VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-011-0042-1
Math. Slovaca **61** (2011), No. 5, 733–746

DENSITIES GENERATED BY EQUIVALENT MEASURES

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(*Communicated by David Buhagiar*)

ABSTRACT. In the family of all measures equivalent to the Borel measure μ defined in a metric space X, we look for measures generating the same density points (preserving μ-density at fixed point $x_0 \in X$).

1. Introduction

In the paper we consider the density points of a Borel set defined by equivalent Borel measures in a metric space. For the convenience of a reader let us repeat some basic definitions and considerations from our previous paper. The $\lim\limits_{h\to 0^+} \frac{\lambda(E\cap[x_0-h,x_0+h])}{2h}$ is called *the density of a Lebesgue measurable set* $E \subset \mathbb{R}$ at a point $x_0 \in \mathbb{R}$ with respect to Lebesgue measure λ. If the limit is equal to one, we say that x_0 is *a density point* of the set E. The notion of density point, defined at the beginning of XX century, has been studied and developed extensively since the notion of density topology was introduced by Haupt and Pauc in 1952 [HP]. It is known that a density point can be described using only σ-algebra of measurable sets and σ-ideal of null sets. Namely:

Theorem 1. ([PWW]) *Zero is a density point of a measurable set E with respect to Lebesgue measure if and only if any subsequence of the sequence of characteristic functions of sets $nE \cap [-1,1]$ contains a subsequence which converges to $\chi_{[-1,1]}$ almost everywhere.*

In the whole paper we shall assume that X is a metric space, μ is a Borel measure on X. We assume that $x_0 \in X$ belongs to the support of μ i.e. for any closed ball B_r with the center x_0 and arbitrary radius $r > 0$, we have $\mu(B_r) > 0$

2010 Mathematics Subject Classification: Primary 28A05; Secondary 28A25, 28A33.
Keywords: density point, equivalent measures.

and μ is finite on some ball B_{r_0}. The simple generalization of a notion of the density with respect to Lebesgue measure leads us to the following definition:

Definition 1. The limit

$$d = \lim_{r \to 0^+} \frac{\mu(A \cap B_r)}{\mu(B_r)}$$

is called the density of a Borel set A at a point x_0 with respect to measure μ. If $d = 1$ we say that x_0 is a μ-density point of the set A (or x_0 is a density point of A with respect to the measure μ) and a μ-dispersion point of the complement of A.

Recall that measure ν is *absolutely continuous* with respect to measure μ ($\nu \ll \mu$), defined on the same σ-algebra $\mathscr{F}$, if from the fact $\mu(A) = 0$ it follows that $\nu(A) = 0$. We say that measures μ and ν are *equivalent* if $\mu \ll \nu$ and $\nu \ll \mu$. It is well known that if μ and ν are σ-finite measures on $(X, \mathscr{F})$ and $\nu \ll \mu$, then there exists a nonnegative $\mathscr{F}$-measurable function f (called the *Radon-Nikodym derivative* of ν with respect to μ and denoted by $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$) such that for any $A \in \mathscr{F}$ we have $\nu(A) = \int_A f \,\mathrm{d}\mu$.

Since equivalent measures have the same σ-algebras of measurable sets and σ-ideals of null sets, there is a natural question of densities generated by equivalent measures.

In [B] it is shown that if $\nu = \int f \,\mathrm{d}\mu$ and

$$0 < m \leqslant f(x) \leqslant M \quad \mu\text{-almost everywhere on some neighbourhood of } x_0 \qquad (*)$$

then, for every measurable set A,

$$x_0 \text{ is a } \mu\text{-density point of } A \text{ if and only if } x_0 \text{ is a } \nu\text{-density point of } A \quad (**)$$

In [BFP] we have shown that conditions $(*)$ and $(**)$ are not equivalent. We have proved that sets of density points with respect to equivalent measures can be essentially different. In particular, a density point of a set A with respect to some Borel measure μ can be even a dispersion point of a set with respect to measure ν equivalent to μ. We have also considered a density with respect to the measure which is a limit of a sequence of equivalent measures. It appears that even the uniform (with respect to Borel subsets) convergence of measures does not preserve density points.

Assume that f is a μ-measurable nonnegative real function defined on X. From now on we will denote by μ_f the measure defined by the Radon-Nikodym type formula $\mu_f(A) = \int_A f \,\mathrm{d}\mu$.

Definition 2. We shall say that a measure μ_f preserves μ-density at x_0 if, for any Borel set $B \subset X$ such that x_0 is a μ-density point of B, x_0 is also a μ_f-density point of B.

The following problem seems to be natural:

– Assume that μ_f preserves μ-density at x_0. For which functions g's we have the same property?
– For which functions f's, μ_f preserves μ-density at x_0?

The aim of this paper is to present answers to these questions by specifying some sufficient conditions.

