VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-011-0059-5
Math. Slovaca **61** (2011), No. 6, 933–940

ON $\mathcal{I}$-CONVERGENCE IN RANDOM 2-NORMED SPACES

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(Communicated by Wladyslaw Wilczynski)

ABSTRACT. Recently the concepts of statistical convergence and ideal convergence have been studied in 2-normed and 2-Banach spaces by various authors. In this paper we define and study the notion of ideal convergence in random 2-normed space and construct some interesting examples.

1. Introduction and preliminaries

The idea of statistical convergence was introduced by Fast [4] and since then several generalizations and applications of this concept have been investigated by various authors. One of its generalizations is the ideal convergence or I-convergence which was introduced by Kostyrko et al [11] and studied by Balcerzak et al [2], Das et al [3], and Komisarski [12]. Recently, Karakus [10] studied the concept of statistical convergence in probabilistic normed spaces.

The theory of probabilistic normed spaces was initiated and developed in [1], [14], [18], [19] and further it was extended to random/probabilistic 2-normed spaces [8] by using the concept of 2-norm [7].

Recently, statistical convergence and I-convergence have been studied in 2-Banach and 2-normed spaces in [9] and [17]. In this paper we define and study I-convergence in random 2-normed space which is quite a new and interesting idea to work with. For the study of statistical convergence and I-convergence of double sequences we refere to [3], [13], [15] and [16].

2010 Mathematics Subject Classification: Primary 40A05; Secondary 46A70, 40A99, 46A99.
Keywords: t-norm, random 2-normed space, statitical convergence, I-convergence.

Definition 1.1. ([4, 5]) Let K be a subset of $\mathbb{N}$ the set of natural numbers. Then the *asymptotic density of* K denoted by $\delta(K)$, is defined as

$$\delta(K) = \lim_n \frac{1}{n} |\{k \leq n : \ k \in K\}|,$$

where the vertical bars denote the cardinality of the enclosed set.

A number sequence $x = (x_k)$ is said to be *statistically convergent* to the number ℓ if for each $\varepsilon > 0$, the set $K(\varepsilon) = \big\{k \leq n : \ |x_k - \ell| \geq \varepsilon\big\}$ has asymptotic density zero, i.e.

$$\lim_n \frac{1}{n} \big|\big\{k \leq n : \ |x_k - \ell| \geq \varepsilon\big\}\big| = 0.$$

In this case we write $\text{st-lim}\, x = \ell$.

Definition 1.2. If X is a non-empty set then a family of subsets of X is called an *ideal in* X if and only if

(a) $\emptyset \in I$,

(b) $A, B \in I$ implies $A \cup B \in I$,

(c) For each $A \in I$ and $B \subset A$ we have $B \in I$,

I is called *nontrivial ideal* if $X \in I \neq \emptyset$.

Definition 1.3. Let X be a non-empty set and $P(X)$ be the power set of X. A non-empty family of sets $F \subset P(X)$ is called a *filter on* X if and only if

(a) $\emptyset \notin F$,

(b) $A, B \in F$ implies $A \cap B \in F$,

(c) For each $A \in F$ and $B \supset A$ we have $B \in F$.

Definition 1.4. A non-trivial ideal I in X is called an *admissible ideal* if it is different from $P(\mathbb{N})$ and it contains all singletons i.e., $\{x\} \in I$ for each $x \in X$.

Let $I \subset P(X)$ be a non-trivial ideal. Then a class

$$F(I) = \{M \subset X : \ M = X \setminus A, \ \text{for some} \ A \in I\}$$

is a filter on X, called the *filter associated with the ideal* I.

Definition 1.5. An admissible ideal $I \subset P(\mathbb{N})$ is said to satisfy the *condition*

(AP) if for every sequence $(A_n)_{n \in \mathbb{N}}$ of pairwise disjoint sets from I there are sets $B_n \subset \mathbb{N}$, $n \in \mathbb{N}$, such that the symmetric difference $A_n \triangle B_n$ is a finite set for every n and $\bigcup_{n \in \mathbb{N}} B_n \in I$.

