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ON THE DIFFERENCE EQUATION $y_{n+1}=\frac{\alpha+y_n^p}{\beta y_{n-1}^p}-\frac{\gamma+y_{n-1}^p}{\beta y_n^p}$

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ABSTRACT. In this paper, we investigate the global stability and the periodic nature of solutions of the difference equation

$$y_{n+1}=\frac{\alpha+y_n^p}{\beta y_{n-1}^p}-\frac{\gamma+y_{n-1}^p}{\beta y_n^p}, \qquad n=0,1,2,\dots$$

where $\alpha,\beta,\gamma\in\big(0,\infty\big)$, $\alpha\big(1-p\big)-\gamma>0$, $0<p<1$, every $y_n\neq 0$ for $n=-1,0,1,2,\dots$ and the initial conditions y_{-1}, y_0 are arbitrary positive real numbers. We show that the equilibrium point of the difference equation is a global attractor with a basin that depends on the conditions of the coefficients.

1. Introduction and some basic observations

In [1], the global behavior of the difference equation

$$x_{n+1}=\alpha+\frac{x_{n-1}}{x_n^k}, \qquad n=0,1,2\dots,$$

was investigated by Alaa E. Hamza and A. Morsy, where $\alpha\in\big[0,\infty\big)$ and $k\in\big(0,\infty\big)$. In [2], the global stability, the boundedness character and the periodic nature of the positive solutions of the recursive sequence

$$x_{n+1}=\alpha+\frac{x_{n-1}}{x_n}, \qquad n=0,1,2,\dots,$$

was investigated by Amleh et al., where $\alpha\in\big[0,\infty\big)$ and the initial conditions x_{-1} and x_0 are arbitrary positive real numbers. They showed that a necessary and sufficient condition which every positive solution of the above equation be

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bounded is $\alpha \geq 1$. Furthermore in [3], the global stability and the periodic nature of the positive solutions of the difference equation

$$y_{n+1} = \alpha + \frac{y_{n-1}}{y_n} - \frac{y_n}{y_{n-1}}, \qquad n = 0, 1, 2, \ldots,$$

was investigated, where $\alpha > 0$, the initial conditions y_{-1} and y_0 are arbitrary positive real numbers.

In this paper, we investigate the periodic character and the global stability of solutions of the nonlinear second order rational difference equation

$$y_{n+1} = \frac{\alpha + y_n^p}{\beta y_{n-1}^p} - \frac{\gamma + y_{n-1}^p}{\beta y_n^p}, \qquad n = 0, 1, 2, \ldots, \tag{1}$$

where $\alpha, \beta, \gamma \in (0, \infty)$, the initial conditions y_{-1} and y_0 are arbitrary positive real numbers and every $y_n \neq 0$ for $n = -1, 0, 1, 2, \ldots$, $\alpha(1 - p) - \gamma > 0$ and $0 < p < 1$.

Other nonlinear, rational difference equations were investigated in [4]–[12].

We first recall some results which will be useful in the sequel.

Let $I \subset R$ and let $f \colon I \times I \to I$ be a continuous function. Consider the difference equation

$$y_{n+1} = f(y_n, y_{n-1}), \qquad n = 0, 1, 2, \ldots, \tag{2}$$

where the initial conditions $y_{-1}, y_0 \in I$. We say that $\overline{y}$ is an equilibrium point of Eq. (2) if

$$\overline{y} = f(\overline{y}, \overline{y}). \tag{3}$$

Let

$$s = \frac{\partial f}{\partial u}(\overline{y}, \overline{y}), \qquad t = \frac{\partial f}{\partial v}(\overline{y}, \overline{y}) \tag{4}$$

denote the partial derivatives of $f(u, v)$ evaluated at an equilibrium point $\overline{y}$ of Eq. (2). Then the equation

$$x_{n+1} = s x_n + t x_{n-1}, \qquad n = 0, 1, 2, \ldots, \tag{5}$$

is called the linearized equation associated with Eq. (2) about the equilibrium point $\overline{y}$ [4].