2. How can we modify the function f?

The partial answer to our first question is:

Theorem 2. *Assume that x_0 is a μ_f-density point of a given set A. Then*

a) *for any function h for which there are m, M such that*

$$0 < m \leqslant h(x) \leqslant M$$

for μ-almost all $x \in X$, x_0 is a μ_g-density point of A for the function $g(x) = f(x) \cdot h(x)$;

b) *for any function α satisfying the inequality*

$$0 < m \leqslant \alpha(x) \leqslant M < 1$$

for some m, M and μ-almost all $x \in X$, x_0 is a μ_g-density point of A, for the function g being a "convex combination" of 1 and f:

$$g(x) = \alpha(x) + (1 - \alpha(x)) \cdot f(x);$$

c) *x_0 is a μ_g-density point of A for $g(x) = \max\bigl(1, f(x)\bigr)$.*

Proof.

a) Due to inequalities

$$mf(x) \leqslant h(x) \cdot f(x) \leqslant Mf(x)$$

for μ-almost all x, we have for any Borel set C

$$m\mu_f(C) \leqslant \mu_g(C) \leqslant M\mu_f(C).$$

Therefore, denoting by A' the set $X \backslash A$,

$$0 \leqslant \frac{\mu_g(A' \cap B_r)}{\mu_g(B_r)} \leqslant \frac{M \cdot \mu_f(A' \cap B_r)}{m \cdot \mu_f(B_r)}$$

for any positive r. From

$$\lim_{r \to 0^+} \frac{\mu_f\,(A' \cap B_r)}{\mu_f(B_r)} = 0,$$

we obtain

$$\lim_{r \to 0^+} \frac{\mu_g\,(A' \cap B_r)}{\mu_g(B_r)} = 0.$$

b) By our assumptions $g(x)$ is positive for μ-almost all x and both limits $\lim\limits_{r\to 0^+} \frac{\mu(A'\cap B_r)}{\mu(B_r)}$ and $\lim\limits_{r\to 0^+} \frac{\mu_f(A'\cap B_r)}{\mu_f(B_r)}$ are equal to zero. Moreover, for any $r > 0$,

$$
\begin{aligned}
0 \leqslant \frac{\mu_g(A'\cap B_r)}{\mu_g(B_r)} &= \frac{\int\limits_{A'\cap B_r} [\alpha + (1-\alpha)f]\,\mathrm{d}\mu}{\int\limits_{B_r} [\alpha + (1-\alpha)f]\,\mathrm{d}\mu} \\
&\leqslant \frac{M\mu(A'\cap B_r)}{m\mu(B_r) + (1-M)\mu_f(B_r)} + \frac{(1-m)\mu_f(A'\cap B_r)}{m\mu(B_r) + (1-M)\mu_f(B_r)} \\
&\leqslant \frac{M\mu(A'\cap B_r)}{m\mu(B_r)} + \frac{(1-m)\mu_f(A'\cap B_r)}{(1-M)\mu_f(B_r)}.
\end{aligned}
$$

Hence

$$\lim_{r\to 0^+} \frac{\mu_g(A'\cap B_r)}{\mu_g(B_r)} = 0.$$

c) By assumptions we have $\lim\limits_{r\to 0^+} \frac{\mu(A'\cap B_r)}{\mu(B_r)} = 0$ and $\lim\limits_{r\to 0^+} \frac{\mu_f(A'\cap B_r)}{\mu_f(B_r)} = 0$. Let us decompose any ball B_r into two disjoint sets

$$B_r^+ = \{x \in B_r : \ f(x) \geqslant 1\} \qquad \text{and} \qquad B_r^- = \{x \in B_r : \ f(x) < 1\}.$$

For any $r > 0$

$$
\begin{aligned}
0 \leqslant \frac{\mu_g(A'\cap B_r)}{\mu_g(B_r)} &= \frac{\int\limits_{A'\cap B_r} \max(1,f)\,\mathrm{d}\mu}{\int\limits_{B_r} \max(1,f)\,\mathrm{d}\mu} \\
&= \frac{\int\limits_{A'\cap B_r^+} \max(1,f)\,\mathrm{d}\mu + \int\limits_{A'\cap B_r^-} \max(1,f)\,\mathrm{d}\mu}{\int\limits_{B_r^+} \max(1,f)\,\mathrm{d}\mu + \int\limits_{B_r^-} \max(1,f)\,\mathrm{d}\mu} \\
&= \frac{\int\limits_{A'\cap B_r^+} f\,\mathrm{d}\mu}{\int\limits_{B_r^+} f\,\mathrm{d}\mu + \int\limits_{B_r^-} 1\,\mathrm{d}\mu} + \frac{\mu(A'\cap B_r^-)}{\int\limits_{B_r^+} f\,\mathrm{d}\mu + \int\limits_{B_r^-} 1\,\mathrm{d}\mu} \\
&\leqslant \frac{\int\limits_{A'\cap B_r} f\,\mathrm{d}\mu}{\int\limits_{B_r^+} f\,\mathrm{d}\mu + \int\limits_{B_r^-} f\,\mathrm{d}\mu} + \frac{\mu(A'\cap B_r)}{\int\limits_{B_r^+} 1\,\mathrm{d}\mu + \int\limits_{B_r^-} 1\,\mathrm{d}\mu} \\
&= \frac{\mu_f(A'\cap B_r)}{\mu_f(B_r)} + \frac{\mu(A'\cap B_r)}{\mu(B_r)}.
\end{aligned}
$$

Hence

$$\lim_{r\to 0^+} \frac{\mu_g(A'\cap B_r)}{\mu_g(B_r)} = 0.$$

□

The same argument as in the part b) of the proof of the last theorem gives the following fact:

Remark 1. Suppose that x_0 is a density point of a Borel set A with respect to measures μ_{f_1} and μ_{f_2}. Then

(a) x_0 is a density point of A with respect to μ_g where g is a convex combination of f_1 and f_2

$$g(x) = \alpha f_1(x) + (1-\alpha) f_2(x),$$

where $\alpha \in (0,1)$;

(b) x_0 is a density point of A with respect to μ_g for the function g given by the formula

$$g(x) = \alpha(x) f_1(x) + (1-\alpha(x)) f_2(x),$$

where α is a function satisfying conditions from the part b) of Theorem 2.