Definition 1.6. Let I be a non trivial ideal of $\mathbb{N}$. A sequence $x = (x_k)$ is said to be *I-convergent to* $L \in X$ if for each $\varepsilon > 0$,

$$\{k \in \mathbb{N} : \ |x_k - \ell| \geq \varepsilon\} \in I.$$

In this case we write $I\text{-}\lim x = L$.

2. Random 2-normed space and $\mathcal{I}$-convergence

Definition 2.1. A function $f\colon \mathbb{R} \to \mathbb{R}_0^+$ is called a *distribution function* if it is a non-decreasing and left continuous with $\inf\limits_{t\in\mathbb{R}} f(t) = 0$ and $\sup\limits_{t\in\mathbb{R}} f(t) = 1$. By D^+, we denote the set of all distribution functions such that $f(0) = 0$.

If $a \in \mathbb{R}_0^+$, then $H_a \in D^+$, where

$$H_a(t) := \begin{cases} 1, & \text{if } \ t > a; \\ 0, & \text{if } \ t \leq a. \end{cases}$$

It is obvious that $H_0 \geq f$ for all $f \in D^+$.

A *t-norm* is a continuous mapping $*\colon [0,1] \times [0,1] \to [0,1]$ such that $\big([0,1], *\big)$ is abelian monoid with unit one and $c * d \geq a * b$ if $c \geq a$ and $d \geq b$ for all $a, b, c \in [0,1]$. A *triangle function* τ is a binary operation on D^+ which is commutative, associative and $\tau(f, H_0) = f$ for every $f \in D^+$.

In [7], Gähler introduced the following concept of 2-normed space.

Definition 2.2. A 2*-normed space* is a pair $\big(X, \|\cdot,\cdot\|\big)$, where X is a linear space of greater than one and $\|\cdot,\cdot\|\colon X \times X \to \mathbb{R}$ such that

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent,

(ii) $\|x, y\| = \|y, x\|$,

(iii) $\|\alpha x, y\| = |\alpha| \|x, y\|$ for $\alpha \in \mathbb{R}$, and

(iv) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for all $x, y, z \in X$.

Example 2.1. The usual 2-norm on $X = \mathbb{R}^2$ is $\|x, y\| = |x_1 y_2 - x_2 y_1|$, $x = (x_1, x_2)$, $y = (y_1, y_2)$.

Recently, Goleţ [8] used the idea of 2-normed space to define the random 2-normed space.

Definition 2.3. Let X is a linear space of greater than one, τ a triangle, and $\mathcal{F}\colon X \times X \to D^+$. Then $\mathcal{F}$ is called a *probabilistic* 2*-norm* and $(X, \mathcal{F}, \tau)$ a probabilistic 2-normed space if the following conditions are satisfied:

(2.3.1) $\mathcal{F}(x, y; t) = H_0(t)$ if x and y are linearly dependent, where $\mathcal{F}(x, y; t)$ denotes the value of $\mathcal{F}(x, y)$ at $t \in \mathbb{R}$,

(2.3.2) $\mathcal{F}(x, y; t) \neq H_0(t)$ if x and y are linearly independent,

(2.3.3) $\mathcal{F}(x, y; t) = \mathcal{F}(y, x; t)$ for all $x, y \in X$,

(2.3.4) $\mathcal{F}(\alpha x, y; t) = \mathcal{F}\left(x, y; \frac{t}{|\alpha|}\right)$ for every $t > 0$, $\alpha \neq 0$ and $x, y \in X$,

(2.3.5) $\mathcal{F}(x + y, z; t) \geq \tau\big(\mathcal{F}(x, z; t), \mathcal{F}(y, z; t)\big)$ whenever $x, y, z \in X$.

If (2.3.5) is replaced by

(2.3.5)′ $\mathcal{F}(x + y, z; t_1 + t_2) \geq \mathcal{F}(x, z; t_1) * \mathcal{F}(y, z; t_2)$ for all $x, y, z \in X$ and $t_1, t_2 \in \mathbb{R}_0^+$;

then $(X, \mathcal{F}, *)$ is called a *random* 2*-normed space* (for short, RTNS).