Theorem 1.1 (**Linearized Stability**). ([1, 4]) *Let* Eq. (5) *be the linearized equation associated with the* Eq. (2) *about the equilibrium point* $\overline{y}$. *The characteristic equation associated with* (5) *is*

$$\lambda^2 - s\lambda - t = 0. \tag{6}$$

(i) *If both roots of the quadratic equation* (6) *lie in the unit disk* $|\lambda| < 1$, *then the equilibrium point* $\overline{y}$ *of* Eq. (2) *is locally asymptotically stable.*

(ii) *If at least one of the roots of* Eq. (6) *has absolute value greater than one, then the equilibrium point* $\overline{y}$ *of* Eq. (2) *is unstable.*

(iii) *A necessary and sufficient condition for both roots of* Eq. (6) *to lie in the open unit disk* $|\lambda| < 1$, *is*

$$|s| < 1 - t < 2. \tag{7}$$

In this case the locally asymptotically stable equilibrium point $\overline{y}$ *is also called a sink.*

(iv) *A necessary and sufficient condition for both roots of* Eq. (6) *to have absolute value greater than one is*

$$|t| > 1 \qquad \text{and} \qquad |s| < |1 - t|. \tag{8}$$

In this case $\overline{y}$ *is called a repeller.*

(v) *A necessary and sufficient condition for one root of* Eq. (6) *to have absolute value greater than one and for the other to have absolute value less than one is*

$$s^2 + 4t > 0 \qquad \text{and} \qquad |s| > |1 - t|. \tag{9}$$

In this case the unstable equilibrium point $\overline{y}$ *is called a saddle point.*

(vi) *A necessary and sufficient condition that a root of* Eq. (6) *has absolute value equal to one is*

$$|s| = |1 - t| \tag{10}$$

or

$$t = -1 \qquad \text{and} \qquad |s| \leq 2. \tag{11}$$

In this case the equilibrium point $\overline{y}$ *is called non-hyperbolic point.*

2. The local stability of the equilibrium point of Eq. (1)

Consider the recursive sequence (1) where $\alpha, \beta, \gamma \in (0, \infty)$, the initial conditions y_{-1} and y_0 are arbitrary positive real numbers and every $y_n \neq 0$ for $n = -1, 0, 1, 2, \ldots$, $\alpha(1-p) - \gamma > 0$ and $0 < p < 1$.

In this section, we discuss the local stability of the equilibrium point $\overline{y}$ and period two solutions of Eq. (1).

The equilibrium point of Eq. (1) is

$$\overline{y} = \left(\frac{\alpha - \gamma}{\beta}\right)^{\frac{1}{p+1}} \tag{12}$$

and the linearized equation associated with Eq. (1) about the equilibrium point $\overline{y}$ is

$$x_{n+1} - \left(\frac{2p}{\beta\overline{y}} + \frac{p\gamma}{\alpha - \gamma}\right)x_n + \left(\frac{2p}{\beta\overline{y}} + \frac{p\alpha}{\alpha - \gamma}\right)x_{n-1} = 0, \qquad n = 0, 1, \ldots, \tag{13}$$

Hence its characteristic equation is

$$\lambda^2 - \left(\frac{2p}{\beta\overline{y}} + \frac{p\gamma}{\alpha-\gamma}\right)\lambda + \left(\frac{2p}{\beta\overline{y}} + \frac{p\alpha}{\alpha-\gamma}\right) = 0. \tag{14}$$

By using Theorem 1.1, we have the following results.

Theorem 2.1. *Suppose that* $\alpha, \beta, \gamma \in (0,\infty)$, $\alpha(1-p) - \gamma > 0$ *and* $0 < p < 1$. *Then the following statements are true.*

(i) *The positive equilibrium point* $\overline{y}$ *of* Eq. (1) *is locally asymptotically stable if*

$$\beta^{\frac{p}{p+1}} > \frac{2p(\alpha-\gamma)^{\frac{p}{p+1}}}{[\alpha(1-p)-\gamma]}. \tag{15}$$

(ii) *The positive equilibrium point* $\overline{y}$ *of* Eq. (1) *is unstable and in fact is a repeller point if*

$$\beta^{\frac{p}{p+1}} < \frac{2p(\alpha-\gamma)^{\frac{p}{p+1}}}{[\alpha(1-p)-\gamma]}. \tag{16}$$

(iii) *The positive equilibrium point* $\overline{y}$ *of* Eq. (1) *is non-hyperbolic point if*

$$\beta^{\frac{p}{p+1}} = \frac{2p(\alpha-\gamma)^{\frac{p}{p+1}}}{[\alpha(1-p)-\gamma]}. \tag{17}$$

P r o o f.

