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# SEPARATING MAPS ON WEIGHTED FUNCTION ALGEBRAS ON TOPOLOGICAL GROUPS

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ABSTRACT. Let $G_1$ and $G_2$ be locally compact groups and let $\omega_1$ and $\omega_2$ be weight functions on $G_1$ and $G_2$, respectively. For $i = 1, 2$, let also $C_0(G_i, 1/\omega_i)$ be the algebra of all continuous complex-valued functions $f$ on $G_i$ such that $f/\omega_i$ vanish at infinity, and let $H\colon C_0(G_1, 1/\omega_1) \to C_0(G_2, 1/\omega_2)$ be a separating map; that is, a linear map such that $H(f)H(g) = 0$ for all $f, g \in C_0(G_1, 1/\omega_1)$ with $fg = 0$. In this paper, we study conditions under which $H$ can be represented as a weighted composition map; i.e., $H(f) = \varphi(f \circ h)$ for all $f \in C_0(G_1, 1/\omega_1)$, where $\varphi\colon G_2 \to \mathbb{C}$ is a non-vanishing continuous function and $h\colon G_2 \to G_1$ is a topological isomorphism. Finally, we offer a statement equivalent to that $h$ is also a group homomorphism.



## 1. Introduction

Let $G_1$ and $G_2$ be locally compact groups with left Haar measures $\lambda_1$ and $\lambda_2$, respectively. For $i = 1, 2$, let $\omega_i$ be a weight function on $G_i$; that is, a continuous function $\omega_i\colon G_i \to (0, \infty)$ such that $\omega_i(xy) \le \omega_i(x)\,\omega_i(y)$ for all $x, y \in G_i$; let also $C_0(G_i, 1/\omega_i)$ denotes the Banach algebra of all continuous complex-valued functions $f$ on $G_i$ such that $f/\omega_i$ vanish at infinity, equipped with the norm

$$\|f\|_{\infty,\omega_i} = \left\|\frac{f}{\omega_i}\right\|_\infty$$

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and the operations

$$(cf + g)(x) = cf(x) + g(x),$$
$$(f \cdot_{\omega_i} g)(x) = \frac{f(x)g(x)}{\omega_i(x)}$$

for all $f, g \in C_0(G_i, 1/\omega_i)$, $x \in G_i$ and $c \in \mathbb{C}$. For measurable functions $f$ and $g$ on $G_i$, the convolution product $*$ is defined as

$$(f * g)(x) = \int_{G_i} f(y)\, g(y^{-1}x)\, \mathrm{d}\lambda_i(y)$$

for all $x \in G$ whenever the right hand side exists.

A linear map $H\colon C_0(G_1, 1/\omega_1) \to C_0(G_2, 1/\omega_2)$ is called *separating* if for each $f, g \in C_0(G_1, 1/\omega_1)$,

$$f \cdot_{\omega_1} g = 0 \implies H(f) \cdot_{\omega_2} H(g) = 0.$$

Separating maps were considered by Beckenstein, Narici and Todd [3] for the algebra of complex-valued continuous functions defined on a compact Hausdorff space. The main goal of studies in the field was to prove automatic continuity for separating maps. As a result, some topological links between underlying spaces are deduced, and weighted composition type representations for separating maps are obtained. In recent years, considerable attention has been given to separating maps; see for example [1], [2], and [4] on Banach lattices, [3] and [6] on spaces of continuous functions, and [5] on group algebras of locally compact Abelian groups.

In this paper, we study separating maps between weighted function algebras $C_0(G_1, 1/\omega_1)$ and $C_0(G_2, 1/\omega_2)$. We give some conditions under which a separating map $H\colon C_0(G_1, 1/\omega_1) \to C_0(G_2, 1/\omega_2)$ can be represented as a *weighted composition map*; that is,

$$H(f)(x) = \varphi(x)\,(f \circ h)(x)$$

for all $f \in C_0(G_1, 1/\omega_1)$ and $x \in G_2$, where $\varphi$ is a complex-valued continuous function on $G_2$ and $h\colon G_2 \to G_1$ is a topological isomorphism. In this case, we offer a necessary and sufficient condition for that $h$ is also a group homomorphism. To that end, we introduce and study certain linear maps from $C_0(G_1, 1/\omega_1)$ into $C_0(G_2, 1/\omega_2)$.

## 2. The results

We commence with the definition of convolution quasi-homomorphism.

**Definition 2.1.** Let $G_1$ and $G_2$ be locally compact groups and $\omega_1$ and $\omega_2$ be weight functions on $G_1$ and $G_2$, respectively. We say that a linear map $H\colon C_0(G_1, 1/\omega_1) \to C_0(G_2, 1/\omega_2)$ is a *convolution quasi-homomorphism* if

$$(Hf * Hg)(x) = 0$$

for all $x \in G_2$ and $f, g \in C_0(G_1, 1/\omega_1)$ with $f, g \geq 0$ such that

$$H(f * g)(x) = 0,$$

whenever the two convolution products make sense.

