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NEIGHBORHOODS AND CONVERGENCE WITH RESPECT TO A CLOSURE OPERATOR

Josef Šlapal

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ABSTRACT. We study neighborhoods with respect to a categorical closure operator. In particular, we discuss separation and compactness obtained from neighborhoods in a natural way and compare them with the usual closure separation and closure compactness. We also introduce a concept of convergence based on using centered systems of subobjects, which naturally generalizes the classical filter convergence in topological spaces. We investigate behavior of the convergence introduced and show, among others, that it relates to the separation and compactness in natural ways.

1. Introduction

The theory of categorical closure operators was founded by D. Dikranjan and E. Giuli in [15] and then developed by these authors and W. Tholen in [16]. Categories with a closure operator generalize the category Top of topological spaces and continuous maps and, therefore, there is a natural problem of extending classical topological concepts from topological spaces to objects of these categories. A number of recent papers on the theory of categorical closure operators are devoted to the study of these extended concepts. For example, separation and compactness are studied in [6], [10], [12] and [17], connectedness in [3], [7], [9] and [11], openness in [23] and quotient maps in [13]. In the present paper we study another concept with respect to a categorical closure operator, namely neighborhoods.

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Many topological concepts may simply be defined with the help of the Kuratowski closure operator, without using open sets or neighborhoods. But, on the other hand, neighborhoods may be used as a basic concept for introducing and studying topological spaces — cf. [14]. And neighborhoods become even needed when, for instance, we want to introduce convergence in a natural way. For this reason, a concept of a neighborhood with respect to a categorical closure operator was defined in [21] and then studied in [22]. In [22], concepts of separation and compactness obtained from neighborhoods in a natural way are introduced and studied. In the present paper, we continue the study from [22] an compare the separation and compactness with the usual closure separation and closure compactness. We then use neighborhoods for introducing convergence in a natural way. The convergence is expressed with the help of centered systems of subobjects and we show that it behaves analogously to the topological convergence of filters. We also show that the convergence relates the separation and compactness in the usual way.

2. Preliminaries

The present paper is a continuation of [22]. To make it self-contained, we repeat definitions of all concepts used and recall all relevant results (without proofs) and examples from [22].

For the general categorical terminology used see [1] and for that concerning categorical closure operators see [4] and [18]. The lattice-theoretic concepts and results used are taken from [24] and the topological ones from [19]. Let $\mathscr{X}$ be a finitely complete category with a proper $(\mathscr{E}, \mathscr{M})$-factorization structure for morphisms (here, the properness means that $\mathscr{E}$ is a class of epimorphisms and $\mathscr{M}$ is a class of monomorphisms in $\mathscr{X}$). For simplicity, $\mathscr{X}$ is assumed to have the properties that multiple pullbacks of arbitrary large families of $\mathscr{M}$-morphisms with a common codomain exist (and hence belong to $\mathscr{M}$) and that $\mathscr{E}$ is stable under pullbacks along $\mathscr{M}$-morphisms. Given an $\mathscr{X}$-object X, each $\mathscr{M}$-morphism with the codomain X is called a *subobject* of X. We denote by $\operatorname{sub} X$ the *subobject lattice* of X, i.e., the possibly large complete lattice of all (isomorphism classes of) subobjects of X. As usual, we identify isomorphism classes of subobjects of X with their representatives. So, each subobject of X is considered to be an element of $\operatorname{sub} X$, and we write $m = n$ instead of $m \simeq n$ for subobjects m, n of X. In the same way, by saying that m and n are different, in symbols $m \neq n$, we mean that m and n are not isomorphic. The joins and meets in $\operatorname{sub} X$ are denoted by the usual symbols $\vee$, $\bigvee$ and $\wedge$, $\bigwedge$, respectively. The least element of $\operatorname{sub} X$ is denoted by o_X (of course, the identity morphism id_X is the greatest element of $\operatorname{sub} X$). If $\mathrm{id}_X = o_X$, then the $\mathscr{X}$-object X is called *trivial*.

For any $m \in \operatorname{sub} X$, $\overline{m}$ denotes the pseudocomplement of m — provided it exists. Recall that, in a (possibly large) lattice L with a least element 0, an element $\overline{x} \in L$ is said to be a pseudocomplement of an element $x \in L$ if $x \wedge y = 0 \iff y \leq \overline{x}$ is valid whenever $y \in L$. It immediately follows that $x \leq \overline{\overline{x}}$ and that $x \leq y \Longrightarrow \overline{y} \leq \overline{x}$, hence $\overline{x} = \overline{\overline{\overline{x}}}$ whenever $x, y \in L$ and the corresponding pseudocomplements exist. An element $x \in L$ is said to be *pseudocomplementable* if it has a pseudocomplement, and the lattice L is called *pseudocomplemented* provided that all of its elements are pseudocomplementable. If L is pseudocomplemented and such that $\overline{\overline{x}} = x$ for every $x \in L$, then L is a Boolean algebra (with $\overline{x}$ the complement of x) — see [24]. Recall also that a lattice L with a least element 0 is said to be *atomic* if, for each element $x \in L$, $x \neq 0$, there is an atom p of L such that $p \leq x$, and it is said to be *atomistic* provided that each element of L is even a join of a class of atoms of L. Note that every atom $p \in \operatorname{sub} X$ has the property $p \leq m$ or $p \leq \overline{m}$ whenever $m \in \operatorname{sub} X$ is a pseudocomplementable subobject (because $p \not\leq m \Longrightarrow p \wedge m = o_X \Longrightarrow p \leq \overline{m}$).