(i) To show the positive equilibrium point $\overline{y}$ of Eq. (1) is locally asymptotically stable, we will show that

$$\left|\frac{2p}{\beta\overline{y}} + \frac{p\gamma}{\alpha-\gamma}\right| < 1 + \frac{2p}{\beta\overline{y}} + \frac{p\alpha}{\alpha-\gamma} < 2. \tag{18}$$

Since $\alpha(1-p) - \gamma > 0$, then $\alpha > \gamma$. So from $\alpha > \gamma$ and $0 < p < 1$, we can write

$$\gamma p < (\alpha-\gamma) + \alpha p$$

from which we have

$$\frac{\gamma p}{\alpha-\gamma} < 1 + \frac{\alpha p}{\alpha-\gamma}$$

and

$$\frac{2p}{\beta\overline{y}} + \frac{\gamma p}{\alpha-\gamma} < 1 + \frac{2p}{\beta\overline{y}} + \frac{\alpha p}{\alpha-\gamma}. \tag{19}$$

By (19), we get

$$\left|\frac{2p}{\beta\overline{y}} + \frac{\gamma p}{\alpha-\gamma}\right| < 1 + \frac{2p}{\beta\overline{y}} + \frac{\alpha p}{\alpha-\gamma}. \tag{20}$$

On the other hand, from (15) we can write

$$2p(\alpha-\gamma)^{1-\frac{1}{p+1}} < \beta^{\frac{p}{p+1}}\left[\alpha(1-p)-\gamma\right]$$

and

$$2p(\alpha-\gamma)<\beta\left[\alpha(1-p)-\gamma\right]\left(\frac{\alpha-\gamma}{\beta}\right)^{\frac{1}{p+1}}. \tag{21}$$

$\alpha(1-p)-\gamma>0$, $0<p<1$ and by the definition of equilibrium point in (12) are taken into account that it is obtained

$$2p(\alpha-\gamma)<\left[\alpha(1-p)-\gamma\right]\beta\overline{y}$$

and

$$2p(\alpha-\gamma)+\alpha\beta p\overline{y}<(\alpha-\gamma)\beta\overline{y}. \tag{22}$$

Hence, using (22), we get

$$1+\frac{2p}{\beta\overline{y}}+\frac{p\alpha}{\alpha-\gamma}<2. \tag{23}$$

From (20) and (23), it is obtained (18). This completes the proof of the first part in the theorem (see Example 2.2 and Figure 1).

(ii) To show the positive equilibrium $\overline{y}$ of Eq. (1) is unstable (and in fact is a repeller point), Theorem 1.1 and (16) should be taken into account and

$$\left|-\left(\frac{2p}{\beta\overline{y}}+\frac{\alpha p}{\alpha-\gamma}\right)\right|>1 \qquad \text{and} \qquad \left|\frac{2p}{\beta\overline{y}}+\frac{\gamma p}{\alpha-\gamma}\right|<\left|1+\frac{2p}{\beta\overline{y}}+\frac{\alpha p}{\alpha-\gamma}\right| \tag{24}$$

should be indicated. In view of $\alpha>\gamma$ and $0<p<1$, we get

$$\gamma p<(\alpha-\gamma)+\alpha p$$

from which we have

$$\frac{\gamma p}{\alpha-\gamma}<1+\frac{\alpha p}{\alpha-\gamma}$$

and

$$\frac{2p}{\beta\overline{y}}+\frac{\gamma p}{\alpha-\gamma}<1+\frac{2p}{\beta\overline{y}}+\frac{\alpha p}{\alpha-\gamma}. \tag{25}$$

By (25), we get

$$\left|\frac{2p}{\beta\overline{y}}+\frac{\gamma p}{\alpha-\gamma}\right|<\left|1+\frac{2p}{\beta\overline{y}}+\frac{\alpha p}{\alpha-\gamma}\right|. \tag{26}$$