We first state and prove the following key lemma.

**Lemma 2.2.** *Let $G_1$ and $G_2$ be locally compact groups and $\omega_1$ and $\omega_2$ be weight functions on $G_1$ and $G_2$, respectively. Let $H\colon C_0(G_1, 1/\omega_1) \to C_0(G_2, 1/\omega_2)$ be the map defined by $Hf = \varphi(f \circ h)$ for $f \in C_0(G_1, 1/\omega_1)$, where $\varphi\colon G_2 \to \mathbb{C}$ is a non-vanishing continuous function and $h\colon G_2 \to G_1$ is a continuous map. Then the following assertions are equivalent.*

(a) *The map $h$ is a group homomorphism.*

(b) *The linear map $H$ is a convolution quasi-homomorphism.*

Proof. First, we show that (a) implies (b). Suppose that $f, g \in C_0(G_1, 1/\omega_1)$, $f * g \in C_0(G_1, 1/\omega_1)$ and $H(f) * H(g) \in C_0(G_2, 1/\omega_2)$. If $x \in G_2$ with $H(f * g)(x) = 0$, then

$$\varphi(x)\,(f * g)(h(x)) = 0.$$

Since $\varphi$ is non-vanishing, we have $(f * g)(h(x)) = 0$; that is,

$$\int_{G_1} f(z)\, g(z^{-1}h(x))\, \mathrm{d}\lambda_1(z) = 0,$$

where $\lambda_1$ is a left Haar measure on $G_1$. Hence

$$f(z)\, g(z^{-1}h(x)) = 0 \qquad (z \in G_1).$$

On the other hand,

$$\begin{aligned}(Hf * Hg)(x) &= \int_{G_2} Hf(y)\, Hg(y^{-1}x)\, \mathrm{d}\lambda_2(y)\\ &= \int_{G_2} \varphi(y)\, \varphi(y^{-1}x)\, f(h(y))\, g(h(y^{-1}x))\, \mathrm{d}\lambda_2(y),\end{aligned}$$

where $\lambda_2$ is a left Haar measure on $G_2$. Since $h$ is a homomorphism we have

$$(Hf * Hg)(x) = 0.$$

To prove that (b) implies (a), suppose that there exist $x, y \in G_2$ such that

$$h(xy) \neq h(x)h(y).$$

Then we can choose open neighbourhoods $U$ and $V$ of $h(x)$ and $h(y)$ respectively such that $h(xy) \notin UV$. Moreover, we can find positive functions $f, g \in C_0(G_1, 1/\omega_1)$ with

$$f(h(x)) > 0, \quad g(h(y)) > 0, \qquad \operatorname{supp}(f) \subseteq U, \quad \operatorname{supp}(g) \subseteq V.$$

Let us first assume that $\varphi$ is real-valued. Without loss of generality, we can suppose $\varphi(x) > 0$ and $\varphi(y) < 0$. Hence, there exists a compact symetric neighbourhood $W$ of the identity element $G_2$ such that $\varphi > 0$ on $xW$ and $\varphi < 0$ on $Wy$.

We can find a positive function $p \in C_0(G_1, 1/\omega_1)$ such that $p(h(x)) = 1$ and $p$ vanishes outside $h(xW)$. In the same way, there exists a positive function $q \in C_0(G_1, 1/\omega_1)$ with $q(h(y)) = 1$ and $q$ vanishes outside $h(Wy)$. We therefore have

$$((pf) * (qg))(h(xy)) = \int_{G_1} p(z)\, f(z)\, q(z^{-1}h(xy))\, g(z^{-1}h(xy))\, \mathrm{d}\lambda_1(z).$$

Now, if $z \in U$ and $z^{-1}h(xy) \in V$, then $h(xy) \in UV$. This contradiction shows that

$$((pf) * (qg))(h(xy)) = 0.$$

Thus

$$H((pf) * (qg))(xy) = \varphi(xy)\, ((pf) * (pg))(h(xy)) = 0.$$

On the other hand,

$$\begin{aligned}
&(H(pf) * H(qg))(xy) \\
&= \int_{G_2} \varphi(z)\, \varphi(z^{-1}xy)\, p(h(z))\, q(h(z^{-1}xy))\, f(h(z))\, g(h(z^{-1}xy))\, \mathrm{d}\lambda_2(z) \\
&= \int_{xW} \varphi(z)\, \varphi(z^{-1}xy)\, p(h(z))\, q(h(z^{-1}xy))\, f(h(z))\, g(h(z^{-1}xy))\, \mathrm{d}\lambda_2(z).
\end{aligned}$$

By choosing a suitable open neighborhood of the identity element of $G_2$ contained in $W$, we get

$$(H(pf) * H(qg))(xy) \neq 0.$$

This contradicts the fact that $H$ is a convolution quasi-homomorphism. So, the proof of the real case is complete.