Remark 2.1. Note that every 2-normed space $(X, \|\cdot, \cdot\|)$ can be made a random 2-normed space in a natural way, by setting

(i) $\mathcal{F}(x, y; t) = H_0(t - \|x, y\|)$, for every $x, y \in X$, $t > 0$ and $a * b = \min\{a, b\}$, $a, b \in [0, 1]$;

(ii) $\mathcal{F}(x, y; t) = \frac{t}{t + \|x, y\|}$ for every $x, y \in X$, $t > 0$ and $a * b = ab$ for $a, b \in [0, 1]$.

Quite recently, Mursaleen and Mohiuddine [16] studied the concept of ideal convergence in intuitionistic fuzzy normed space. Now we define the concept of convergence and I-convergence and I^*-convergence in RTN space:

Definition 2.4. Let $(X, \mathcal{F}, *)$ be a RTN space. A sequence $x = (x_k)$ is said to be *convergent* (or $\mathcal{F}$-convergent) to ξ in $(X, \mathcal{F}, *)$ if for every $\varepsilon > 0$, $\theta \in (0, 1)$, there exists a positive integer k_0 such that $\mathcal{F}(x_k - \xi, z; \varepsilon) > 1 - \theta$ whenever $k \geq k_0$. In this case we write $\mathcal{F}\text{-}\lim_k x_k = \xi$, and ξ is called the $\mathcal{F}$-limit of $x = (x_k)$.

Definition 2.5. Let $(X, \mathcal{F}, *)$ be a RTN space. A sequence $x = (x_k)$ is said to be *I-convergent to* ξ in $(X, \mathcal{F}, *)$ if for every $\varepsilon > 0$, $\theta \in (0, 1)$ and nonzero $z \in X$

$$\big\{k \in \mathbb{N} :\ \mathcal{F}(x_k - \xi, z; \varepsilon) \leq 1 - \theta\big\} \in I,$$

or equivalently

$$\big\{k \in \mathbb{N} :\ \mathcal{F}(x_k - \xi, z; \varepsilon) > \theta\big\} \in F(I).$$

and we say that x is *$I(RTN)$-convergent to* ξ.
In this case we write $I(RTN)\text{-}\lim x = \xi$, and ξ is called the $I(RTN)$-limit of x.

Definition 2.6. Let $(X, \mathcal{F}, *)$ be a RTN space. We say that a sequence $x = (x_k)$ of elements in X is *I^*-convergent* to $\xi \in X$ in $(X, \mathcal{F}, *)$ if there exists a subset $K = \{k_m : k_1 < k_2 < \cdots\}$ of $\mathbb{N}$ such that $K \in F(I)$ (i.e. $\mathbb{N} \backslash K \in I$) and $\nu\text{-}\lim\limits_m x_{k_m} = \xi$. In this case we write $I^*(RTN)\text{-}\lim x = \xi$, and ξ is called the *$I^*(RTN)$-limit of x.*

3. Main results

We prove the following results:

Theorem 3.1. *Let $(X, \mathcal{F}, *)$ be a RTN space. If a sequence $x = (x_k)$ is $I(RTN)$-convergent, then $I(RTN)$-limit is unique.*

P r o o f. Suppose that $I(RTN)\text{-}\lim x = \xi_1$ and $I(RTN)\text{-}\lim x = \xi_2$. Given $\varepsilon > 0$ choose $r > 0$ such that $(1-r) * (1-r) > 1 - \varepsilon$. Then, for any $t > 0$ and nonzero $z \in X$, define the following sets as:

$$K_{\mathcal{F}_1}(r,t) = \big\{n \in \mathbb{N} : \ \mathcal{F}(x_n - \xi_1, z; t) \le 1 - r\big\},$$
$$K_{\mathcal{F}_2}(r,t) = \big\{n \in \mathbb{N} : \ \mathcal{F}(x_n - \xi_2, z; t) \le 1 - r\big\}.$$