Remark 2. Assume that μ_f preserves μ-density at x_0. Then, for any function g constructed as in Theorem 2 or Remark 1, the measure μ_g preserves μ-density at x_0, too.

However, for the function $g(x) = \min\bigl(1, f(x)\bigr)$, the thesis of Theorem 2 is not true. Before we construct an appropriate example let us define a useful class of sets of the form

$$A = \bigcup_{n=1}^{\infty} (B_{b_n} \backslash B_{a_n})$$

where $0 < b_{n+1} < a_n < b_n$ for any $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_n = 0$. We shall call them "ring sets". It is not difficult to check that:

Lemma 1. *For any "ring set" $A = \bigcup_{n=1}^{\infty} (B_{b_n} \backslash B_{a_n})$ we have*

$$\limsup_{r\to 0^+} \frac{\mu(A\cap B_r)}{\mu(B_r)} = \limsup_{n\to\infty} \frac{\mu(A\cap B_{b_n})}{\mu(B_{b_n})}$$

and

$$\liminf_{r\to 0^+} \frac{\mu(A\cap B_r)}{\mu(B_r)} = \liminf_{n\to\infty} \frac{\mu(A\cap B_{a_n})}{\mu(B_{a_n})}.$$

Example 1. Let us consider the space $[0,\infty)$ with the Lebesgue measure λ. Denote by a_n, b_n and c_n numbers $a_n = 2^{-n} - 4^{-n}$, $b_n = 2^{-n}$ and $c_n = a_n - 4^{-n} = b_n - 2\cdot 4^{-n}$. We define on $[0,\infty)$ the nonnegative measurable function f by the formula

$$f(x) = \begin{cases} 1 & \text{for } x \in \{0\} \cup \bigcup_{n=1}^{\infty} [a_n, b_n], \\ \frac{1}{2^{n-1}-1} & \text{for } x \in (b_{n+1}, c_n],\ n \in \mathbb{N}, \\ 2^{n-1} & \text{for } x \in (c_n, a_n),\ n \in \mathbb{N}. \end{cases}$$

We claim that zero is a dispersion point of the set

$$A = \bigcup_{n=1}^{\infty} (a_n, b_n]$$

with respect to λ as well as to λ_f and it is not a dispersion point of A with respect to λ_g, for $g = \min(1, f)$.

Easily, from Lemma 1, we obtain that

$$\begin{aligned}\limsup_{h\to 0^+} \frac{\lambda\left(A \cap [0,h]\right)}{h} &= \limsup_{n\to\infty} \frac{\lambda\left(A \cap [0,b_n]\right)}{b_n} \\ &= \lim_{n\to\infty} 2^n \sum_{k=n}^{\infty} 4^{-k} = \lim_{n\to\infty} \frac{4}{3} \cdot 2^{-n} = 0.\end{aligned}$$

Thus, zero is a dispersion point of A with respect to λ. To obtain that zero is a dispersion point of A with respect to λ_f, observe that

$$\lambda_f\left((c_n, a_n)\right) = 4^{-n} \cdot 2^{n-1} = 2^{-n-1} = \lambda([b_{n+1}, b_n])$$

and

$$\lambda_f((a_n, b_n]) = \lambda_f([a_n, b_n]) = \lambda([a_n, b_n]) = \lambda((a_n, b_n]).$$

Hence, for any $n \in \mathbb{N}$,

$$\lambda_f([0, b_n]) > \lambda([0, b_n]) \qquad \text{and} \qquad \lambda_f(A \cap [0, b_n]) = \lambda(A \cap [0, b_n]).$$

Consequently,

$$\lim_{h\to 0^+} \frac{\lambda_f\left(A \cap [0,h]\right)}{\lambda_f([0,h])} = \lim_{n\to\infty} \frac{\lambda_f\left(A \cap [0,b_n]\right)}{\lambda_f([0,b_n])} = 0.$$

Clearly,

$$g(x) = \begin{cases} 1 & \text{for } x \in \{0\} \cup \bigcup\limits_{n=1}^{\infty} (c_n, b_n], \\ \frac{1}{2^{n-1}-1} & \text{for } x \in (b_{n+1}, c_n], \quad n \in \mathbb{N}. \end{cases}$$

For any $k \in \mathbb{N}$,

$$\begin{aligned}\lambda_g((b_{k+1}, c_k]) &= (2^{-k} - 2 \cdot 4^{-k} - 2^{-k-1}) \cdot \frac{1}{2^{k-1}-1} \\ &= 4^{-k}(2^{k-1} - 2) \cdot \frac{1}{2^{k-1}-1} < 4^{-k}.\end{aligned}$$

Therefore,

$$\lambda_g((b_{k+1}, b_k]) < 3 \cdot \lambda_g([a_k, b_k])$$

and, for any $n \in \mathbb{N}$,

$$\frac{\lambda_g(A \cap [0, b_n])}{\lambda_g([0, b_n])} > \frac{1}{3}$$

what means that zero is not a dispersion point of A with respect to λ_g.