On the other hand, from (16) we can write

$$2p(\alpha-\gamma)^{1-\frac{1}{p+1}}>\beta^{\frac{p}{p+1}}\left[\alpha(1-p)-\gamma\right]$$

and

$$2p(\alpha-\gamma)>\beta\left[\alpha(1-p)-\gamma\right]\left(\frac{\alpha-\gamma}{\beta}\right)^{\frac{1}{p+1}}. \tag{27}$$

$\alpha(1-p)-\gamma>0$, $0<p<1$ and (12) are taken into account that it is obtained

$$2p(\alpha-\gamma)>\left[\alpha(1-p)-\gamma\right]\beta\overline{y}$$

and

$$2p(\alpha-\gamma)+\alpha\beta p\overline{y}>(\alpha-\gamma)\beta\overline{y}. \tag{28}$$

Using (28), we obtain

$$\Big| -\Big(\frac{2p}{\beta\overline{y}}+\frac{\alpha p}{\alpha-\gamma}\Big)\Big| > 1. \tag{29}$$

From (26) and (29), it is obtained (24). This completes the proof of the second part in the theorem (see Example 2.3 and Figure 2).

(iii) Let (17) holds. In this case we get

$$2p\big(\alpha-\gamma\big)^{1-\frac{1}{p+1}} = \beta^{\frac{p}{p+1}}\big[\alpha\big(1-p\big)-\gamma\big]$$

and

$$2p\big(\alpha-\gamma\big) = \beta\big[\alpha\big(1-p\big)-\gamma\big]\Big(\frac{\alpha-\gamma}{\beta}\Big)^{\frac{1}{p+1}}. \tag{30}$$

Since $\alpha\big(1-p\big)-\gamma<0$, $0<p<1$ and from (12), we have

$$2p\big(\alpha-\gamma\big)+\alpha\beta p\overline{y} = \big(\alpha-\gamma\big)\beta\overline{y}. \tag{31}$$

By using (31), we get

$$-\Big(\frac{2p}{\beta\overline{y}}+\frac{\alpha p}{\alpha-\gamma}\Big) = -1 \tag{32}$$

and it is obvious that $t=-1$.

Secondly from our hypothesis, since $0<p<1$, we can easily see that

$$\frac{\big(\alpha-\gamma\big)-p\big(\alpha-\gamma\big)}{\big(\alpha-\gamma\big)} < 2. \tag{33}$$

From (33), we have the following inequality

$$\frac{2p}{\Big\{\frac{(2p)^{p+1}(\alpha-\gamma)^{p}(\alpha-\gamma)}{[\alpha(1-p)-\gamma]^{p+1}}\Big\}^{\frac{1}{p+1}}}+\frac{\gamma p}{\alpha-\gamma} < 2. \tag{34}$$

Furthermore, considering (12), (17) and (34) together, we get that

$$\Big|\frac{2p}{\beta\overline{y}}+\frac{\gamma p}{\alpha-\gamma}\Big| < 2. \tag{35}$$

Thus, we obtain $|s|<2$. Moreover, we have $t=-1$. This completes the proof. □

Example 2.2. For $\alpha=10$, $\beta=4$, $\gamma=2$, $p=0.5$, $y_{-1}=2$ and $y_0=5$ the graph of the first 100 iterations of Eq. (1) is given in Figure 1. The graph suggests that the solutions of Eq. (1) is converging to a stable equilibrium value of about $\overline{y}$. If this converge continues in the form of a limit as values of y_n can be predicted to be at or extremely near the equilibrium value.