Since $I(RTN)\text{-}\lim x = \xi_1$, we have $K_{\mathcal{F}_1}(r,t) \in I$. Similarly $I(RTN)\text{-}\lim x = \xi_2$ implies that $K_{\mathcal{F}_2}(r,t) \in I$. Now let $K_{\mathcal{F}}(r,t) = K_{\mathcal{F}_1}(r,t) \cup K_{\mathcal{F}_2}(r,t)$. Then from the definition of I, $K_{\mathcal{F}}(r,t) \in I$ and hence its compliment $K^C_{\mathcal{F}}(r,t)$ is a nonempty set which belongs to $F(I)$. Now if $k \in \mathbb{N} \backslash K_{\mathcal{F}}(r,t)$, then we have $\mathcal{F}(\xi_1 - \xi_2, z; t) \ge \mathcal{F}\big(x_n - \xi_1, z; \frac{t}{2}\big) * \mathcal{F}\big(x_n - \xi_2, z; \frac{t}{2}\big) > (1-r) * (1-r) > 1 - \varepsilon$.

Since $\varepsilon > 0$ was arbitrary, we get $\mathcal{F}(\xi_1 - \xi_2, z; t) = 1$ for all $t > 0$ and nonzero $z \in X$. Hence $\xi_1 = \xi_2$. $\square$

Theorem 3.2. *Let $(X, \mathcal{F}, *)$ be a RTN space. If $\mathcal{F}\text{-}\lim x = \xi$ then $I(RTN)\text{-}\lim x$ $= \xi$. But converse need not be true in general.*

P r o o f. Let $\mathcal{F}\text{-}\lim x = \xi$. Then for every $\varepsilon > 0$, $t > 0$ and nonzero $z \in X$, there is a positive integer N such that

$$\mathcal{F}(x_n - \xi, z; t) > 1 - \varepsilon$$

for all $n \ge N$. Since the set $A(\varepsilon) := \big\{k \in \mathbb{N} : \ \mathcal{F}(x_k - \xi, z; t) \le 1 - \varepsilon\big\} \subset \big\{1, 2, 3, \dots\big\}$ and the ideal I is admissible, $A(\varepsilon) \in I$. Hence $I(RTN)\text{-}\lim x = \xi$. $\square$

The following example shows that the converse need not be true.

Example 3.1. Let $X = \mathbb{R}^2$, with the 2-norm $\|x, z\| = |x_1 z_2 - x_2 z_1|$, $x = (x_1, x_2), z = (z_1, z_2)$, and $a * b = ab$ for all $a, b \in [0, 1]$. Let $\mathcal{F}(x, z; t) = \frac{t}{t+\|x,z\|}$ for every $x, z \in X$, $z_2 \neq 0$, and $t > 0$. Now we define a sequence $x = (x_k)$ by

$$x_k := \begin{cases} (k, 0); & \text{if } k = n^2, \ n \in \mathbb{N}, \\ (0, 0); & \text{otherwise}; \end{cases}$$

and write

$$K_n(\varepsilon, t) := \big\{k \leq n : \ \mathcal{F}(x_k - \xi, z; t) \leq 1 - \varepsilon\big\}, \qquad 0 < \varepsilon < 1; \ \ \xi = (0, 0).$$

We see that

$$\mathcal{F}(x_k - \xi, z; t) := \begin{cases} \frac{t}{t+kz_2}, & \text{if } k = n^2, \ n \in \mathbb{N}; \\ 1, & \text{otherwise}; \end{cases}$$

and hence

$$\lim_k \mathcal{F}(x_k - \xi, z; t) := \begin{cases} 0, & \text{if } k = n^2, n \in \mathbb{N}; \\ 1, & \text{otherwise}; \end{cases}$$

Therefore $x = (x_k)$ is not convergent in $(X, \mathcal{F}, *)$. But if we take $I = I_\delta = \big\{A \subset \mathbb{N} : \ \delta(A) = 0\big\}$, then since $K_n(\varepsilon, t) \subset \{1, 4, 9, 16, \dots\}$, $\delta(K_n(\varepsilon, t)) = 0$, that is $I(RTN)\text{-}\lim x = \xi$.

Theorem 3.3. *Let $(X, \mathcal{F}, *)$ be a RTN space. Let $I(RTN)\text{-}\lim x = \xi_1$ and $I(RTN)\text{-}\lim y = \xi_2$. Then*

(i) $I(RTN)\text{-}\lim(x \pm y) = \xi_1 \pm \xi_2$.