It is not difficult to check that the measure λ_f defined in the above example does not preserve λ-density at zero. Indeed, consider the set

$$B = \bigcup_{n=1}^{\infty} (c_n, a_n].$$

For any natural integer n

$$\frac{\lambda(B \cap [0, a_n])}{a_n} = \frac{1}{2^{-n} - 4^{-n}} \cdot \sum_{k=n}^{\infty} 4^{-k} = \frac{1}{2^{-n}(1 - 2^{-n})} \cdot \frac{4^{-n}}{1 - \frac{1}{4}} = \frac{4}{3} \cdot \frac{2^{-n}}{1 - 2^{-n}}$$

and, consequently,

$$\lim_{h \to 0^+} \frac{\lambda\left(B \cap [0, h]\right)}{h} = \lim_{n \to \infty} \frac{\lambda\left(B \cap [0, a_n]\right)}{a_n} = 0.$$

On the other hand, for any $k \in \mathbb{N}$,

$$\begin{aligned} \lambda_f((b_{k+1}, c_k]) &= (2^{-k} - 2 \cdot 4^{-k} - 2^{-k-1}) \frac{1}{2^{k-1} - 1} < 4^{-k} \\ &= \lambda_f([a_k, b_k]) < \lambda_f((c_k, a_k]) \end{aligned}$$

and, for any $n \in \mathbb{N}$,

$$\begin{aligned} \lambda_f([0, a_n]) &= \sum_{k=n}^{\infty} \lambda_f(a_{k+1}, a_k]) \\ &= \sum_{k=n}^{\infty} (\lambda_f(a_{k+1}, b_{k+1}]) + \lambda_f((b_{k+1}, c_k]) + \lambda_f((c_k, a_k]) \\ &< 3 \sum_{k=n}^{\infty} \lambda_f(c_k, a_k]) = 3\lambda_f(B \cap [0, a_n]). \end{aligned}$$

It means that $\limsup\limits_{h \to 0^+} \frac{\lambda_f(B \cap [0,h])}{\lambda_f([0,h])} \geqslant \frac{1}{3}$. Therefore zero is not a λ_f-density point of B', althought it is a λ-density point of this set.

In the next theorem we prove that μ_g with $g(x) = \min(1, f(x))$ preserves μ-density at x_0 provided that for any Borel set $A \subset X$ and decreasing to zero sequence (h_n), from the fact that $\lim\limits_{n \to \infty} \frac{\mu(A \cap B_{h_n})}{\mu(B_{h_n})} = 0$ it follows that $\lim\limits_{n \to \infty} \frac{\mu_f(A \cap B_{h_n})}{\mu_f(B_{h_n})} = 0$.

Theorem 3. *Assume that for any Borel set $A \subset X$ and any decreasing to zero sequence (h_n), the equality*

$$\lim_{n \to \infty} \frac{\mu\left(A \cap B_{h_n}\right)}{\mu(B_{h_n})} = 0$$

implies the equality

$$\lim_{n \to \infty} \frac{\mu_f\left(A \cap B_{h_n}\right)}{\mu_f(B_{h_n})} = 0$$

and $g = \min(1, f)$. Then, for any Borel set $A \subset X$ and any decreasing to zero sequence (h_n), the equality

$$\lim_{n\to\infty} \frac{\mu\left(A \cap B_{h_n}\right)}{\mu(B_{h_n})} = 0$$

implies the equality

$$\lim_{n\to\infty} \frac{\mu_g\left(A \cap B_{h_n}\right)}{\mu_g(B_{h_n})} = 0.$$

P r o o f. Suppose, on the contrary, that there exists a Borel set $A \subset X$ and a decreasing to zero sequence $(h_n) \searrow 0$ such that

$$\lim_{n\to\infty} \frac{\mu\left(A \cap B_{h_n}\right)}{\mu(B_{h_n})} = 0 \qquad \text{and} \qquad \limsup_{n\to\infty} \frac{\mu_g\left(A \cap B_{h_n}\right)}{\mu_g(B_{h_n})} > 0. \tag{1}$$

Then there exist a positive number α and a subsequence (h_{n_k}) of the sequence (h_n) such that for every $k \in \mathbb{N}$

$$\frac{\mu_g\left(A \cap B_{h_{n_k}}\right)}{\mu_g(B_{h_{n_k}})} > \alpha.$$

For the simplicity, we assume that

$$\frac{\mu_g\left(A \cap B_{h_n}\right)}{\mu_g(B_{h_n})} > \alpha \qquad \text{for every} \quad n \in \mathbb{N}. \tag{2}$$

Since $\lim_{n\to\infty} \frac{\mu(A\cap B_{h_n})}{\mu(B_{h_n})} = 0$, we have $\lim_{n\to\infty} \frac{\mu_f(A\cap B_{h_n})}{\mu_f(B_{h_n})} = 0$. Hence, there is a positive integer n_0 such that for $n \geqslant n_0$

$$\frac{\mu\left(A \cap B_{h_n}\right)}{\mu(B_{h_n})} < \frac{\alpha}{2} \qquad \text{and} \qquad \frac{\mu_f\left(A \cap B_{h_n}\right)}{\mu_f(B_{h_n})} < \frac{\alpha}{2}. \tag{3}$$