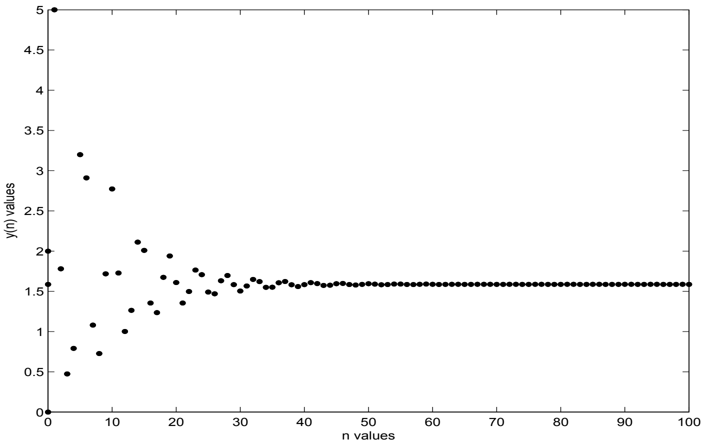

FIGURE 1. Graph of the iteration solutions of Eq. (1) for $\alpha = 10$, $\beta = 4$, $\gamma = 2$, $p = 0.5$, $y_{-1} = 2$ and $y_0 = 5$

Example 2.3. Let $\alpha = 10$, $\beta = 1/9$, $\gamma = 2$, $p = 0.5$, $y_{-1} = 2$ and $y_0 = 5$. Not that the values of α, β and γ verify (11). The graph of the first 100 iterations of Eq. (1) is given in Figure 2. It does not contain any of the predictable patterns of solutions. Absent of any pattern or repetition, the general long-term behavior and specific values of y_n for large n are impossible to predict from the graph.

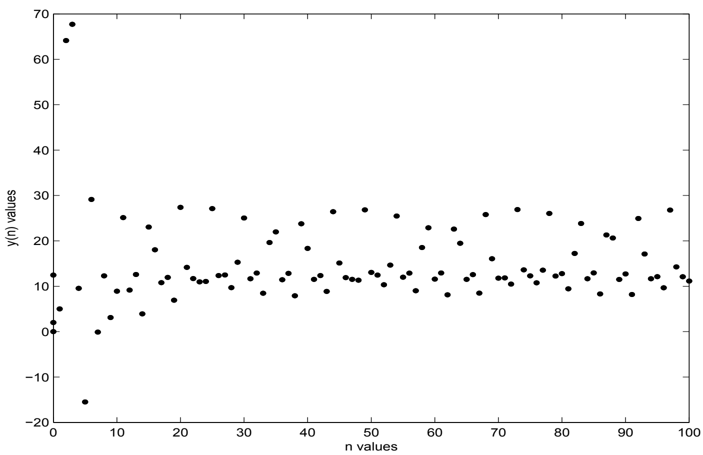

FIGURE 2. Graph of the iteration solutions of Eq. (1) for $\alpha = 10$, $\beta = 1/9$, $\gamma = 2$, $p = 0.5$, $y_{-1} = 2$ and $y_0 = 5$

3. Periodic nature of Eq. (1)

In this section, periodic structure of the solutions of Eq. (1) is emphasized.

Theorem 3.1. Eq. (1) *has a period two solution* $\{y_n\}_{n=-1}^{\infty}$ *if and only if* (y_{-1}, y_0) *is a solution of the system,*

$$x = \frac{\alpha + y^p}{\beta x^p} - \frac{\gamma + x^p}{\beta y^p}, \qquad y = \frac{\alpha + x^p}{\beta y^p} - \frac{\gamma + y^p}{\beta x^p} \tag{36}$$

Furthermore if $y_{-1} \neq y_0$, *then* $\{y_n\}_{n=-1}^{\infty}$ *is a prime period two solution.*

Proof. First, let $\{y_n\}_{n=-1}^{\infty}$ be a period two solution of Eq. (1), then we have

$$\begin{aligned} y_{-1} = y_1 &= \frac{\alpha + y_0^p}{\beta y_{-1}^p} - \frac{\gamma + y_{-1}^p}{\beta y_0^p} \\ y_0 = y_2 &= \frac{\alpha + y_{-1}^p}{\beta y_0^p} - \frac{\gamma + y_0^p}{\beta y_{-1}^p}. \end{aligned}$$

So, (y_{-1}, y_0) is a solution of the system (36).