(ii) $I(RTN)\text{-}\lim \alpha x = \alpha \xi_1$, $\alpha \in \mathbb{R}$.

Easy to prove.

Theorem 3.4. *Let $(X, \mathcal{F}, *)$ be a RTN space. If $I^*(RTN)\text{-}\lim x = \xi$ then $I(RTN)\text{-}\lim x = \xi$.*

Proof. Suppose that $I^*(RTN)\text{-}\lim x = \xi$. Then $K = \big\{k_m : \ k_1 < k_2 < \cdots\big\} \in F(I)$ (i.e. $\mathbb{N} \backslash K = H$ (say) $\in I$) such that $\mathcal{F}\text{-}\lim_m x_{k_m} = \xi$. But then for each $\varepsilon > 0$ and $t > 0$ there exists a positive integer N such that $\mathcal{F}(x_{k_m} - \xi, z; t) > 1 - \varepsilon$ for all $m > N$. Since $\big\{k_m \in K : \ \mathcal{F}(x_{k_m} - \xi, z; t) \leq 1 - \varepsilon\big\}$ is contained in $\{k_1 < k_2 < \cdots < k_{N-1}\}$ and the ideal I is admissible, we have

$$\big\{k_m \in K : \ \mathcal{F}(x_{k_m} - \xi, z; t) \leq 1 - \varepsilon\big\} \in I.$$

Hence

$$\big\{k \in \mathbb{N} : \ \mathcal{F}(x_k - \xi, z; t) \leq 1 - \varepsilon\big\} \subseteq H \cup \{k_1 < k_2 < \cdots < k_{N-1}\} \in I$$

for all $\varepsilon > 0$ and $t > 0$. Therefore, we conclude that $I(RTN)\text{-}\lim x = \xi$. $\square$

Remark 3.1. The following example shows that the converse of Theorem 3.4 need not be true.

Example 3.2. Let $X = \mathbb{R}^2$, with the 2-norm $\|x, z\| = |x_1z_2 - x_2z_1|$, $x = (x_1, x_2)$, $z = (z_1, z_2)$, and $a * b = ab$ for all $a, b \in [0, 1]$. Let $\mathcal{F}(x, z; t) = \frac{t}{t+\|x,y\|}$ for every $x, y \in X$, $t > 0$.

Then $(X, \mathcal{F}, *)$ is a RTN space. Let $\mathbb{N} = \bigcup\limits_j \triangle_j$ be a decomposition of $\mathbb{N}$ such that for any $n \in \mathbb{N}$ each Δ_j contains infinitely many j's where $j \geq n$ and $\Delta_j \cap \Delta_n = \emptyset$ for $j \neq n$. Now we define a sequence $x = (x_n)$ by $x_n = \left(\frac{1}{j}, 0\right)$ if $n \in \triangle_j$. Then $\mathcal{F}(x_n, z; t) = \frac{t}{t+\|x_n,y\|} \to 1$ as $n \to \infty$. Hence $I(RTN)\text{-}\lim x = 0$.

Now suppose that $I^*(RTN)\text{-}\lim\limits_n x_n = 0$. Then there exists a subset $K = \{n_j : n_1 < n_2 < \cdots\}$ of $\mathbb{N}$ such that $K \in F(I)$ (i.e. $\mathbb{N} \setminus K = H$ (say) $\in I$) such that $\mathcal{F}\text{-}\lim\limits_j x_{n_j} = 0$. Since $K \in F(I)$ there is a set $H \in I$ such that $K = \mathbb{N} \backslash H$. Now, from the definition of I, there exists, say $p \in \mathbb{N}$ such that $H \subset \left(\bigcup\limits_{n=1}^{p} \Delta_n\right)$. But $\Delta_{p+1} \subset K$, and therefore $x_{n_j} = \left(\frac{1}{p+1}, 0\right) > (0, 0)$ for infinitely many n_j's from K which contradicts $\mathcal{F}\text{-}\lim\limits_j x_{n_j} = 0$. Therefore the assumption $I^*(RTN)\text{-}\lim\limits_n x_n = 0$ leads to the contradiction. Hence the converse of the theorem need not be true.

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Received 12. 6. 2009
Accepted 18. 9. 2010

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