Let $C = \bigl\{x \in X : \ g(x) = 1\bigr\} = \bigl\{x \in X : \ f(x) \geqslant 1\bigr\}$. We shall show that for any $n \geqslant n_0$

$$\frac{\mu_f\left(C \cap B_{h_n}\right)}{\mu_f(B_{h_n})} > \frac{1}{2}. \tag{4}$$

Using (3) we obtain

$$\mu_f(B_{h_n}) > \frac{2}{\alpha}\mu_f(A \cap B_{h_n}) \geqslant \frac{2}{\alpha}\mu_g(A \cap B_{h_n})$$

and, from (2), we have

$$\mu_g(B_{h_n}) < \frac{1}{2}\mu_f(B_{h_n}).$$

Moreover,

$$\begin{aligned}\mu_f(B_{h_n}) &= \mu_f(C \cap B_{h_n}) + \mu_f(C' \cap B_{h_n}) \\ &= \mu_f(C \cap B_{h_n}) + \mu_g(C' \cap B_{h_n})\end{aligned}$$

$$\leqslant \mu_f(C \cap B_{h_n}) + \mu_g(B_{h_n}) < \mu_f(C \cap B_{h_n}) + \frac{1}{2}\mu_f(B_{h_n}).$$

Therefore $\mu_f(C \cap B_{h_n}) > \frac{1}{2}\mu_f(B_{h_n})$ which gives (4).

Now we shall prove that

$$\lim_{n\to\infty} \frac{\mu\left(C \cap B_{h_n}\right)}{\mu(B_{h_n})} = 0. \tag{5}$$

Fix $\varepsilon > 0$. Because of (1) there is a positive integer N such that for any $n \geqslant N$

$$\frac{\mu\left(A \cap B_{h_n}\right)}{\mu(B_{h_n})} < \varepsilon \cdot \alpha.$$

Using (2) and the definition of the function g we obtain

$$\alpha < \frac{\mu_g\left(A \cap B_{h_n}\right)}{\mu_g(B_{h_n})} \leqslant \frac{\mu\left(A \cap B_{h_n}\right)}{\mu_g(B_{h_n})} = \frac{\mu\left(A \cap B_{h_n}\right)}{\mu(B_{h_n})} \cdot \frac{\mu\left(B_{h_n}\right)}{\mu_g(B_{h_n})} < \varepsilon \cdot \alpha \cdot \frac{\mu\left(B_{h_n}\right)}{\mu_g(B_{h_n})}.$$

It follows that $\mu_g(B_{h_n}) < \varepsilon\mu(B_{h_n})$. Since $\mu_g(B_{h_n}) \geqslant \mu_g(C \cap B_{h_n}) = \mu(C \cap B_{h_n})$, we have $\mu_g(C \cap B_{h_n}) < \varepsilon\mu(B_{h_n})$ for any $n \geqslant N$. This proves (5).

Conditions (4) and (5) condradict the assumption of our theorem, which ends the proof. □

It is obvious that if μ_f satisfies the assumptions of the previous theorem then μ_f preserves μ-density at x_0. However, there is a metric space X, Borel measure μ on X and a measurable positive function f such that μ_f preserves μ-density at x_0 and does not satisfy the assumptions of Theorem 3.

Example 2. Let $X = \bigcup\limits_{n=1}^{\infty} \left\{\frac{1}{n}\right\} \cup \{0\}$ with natural metric, and μ be a measure defined as follows $\mu(\{0\}) = 0$ and $\mu\big(\left\{\frac{1}{n}\right\}\big) = \frac{1}{2^n}$ for $n \in \mathbb{N}$.

Observe first that zero is a μ-density point of $A \subset X$ if and only if the set $X \backslash A$ is finite. It is clear that if $X \backslash A$ is finite then zero is a density point of A with respect to any Borel measure ν on X, if only $\nu(\{0\}) = 0$ (and ν is positive on any ball B_r). Suppose now, that there is a decreasing to zero sequence (x_n) such that $x_n \in X \backslash A$ for every n. Then, for any n, $\mu(A \cap B_{x_n}) \leqslant \mu(B_{x_n}) - \mu\big(\{x_n\}\big)$. From the definition of μ it appears that

$$\mu(B_{x_n}) = 2\mu(\{x_n\}).$$

Hence

$$\frac{\mu(A \cap B_{x_n})}{\mu(B_{x_n})} < \frac{1}{2}$$

and zero is not a μ-density point of A. Therefore, for any positive function f, the measure μ_f preserves μ-density at zero.