Second, let (y_{-1}, y_0) be a solution of the system (36). Then we show that $y_{n+2} = y_n$ for all $n \geq -1$. Since

$$\begin{aligned} y_1 &= \frac{\alpha + y_0^p}{\beta y_{-1}^p} - \frac{\gamma + y_{-1}^p}{\beta y_0^p} = y_{-1} \\ y_2 &= \frac{\alpha + y_{-1}^p}{\beta y_0^p} - \frac{\gamma + y_0^p}{\beta y_{-1}^p} = y_0 \\ y_3 &= \frac{\alpha + y_0^p}{\beta y_{-1}^p} - \frac{\gamma + y_{-1}^p}{\beta y_0^p} = y_1 \end{aligned}$$

then, $y_3 = y_1$. So $y_{n+2} = y_n$ for $n = 1$. Now let $y_{n+2} = y_n$ for $n = k$. Furthermore for $n = k + 1$, we get

$$\begin{aligned} y_{k+2} &= \frac{\alpha + y_{k+1}^p}{\beta y_k^p} - \frac{\gamma + y_k^p}{\beta y_{k+1}^p} = y_k \\ y_{k+3} &= \frac{\alpha + y_{k+2}^p}{\beta y_{k+1}^p} - \frac{\gamma + y_{k+1}^p}{\beta y_{k+2}^p} = y_{k+1}. \end{aligned}$$

Hence, by induction we see that $y_{n+2} = y_n$ for all $n \geq -1$. □

4. Global asymptotic stability of Eq. (1)

In this section, global structure of the solutions of Eq. (1) is discussed.

Lemma 4.1. *Let* $\alpha, \beta, \gamma \in (0, \infty)$, $\alpha(1-p) - \gamma > 0$, $0 < p < 1$ *and*

$$f(u, v) = \frac{\alpha + u^p}{\beta v^p} - \frac{\gamma + v^p}{\beta u^p}.$$

Then the following statements are true.

(i) $f(x, x)$ *is a decreasing function in* $I = (0, \infty)$.

(ii) *If* $u, v \in I = (0, \infty)$, *then* $f(u, v)$ *is an increasing function in* u, *and a decreasing function in* v.

Proof.

(i) Consider the function

$$f(u, v) = \frac{\alpha + u^p}{\beta v^p} - \frac{\gamma + v^p}{\beta u^p}.$$

Then we can write

$$f(x, x) = \frac{\alpha - \gamma}{\beta x^p}.$$

Let $x_1 < x_2$, then we get

$$\beta x_1^p < \beta x_2^p \qquad \text{and} \qquad \frac{1}{\beta x_1^p} > \frac{1}{\beta x_2^p}.$$

On the other hand, from our hypothesis, we can write

$$\frac{\alpha - \gamma}{\beta x_1^p} > \frac{\alpha - \gamma}{\beta x_2^p}.$$

So, we obtain

$$f(x_1, x_1) > f(x_2, x_2).$$

Thus $f(x, x)$ is a decreasing function in $I = (0, \infty)$.

(ii) First, let $u, v \in I = (0, \infty)$ and $u_1 < u_2$. Then $\alpha + u_1^p < \alpha + u_2^p$ is obvious. Moreover, since $u, v \in I = (0, \infty)$ we obtain

$$\frac{\alpha + u_1^p}{\beta v^p} < \frac{\alpha + u_2^p}{\beta v^p}. \tag{37}$$

Furthermore, since $u, v \in I = (0, \infty)$, we get

$$\frac{1}{\beta u_1^p} > \frac{1}{\beta u_2^p} \qquad \text{and} \qquad -\frac{\gamma + v^p}{\beta u_1^p} < -\frac{\gamma + v^p}{\beta u_2^p}. \tag{38}$$

Considering (37) and (38) together, we get that $f(u_1, v) < f(u_2, v)$. Hence $f(u, v)$ is an increasing function in u.

Second, let $u, v \in I = \big(0, \infty\big)$ and $v_1 < v_2$. Then

$$\beta v_1^p < \beta v_2^p$$

and

$$\frac{1}{\beta v_1^p} > \frac{1}{\beta v_2^p}.$$

On the other hand, since $u, v \in I = \big(0, \infty\big)$, we have

$$\frac{\alpha + u^p}{\beta v_1^p} > \frac{\alpha + u^p}{\beta v_2^p}. \tag{39}$$

Moreover we get

$$-\frac{\gamma + v_1^p}{\beta u^p} > -\frac{\gamma + v_2^p}{\beta u^p}. \tag{40}$$