Now, suppose that $\varphi$ is complex-valued, and write $\varphi = \alpha + \mathrm{i}\beta$, where $\alpha$ and $\beta$ are nonzero continuous real-valued functions. Then

$$\begin{aligned}\varphi(z)\varphi(z^{-1}xy) = (\alpha(z)\alpha(z^{-1}xy) - \beta(z)\beta(z^{-1}xy))\\ + \mathrm{i}(\beta(z)\alpha(z^{-1}xy) + \alpha(z)\beta(z^{-1}xy)).\end{aligned}$$

So, if we set

$$\gamma(z) = \alpha(z)\alpha(z^{-1}xy) - \beta(z)\beta(z^{-1}xy),$$

then we get

$$\varphi(z)\varphi(z^{-1}xy) = \gamma(z) + \mathrm{i}(\beta(z)\alpha(z^{-1}xy) + \alpha(z)\beta(z^{-1}xy)).$$

Without loss of generality, we can assume $\gamma(x) > 0$. So, there exists a compact symmetric neighbourhood $W$ of the identity element $G_2$ such that for all $z \in xW$ with $\gamma(z) > 0$. If we choose functions $p$ and $q$ as in the real case and argue as before, we deduce

$$H((pf) * (qg))(xy) = 0.$$

On the other hand,

$$\begin{aligned}&\mathrm{Re}(H((pf) * (qg))(xy))\\ &= \int_{xW} \gamma(z)\, p(h(z))\, q(h(z^{-1}xy))\, f(h(z))\, g(h(z^{-1}xy))\, \mathrm{d}\lambda_2(z).\end{aligned}$$

Consequently,

$$(H(pf) * H(qg))(xy) \neq 0.$$

This contradiction completes the proof. □

**Theorem 2.3.** *Let $G_1$ and $G_2$ be locally compact groups and $\omega_1$ and $\omega_2$ be weight functions on $G_1$ and $G_2$, respectively. Let $H\colon C_0(G_1, 1/\omega_1) \to C_0(G_2, 1/\omega_2)$ be a bijective separating map. Then the following statements are equivalent.*

(a) *There exists a non-vanishing continuous function $\varphi\colon G_2 \to \mathbb{C}$ and a topological isomorphism $h\colon G_2 \to G_1$ such that $Hf = \varphi(f \circ h)$.*

(b) *The map $H$ is a convolution quasi-homomorphism.*

Proof. First, we show that $H$ is a weighted composition map. To that end, define $K\colon C_0(G_1) \to C_0(G_2)$ by

$$K(g) = \frac{H(g\omega_1)}{\omega_2} \qquad (g \in C_0(G_1)).$$

It is clear that $K$ is a bijective separating map between $C_0(G_1)$ and $C_0(G_2)$. So, [6, Theorem 1] implies that $K$ must be a weighted composition map; namely,

$$K(g) = \psi(g \circ k),$$

where $\psi\colon G_2 \to \mathbb{C}$ is a non-vanishing continuous function and $k\colon G_2 \to G_1$ is a homeomorphism. For each $f \in C_0(G_1, 1/\omega_1)$, we have $g := f/\omega_1 \in C_0(G_1)$, and therefore

$$\begin{aligned} H(f) &= H(g\omega_1) \\ &= \omega_2\, K(g) \\ &= \omega_2\, \psi\, (g \circ k) \\ &= \omega_2\, \psi\, ((\frac{f}{\omega_1}) \circ k) \\ &= \frac{\omega_2\, \psi}{\omega_1 \circ k}\, (f \circ k) \\ &= \varphi\, (f \circ h), \end{aligned}$$

where $\varphi := (\omega_2\, \psi)/(\omega_1 \circ k)$ and $h := k$. Thus

$$H(f) = \varphi(f \circ h),$$

where $\varphi\colon G_2 \to \mathbb{C}$ is a non-vanishing continuous function and $h\colon G_2 \to G_1$ is a homeomorphism. Now, the result is a consequence of Lemma 2.2. $\square$

For a complex-valued function $f$ on $G_i$, the set $\{x \in G_i : \ f(x) \neq 0\}$ is called the *cozero set* of $f$ and is denoted by $\mathrm{coz}(f)$.