Let $C = \bigcup\limits_{n=1}^{\infty} \{\frac{1}{2^n}\}$ and $h_n = \frac{1}{2^n+1}$ for $n = 1, 2, \dots$. Then

$$\mu\left(C \cap B_{h_n}\right) = \sum_{k=n+1}^{\infty} \mu\left(\left\{\frac{1}{2^k}\right\}\right) = \sum_{k=n+1}^{\infty} \frac{1}{2^{2^k}} < \frac{2}{2^{2^{n+1}}}$$

and

$$\lim_{n\to\infty} \frac{\mu\left(C \cap B_{h_n}\right)}{\mu(B_{h_n})} < \lim_{n\to\infty} \frac{\frac{2}{2^{2^{n+1}}}}{\frac{2}{2^{n+1}}} = \lim_{n\to\infty} \frac{2^{n+1}}{2^{2^{n+1}}} = 0.$$

Therefore the sequence $\left(\frac{\mu(C\cap B_{h_n})}{\mu(B_{h_n})}\right)$ tends to zero.

Define a function f in such a way that

$$\mu_f\left(\left\{\frac{1}{2^{n+1}}\right\}\right) = \mu(\{h_n\}) \qquad \text{and} \qquad \mu_f(\{h_n\}) = \mu\left(\left\{\frac{1}{2^{n+1}}\right\}\right)$$

for $n \in \mathbb{N}$ and $\mu_f(\{x\}) = \mu(\{x\})$ for other $x \in X$.

Namely, let

$$f(x) = \begin{cases} \frac{2^{2^{n+1}}}{2^{2^n+1}} & \text{for } x = \frac{1}{2^{n+1}},\ n \in \mathbb{N}; \\ \frac{2^{2^n+1}}{2^{2^{n+1}}} & \text{for } x = \frac{1}{2^n+1},\ n \in \mathbb{N}; \\ 1 & \text{for } x \notin \bigcup\limits_{n=2}^{\infty} \{\frac{1}{2^n}\} \cup \bigcup\limits_{n=1}^{\infty} \{\frac{1}{2^n+1}\}. \end{cases}$$

For any n we have $\mu_f(B_{h_n}) = \mu(B_{h_n})$, so

$$\frac{\mu_f\left(C \cap B_{h_n}\right)}{\mu_f(B_{h_n})} > \frac{\mu_f\left(\{\frac{1}{2^{n+1}}\}\right)}{\mu(B_{h_n})} = \frac{\mu(\{h_n\})}{2\mu(\{h_n\})} = \frac{1}{2}$$

and the sequence $\left(\frac{\mu_f(C\cap B_{h_n})}{\mu_f(B_{h_n})}\right)$ does not tend to zero.

3. When μ_f preserves μ-density?

Now, let us turn to our second problem: for which functions f's, μ_f preserves μ-density at x_0. Of course, it is sufficient to assume that there are numbers m and M such that for any Borel set A

$$0 < m \cdot \mu(A) \leqslant \mu_f(A) \leqslant M \cdot \mu(A).$$

(In fact it is equivalent to the condition $(*)$). However, this is not enough if we assume the same inequalities only for balls centered at x_0.

Example 3. Denote by λ the Lebesgue measure on the plane (treated as the complex plane). For $x = re^{i\varphi} \in \mathbb{C}$ let us define

$$f(x) = \begin{cases} 1 & \text{for } r > 1, \\ 2^n - 1 & \text{for } \frac{1}{n+1} < r \leqslant \frac{1}{n} \text{ and } \varphi \in (-\frac{\pi}{2^n}, \frac{\pi}{2^n}), \\ \frac{1}{2^n-1} & \text{for } \frac{1}{n+1} < r \leqslant \frac{1}{n} \text{ and } \varphi \in (-\pi, -\frac{\pi}{2^n}] \cup [\frac{\pi}{2^n}, \pi]. \end{cases}$$

We take origin to be x_0. We will check that measures λ and λ_f are the same on balls centered at origin.

Fix a point $r_0 \in (0,1)$. There is a positive integer j such that $\frac{1}{j+1} < r_0 \leqslant \frac{1}{j}$. Denote by X_j the set $\left\{re^{i\varphi} :\ r \in (0,1), \varphi \in \left(-\frac{\pi}{2^j}, \frac{\pi}{2^j}\right)\right\}$ and observe that

$$\begin{aligned} &\lambda_f\left(B_{r_0} \setminus B_{\frac{1}{j+1}}\right) \\ &= \lambda_f\left(\left(B_{r_0} \setminus B_{\frac{1}{j+1}}\right) \cap X_j\right) + \lambda_f\left(\left(B_{r_0} \setminus B_{\frac{1}{j+1}}\right) \setminus X_j\right) \\ &= \left(2^j - 1\right) \lambda\left(\left(B_{r_0} \setminus B_{\frac{1}{j+1}}\right) \cap X_j\right) + \frac{1}{2^j - 1}\lambda\left(\left(B_{r_0} \setminus B_{\frac{1}{j+1}}\right) \setminus X_j\right) \\ &= \lambda\left(B_{r_0} \setminus B_{\frac{1}{j+1}}\right). \end{aligned}$$

In the same way we check that, for any $n > j$, $\lambda_f\left(B_{\frac{1}{n}} \setminus B_{\frac{1}{n+1}}\right) = \lambda\left(B_{\frac{1}{n}} \setminus B_{\frac{1}{n+1}}\right)$. Consequently,

$$\lambda_f\left(B_{r_0}\right) = \lambda_f\left(B_{r_0} \setminus B_{\frac{1}{j+1}}\right) + \sum_{k=n+1}^{\infty} \lambda_f\left(B_{\frac{1}{n}} \setminus B_{\frac{1}{n+1}}\right) = \lambda\left(B_{r_0}\right).$$

