^C$, $B^C \in F(I)$. This

implies that $B \in I$ which is a contradiction, since $B \notin I$. So that $C \notin I$ and therefore $\Gamma_\nu^I(x) \subset \Gamma_\nu^I(y)$. Similarly we can show that $\Gamma_\nu^I(y) \subset \Gamma_\nu^I(x)$ and hence $\Gamma_\nu^I(x) = \Gamma_\nu^I(y)$.

This completes the proof of the theorem. \square

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**Department of Mathematics*

Aligarh Muslim University

Aligarh 202002

INDIA

E-mail: mursaleenm@gmail.com

***Department of Mathematics*

Faculty of Science

King Abdulaziz University

P.O. Box 80203

Jeddah 21589

SAUDI ARABIA

E-mail: mohiuddine@gmail.com