**Proposition 2.4.** *Let $G_1$ and $G_2$ be locally compact groups and $\omega_1$ and $\omega_2$ be weight functions on $G_1$ and $G_2$, respectively. If $H\colon C_0(G_1, 1/\omega_1) \to C_0(G_2, 1/\omega_2)$ is a surjective linear isometry, then $H$ is a separating map.*

Proof. The result follows from Banach-Stone theorem; see for example [7]. Here, we give a direct proof for the sake of completeness. Suppose that $f, g \in C_0(G_1, 1/\omega_1)$ such that $\mathrm{coz}(f) \cap \mathrm{coz}(g) = \emptyset$. Also, suppose on the contrary that there exists

$$x_0 \in \mathrm{coz}(Hf) \cap \mathrm{coz}(Hg).$$

We can assume $(Hf)(x_0) = \omega_2(x_0)$ and $(Hg)(x_0) = \omega_2(x_0)$. Since $Hf, Hg \in C_0(G_2, 1/\omega_2)$, we can find an open neighbourhood $U$ of $x_0$, contained in $\mathrm{coz}(Hf) \cap \mathrm{coz}(Hg)$ such that

$$|(Hf)(x)| < (3/2)\, \omega_2(x) \quad \text{and} \quad |(Hg)(x)| < (3/2)\, \omega_2(x)$$

for all $x \in U$. By Urysohn's lemma, we can choose $k \in C_0(G_2, 1/\omega_2)$ such that $\mathrm{coz}(k) \subseteq U$,

$$k(x_0) = \big(\|Hf\|_{\infty,\omega_2} + \|Hg\|_{\infty,\omega_2}\big)\, \omega_2(x_0)$$

and

$$\|k\|_{\infty,\omega_2} \leq \|Hf\|_{\infty,\omega_2} + \|Hg\|_{\infty,\omega_2};$$

indeed, there is $\varphi \in C_c(G_2)$ with $\mathrm{coz}(\varphi) \subseteq U$,

$$\varphi(x_0) = \|Hf\|_{\infty,\omega_2} + \|Hg\|_{\infty,\omega_2}$$

and

$$\|\varphi\|_\infty = \|Hf\|_{\infty,\omega_2} + \|Hg\|_{\infty,\omega_2}.$$

So, if we set $k := \omega_2\,\varphi$, then $k$ is the desired function. It is clear that

$$\|Hf + k\|_{\infty,\omega_2} < \|Hf\|_{\infty,\omega_2} + \|Hg\|_{\infty,\omega_2} + 3/2$$

and

$$\|Hg + k\|_{\infty,\omega} < \|Hf\|_{\infty,\omega_2} + \|Hg\|_{\infty,\omega_2} + 3/2.$$

Since for every $x \in G_1$, $f(x) = 0$ or $g(x) = 0$ we get

$$\begin{aligned} \|f + g + H^{-1}k\|_{\infty,\omega_1} &= \max\big\{ \|f + H^{-1}k\|_{\infty,\omega_1}, \|g + H^{-1}k\|_{\infty,\omega_1} \big\} \\ &= \max\big\{ \|Hf + k\|_{\infty,\omega_2}, \|Hg + k\|_{\infty,\omega_2} \big\} \\ &\le \|Hf\|_{\infty,\omega_2} + \|Hg\|_{\infty,\omega_2} + 2. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \|Hf + Hg + k\|_{\infty,\omega_2} &\ge \frac{(Hf)(x_0) + (Hg)(x_0) + k(x_0)}{\omega_2(x_0)} \\ &= 2 + \|Hf\|_{\infty,\omega_2} + \|Hg\|_{\infty,\omega_2}. \end{aligned}$$

Since $H$ is an isometry,

$$\|f + g + H^{-1}k\|_{\infty,\omega_1} \ge \|Hf\|_{\infty,\omega_2} + \|Hg\|_{\infty,\omega_2} + 3.$$

This is contradiction. Consequently, $H$ is a separating map. □

**Corollary 2.5.** *Let $G_1$ and $G_2$ be locally compact groups and $\omega_1$ and $\omega_2$ be weight functions on $G_1$ and $G_2$, respectively. Let $H : C_0(G_1,1/\omega_1) \to C_0(G_2,1/\omega_2)$ be a surjective linear isometry. Then the following assertions are equivalent.*

(a) *There exists a non-vanishing continuous function $\varphi\colon G_2 \to \mathbb{C}$ and a topological isomorphism $h\colon G_2 \to G_1$ such that $Hf = \varphi(f \circ h)$.*

(b) *The map $H$ is a convolution quasi-homomorphism.*

Proof. In view of Proposition 2.4, $H$ is a bijective separating map. So, the result follows from Theorem 2.3. □

We end this paper with the following result.

**Corollary 2.6.** *Let $G_1$ and $G_2$ be locally compact groups and $\omega_1$ and $\omega_2$ be weight functions on $G_1$ and $G_2$, respectively. Let $H : C_0(G_1,1/\omega_1) \to C_0(G_2,1/\omega_2)$ be a surjective linear isometry such that $T(f * g) = Tf * Tg$, whenever two convolution products make sense. Then $G_1$ is topologically isomorphic to $G_2$.*

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