Let

$$A = \bigcup_{n=1}^{\infty} \left\{re^{i\varphi} :\ \frac{1}{n+1} < r \leqslant \frac{1}{n} \text{ and } \varphi \in (-\frac{\pi}{2^n}, \frac{\pi}{2^n})\right\}$$

We will show that zero is a dispersion point of A with respect to λ although it is a density point with respect to λ_f. Indeed, for any $n \in \mathbb{N}$,

$$\lambda(A \cap B_{\frac{1}{n}}) = \sum_{k=n}^{\infty} \frac{\pi}{2^k}\left[\left(\frac{1}{k}\right)^2 - \left(\frac{1}{k+1}\right)^2\right] \leqslant \frac{\pi}{2^n \cdot n^2}$$

and

$$\lambda(B_{1/n}) = \frac{\pi}{n^2}.$$

Hence for $\frac{1}{n+1} < r \leqslant \frac{1}{n}$ we have $\lambda(A \cap B_r) \leqslant \frac{\pi}{2^n \cdot n^2}$ and evidently $\lambda(B_r) = \pi r^2$. Thus

$$0 < \frac{\lambda\left(A \cap B_r\right)}{\lambda(B_r)} < \frac{1}{2^n}$$

and zero is a despersion point of A with respect to λ.

On the other hand, for $\frac{1}{n+1} < r \leqslant \frac{1}{n}$,

$$\begin{aligned}
&\lambda_f(A \cap B_r) \\
&= \lambda_f(A \cap B_{1/n+1}) + \lambda_f(A \cap \left(B_r \setminus B_{1/n+1}\right)) \\
&= \sum_{k=n+1}^{\infty} (2^k - 1)\lambda\left(A \cap \left(B_{1/k} \setminus B_{1/k+1}\right)\right) + (2^n - 1)\,\lambda(A \cap \left(B_r \setminus B_{1/n+1}\right)) \\
&> \frac{2^{n+1}-1}{2^{n+1}}\lambda\left(B_{1/n+1}\right) + \frac{2^n-1}{2^n}\lambda\left(\left(B_r \setminus B_{1/n+1}\right)\right) > \frac{2^n-1}{2^n}\lambda\left(B_r\right).
\end{aligned}$$

Therefore,

$$\frac{\lambda_f\left(A \cap B_r\right)}{\lambda_f(B_r)} = \frac{\lambda_f\left(A \cap B_r\right)}{\lambda(B_r)} > \frac{2^n-1}{2^n}.$$

It means that zero is a density point of A with respect to λ_f.

Now we try to add some additional condition to the comparability of measures μ and μ_f on balls centered at x_0, to obtain a sufficient condition for preservability of μ-density at x_0.

Theorem 4. *Let $F_n = \{x : \ f(x) \geqslant n\}$. Assume that there exist positive numbers M and r_0 such that for any $0 < r \leqslant r_0$ we have $\mu(B_r) \leqslant M\mu_f(B_r)$ and the double limit*

$$\lim_{\substack{r\to 0^+\\ n\to\infty}} \frac{\mu_f\left(F_n \cap B_r\right)}{\mu(B_r)}$$

is equal to zero. Then μ_f preserves μ-density at x_0.

P r o o f. Let $\lim\limits_{r\to 0^+} \frac{\mu\left(A' \cap B_r\right)}{\mu(B_r)} = 0$, where A is a Borel set. Then for every positive integer n and $r \in (0, r_0]$ we have

$$\begin{aligned}
\frac{\mu_f\left(A' \cap B_r\right)}{\mu_f(B_r)} &= \frac{\mu_f\left(A' \cap F_n \cap B_r\right) + \mu_f\left(A' \cap F_n' \cap B_r\right)}{\mu_f(B_r)} \\
&\leqslant \frac{\mu(B_r)}{\mu_f(B_r)}\left(\frac{\mu_f\left(F_n \cap B_r\right)}{\mu(B_r)} + \frac{\mu_f\left(A' \cap F_n' \cap B_r\right)}{\mu(B_r)}\right) \\
&\leqslant M\left(\frac{\mu_f\left(F_n \cap B_r\right)}{\mu(B_r)} + n\frac{\mu\left(A' \cap B_r\right)}{\mu(B_r)}\right).
\end{aligned}$$

Fix $\varepsilon > 0$. There exists a number n_0 and $R > 0$ such that for $n \geqslant n_0$ and $r \leqslant R$ we have

$$\frac{\mu_f\left(F_n \cap B_r\right)}{\mu(B_r)} < \frac{\varepsilon}{2M}.$$

Moreover, there exists $R_1 > 0$ such that for $r \leqslant R_1$

$$\frac{\mu\left(A' \cap B_r\right)}{\mu(B_r)} < \frac{\varepsilon}{2n_0M}.$$

Hence for $r \leqslant \min(R, R_1)$ we have

$$\frac{\mu_f\left(A' \cap B_r\right)}{\mu_f(B_r)} < \varepsilon$$

which ends the proof. □

Finally, let us formulate two symmetric conditions which — taken together — are slightly weaker than $(*)$ and a bit stronger than comparability on balls. Unfortunately, the conditions look rather complicated. Denote