Thus from (39) and (40), it obvious that $f\big(u, v\big)$ is a decreasing function in v. This completes the proof. □

Theorem 4.2. *Let* $\alpha, \beta, \gamma \in \big(0, \infty\big)$, $\alpha\big(1 - p\big) - \gamma > 0$ *and* $0 < p < 1$. *Also we assume that* (15) *holds. Then the positive equilibrium point of* Eq. (1) *is globally asymptotically stable.*

Proof. Let $u, v \in \big(0, \infty\big)$ and set

$$f\big(u, v\big) = \frac{\alpha + u^p}{\beta v^p} - \frac{\gamma + v^p}{\beta u^p}.$$

Then $f\colon \big(0, \infty\big) \times \big(0, \infty\big) \to \big(0, \infty\big)$ is a continuous function and is an increasing function in u, and a decreasing function in v. Let $\mu, \Lambda \in \big(0, \infty\big) \times \big(0, \infty\big)$ is a solution of the system

$$\Lambda = f\big(\Lambda, \mu\big)$$
$$\mu = f\big(\mu, \Lambda\big)$$

then we can write

$$\Lambda = \frac{\alpha + \Lambda^p}{\beta \mu^p} - \frac{\gamma + \mu^p}{\beta \Lambda^p} \qquad \text{and} \qquad \mu = \frac{\alpha + \mu^p}{\beta \Lambda^p} - \frac{\gamma + \Lambda^p}{\beta \mu^p}. \tag{41}$$

From (41), we get

$$\mu + \Lambda = \frac{\alpha - \gamma}{\beta \Lambda^p} + \frac{\alpha - \gamma}{\beta \mu^p} \tag{42}$$

and

$$\mu - \Lambda = \frac{\alpha + \gamma + 2\mu^p}{\beta \Lambda^p} - \frac{\alpha + \gamma + 2\Lambda^p}{\beta \mu^p}. \tag{43}$$

By using (42) and (43) respectively, we have

$$\beta \Lambda^p \mu^p = \big(\alpha - \gamma\big) \frac{\mu^p + \Lambda^p}{\mu + \Lambda} \tag{44}$$

and

$$\big(\mu - \Lambda\big)\beta\Lambda^p\mu^p = \big(\mu^p - \Lambda^p\big)\big[\big(\alpha + \gamma\big) + 2\big(\mu^p + \Lambda^p\big)\big]. \tag{45}$$

By (45), we obtain

$$\big(\mu - \Lambda\big)\Big\{A\big[\big(\alpha + \gamma\big) + 2\big(\mu^p + \Lambda^p\big)\big] - \beta\Lambda^p\mu^p\Big\} = 0 \tag{46}$$

where $A = \mu^{p-1} + \mu^{p-2}\Lambda + \cdots + \mu\Lambda^{p-2} + \Lambda^{p-1}$.

Hence from (44) and (46), we get

$$\frac{\mu - \Lambda}{\mu + \Lambda}\Big\{\big(\mu^p + \Lambda^p + 2B\big)\big[\big(\alpha + \gamma\big) + 2\big(\mu^p + \Lambda^p\big)\big] - \big(\alpha - \gamma\big)\big(\mu^p + \Lambda^p\big)\Big\} = 0.$$

Arranging the last equality, we have

$$\frac{\mu - \Lambda}{\mu + \Lambda}\Big\{2\gamma\big(\mu^p + \Lambda^p\big) + 2\big(\mu^p + \Lambda^p\big)^2 + 2B\big(\alpha + \gamma\big) + 4B\big(\mu^p + \Lambda^p\big)\Big\} = 0$$

where $B = \mu^{p-1}\Lambda + \mu^{p-2}\Lambda^2 + \cdots + \mu^2\Lambda^{p-2} + \mu\Lambda^{p-1}$.

Since $B > 0$ and

$$2\gamma\big(\mu^p + \Lambda^p\big) + 2\big(\mu^p + \Lambda^p\big)^2 + 2B\big(\alpha + \gamma\big) + 4B\big(\mu^p + \Lambda^p\big) > 0,$$

then we get $\mu = \Lambda$. By using [6, Theorem D], $\overline{y}$ is globally asymptotically stable equilibrium point of Eq. (1). This completes the proof. □

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