$$(\triangle) \quad \underset{\varepsilon>0}{\forall} \quad \underset{Z_\varepsilon \in \mathrm{Borel}(X)}{\exists} \quad \underset{M_\varepsilon>0}{\exists} \quad \underset{r_\varepsilon>0}{\exists} \quad \underset{0<r<r_\varepsilon}{\forall} \quad \underset{x \in B_r \setminus Z_\varepsilon}{\forall} \quad \left(\frac{\mu_f\left(Z_\varepsilon \cap B_r\right)}{\mu(B_r)} < \varepsilon \quad \& \quad |f(x)| \leqslant M_\varepsilon\right)$$

and

$$(\triangle') \quad \underset{\varepsilon>0}{\forall} \quad \underset{X_\varepsilon \in \mathrm{Borel}(X)}{\exists} \quad \underset{K_\varepsilon>0}{\exists} \quad \underset{r_\varepsilon>0}{\exists} \quad \underset{0<r<r_\varepsilon}{\forall} \quad \underset{x \in B_r \setminus X_\varepsilon}{\forall} \quad \left(\frac{\mu\left(X_\varepsilon \cap B_r\right)}{\mu_f(B_r)} < \varepsilon \quad \& \quad |f(x)| \geqslant K_\varepsilon\right).$$

Lemma 2. *If the function f satisfies the condition $\triangle$ (Δ') then there exist $K > 0$ ($M > 0$) and $r_0 > 0$ such that*

$$\mu_f(B_r) \leqslant K\mu(B_r) \qquad (\mu(B_r) \leqslant M\mu_f(B_r))$$

for any $0 < r \leqslant r_0$.

P r o o f. Fix $\varepsilon > 0$. Let M_ε, r_ε and Z_ε be choosen in $\triangle$. Let $r_0 = r_\varepsilon$ and $K = M_\varepsilon + \varepsilon$. For any $r \leqslant r_0$

$$\begin{aligned} \frac{\mu_f(B_r)}{\mu(B_r)} &= \frac{1}{\mu(B_r)}\big(\mu_f(B_r \setminus Z_\varepsilon) + \mu_f(B_r \cap Z_\varepsilon)\big) \\ &\leqslant \frac{M_\varepsilon \mu(B_r \setminus Z_\varepsilon)}{\mu(B_r)} + \varepsilon \leqslant M_\varepsilon + \varepsilon = K. \end{aligned}$$

Obviously, the proof for Δ' is analogous. □

Theorem 5. *If the function f satisfies the conditions $\triangle$ and Δ' then x_0 is a μ-density point of a Borel set Z if and only if x_0 is a μ_f-density point of Z.*

P r o o f. Suppose that, for some Borel set A, $\lim\limits_{r \to 0^+} \frac{\mu\left(A' \cap B_r\right)}{\mu(B_r)} = 0$ and fix $\varepsilon > 0$.

By the Lemma 2, there exist positive numbers M and r_0 such that for any $0 < r \leqslant r_0$ we have $\mu(B_r) \leqslant M\mu_f(B_r)$. Let Z_ε, M_ε, r_ε be as in $\triangle$ for some $\varepsilon_0 < \frac{\varepsilon}{2M}$. There exists $R > 0$ such that for $0 < r \leqslant R$ we have

$$\frac{\mu\left(A' \cap B_r\right)}{\mu(B_r)} < \frac{\varepsilon}{2M_\varepsilon \cdot M}.$$

Then for $r < \min(r_0, r_\varepsilon, R)$

$$\begin{aligned}\frac{\mu_f\left(A' \cap B_r\right)}{\mu_f(B_r)} &= \frac{\mu_f\left(A' \cap B_r \cap Z_\varepsilon\right) + \mu_f\left(A' \cap B_r \cap Z'_\varepsilon\right)}{\mu_f(B_r)} \\ &\leqslant \frac{\mu(B_r)}{\mu_f(B_r)}\left(\frac{\mu_f\left(B_r \cap Z_\varepsilon\right)}{\mu(B_r)} + \frac{\mu_f\left(A' \cap B_r \cap Z'_\varepsilon\right)}{\mu(B_r)}\right) \\ &\leqslant M\left(\varepsilon_0 + M_\varepsilon \frac{\mu\left(A' \cap B_r\right)}{\mu(B_r)}\right) < \varepsilon.\end{aligned}$$

The converse implication we obtain in the same way □

REFERENCES

[B] BARTOSZEWICZ, A.: *On density points of subsets of metric space with respect to the measure given by Radon-Nikodym derivative*, Real Anal. Exchange **23** (1997/98), 783–786.

[BFP] BARTOSZEWICZ, A.—FILIPCZAK, M.—POREDA, T.: *On density with respect to equivalent measures*, Demonstratio Math. **43** (2010), 21–28.

[HP] HAUPT, O.—PAUC, H.: *La topologie de Denjoy envisagee comme vraie topologie*, C. R. Math. Acad. Sci. Paris **234** (1952), 390–392.

[PWW] POREDA, W.—WAGNER-BOJAKOWSKA, E.—WILCZYŃSKI, W.: *A category analogue of the density topology*, Fund. Math. **125** (1985), 167–173.

Received 18. 3. 2009
Accepted 24. 9. 2009

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