Given an $\mathscr{X}$-morphism $f\colon X \to Y$ and subobjects $m \in \operatorname{sub} X$ and $n \in \operatorname{sub} Y$, we denote by $f(m)$ the $\mathscr{M}$-part of the $(\mathscr{E}, \mathscr{M})$-factorization of $f \circ m$ and by $f^{-1}(n)$ the inverse image of n (given by the corresponding pullback) along f.

Throughout the paper, we assume that every $\mathscr{X}$-morphism $f\colon X \to Y$ satisfies $f^{-1}(o_Y) = o_X$ (or, equivalently, $\forall m \in \operatorname{sub} X: \;\; f(m) = o_Y \Longrightarrow m = o_X$). This assumption and the stability of $\mathscr{E}$ under pullbacks along $\mathscr{M}$-morphisms result in the following lemma:

Lemma 2.1. ([22]) *Let $X = \prod_{i \in I} X_i$ be a product in $\mathscr{X}$ and $p_i \in \operatorname{sub} X_i$ be an atom for each $i \in I$. If all p_i, $i \in I$, have the same domain (up to isomorphisms), then $\langle p_i;\, i \in I\rangle \in \operatorname{sub} X$ is an atom, too.*

Further, we suppose there is given a concrete category $\mathscr{K}$ over $\mathscr{X}$ with the corresponding underlying functor $|\;|\colon \mathscr{K} \to \mathscr{X}$. As usual, we do not distinguish notationally between $\mathscr{K}$-morphisms and their underlying $\mathscr{X}$-morphisms (i.e., we write f instead of $|f|$ whenever f is a $\mathscr{K}$-morphism). Given a $\mathscr{K}$-object K, by a *subobject* of K we will always mean a subobject of $|K|$ and, correspondingly, we will write briefly $\operatorname{sub} K$ and o_K instead of $\operatorname{sub} |K|$ and $o_{|K|}$, respectively. This will cause no confusion because only the category $\mathscr{X}$, and not $\mathscr{K}$, is assumed to have a subobject structure. The category $\mathscr{K}$ is also supposed to have finite concrete products, and by a (not necessarily finite) product in $\mathscr{K}$ we always mean a concrete one.

Recall that a *closure operator* on $\mathscr{K}$ (with respect to $(\mathscr{E}, \mathscr{M})$) is a family of maps $c = (c_K\colon \operatorname{sub} K \to \operatorname{sub} K)_{K \in \mathscr{K}}$ with the following properties that hold for each $\mathscr{K}$-object K and each $m, p \in \operatorname{sub} K$:

(1) $m \leq c_K(m)$,

(2) $m \leq p \Longrightarrow c_K(m) \leq c_K(p)$,

(3) $f(c_K(m)) \leq c_L(f(m))$ for each $\mathscr{K}$-morphism $f\colon K \to L$.

In fact, the closure operator introduced is a so-called *closure operator with respect to the underlying functor* $|\ |$ — see [5]. It is more general than the classical closure operator introduced in [15], which is obtained when $\mathscr{K} = \mathscr{X}$ and $|\ |$ is the identity functor. Using the above concept of a closure operator, we substantially reduce the restriction given by the assumption that $f^{-1}(o_Y) = o_X$ for every $\mathscr{X}$-morphism $f\colon X \to Y$. For example, the category TopGrp of topological groups does not satisfy this assumption and, therefore, we cannot consider the classical Kuratowski closure operator on TopGrp in our setting. But we may work with the more common Kuratowski closure operator on TopGrp with respect to the forgetful functor $|\ |\colon$ TopGrp $\to$ Set.

The closure operator c is called

(a) *grounded* if $c_K(o_K) = o_K$ for each $K \in \mathscr{K}$,

(b) *idempotent* if $c_K(c_K(m)) = c_K(m)$ for each $K \in \mathscr{K}$ and each $m \in \operatorname{sub} K$,

(c) *additive* if $c_K(m \vee p) = c_K(m) \vee c_K(p)$ for each $K \in \mathscr{K}$ and each $m, p \in \operatorname{sub} K$.

A $\mathscr{K}$-morphism $f\colon K \to L$ is called *c-preserving* if $f(c_K(m)) = c_L(f(m))$ whenever $m \in \operatorname{sub} K$. Thus, if f is c-preserving, then it maps c-closed subobjects to c-closed subobjects, and vice versa provided that c is idempotent.

Given a $\mathscr{K}$-object K, a subobject $m \in \operatorname{sub} K$ is said to be

(α) *c-closed* if $c_K(m) = m$,

(β) *c-dense* if $c_K(m) = \mathrm{id}_K$,

(γ) c-separated if the diagonal morphism $\delta_K\colon |K| \to |K| \times |K|$ is c-closed,

(β) c-compact if the projection $\mathrm{pr}_L\colon K \times L \to L$ is c-preserving for every $\mathscr{K}$-object L.

Throughout the paper, we assume there is given a closure operator $c = (c_K)_{K\in\mathscr{K}}$ on $\mathscr{K}$.

Example 2.2. Basic examples of the above introduced category $\mathscr{K}$ with a closure operator are certain topological constructs with $\mathscr{X} =$ Set where $|\ |\colon \mathscr{K} \to$ Set is the forgetful functor and the (surjections, injections)-factorization structure for morphisms is considered in the base category Set. A number of such examples are given in [4], [5], [11], [12], [18]. Among them, of course, the most natural one is $\mathscr{K} =$ Top, i.e., the construct of topological spaces and continuous maps, with c the Kuratowski closure operator. In what follows, if $\mathscr{K} =$ Top or a (full) subconstruct of Top is taken as an example of $\mathscr{K}$, we always mean that c is just the Kuratowski closure operator (of course, there are also other closure operators on Top — see [18]). Further examples can be found among concrete categories over topological constructs (with a singleton fibre of the empty set) which always have the (surjections,embeddings)-factorization structure for morphisms. Such an example is given by the category TopGrp of topological groups (and continuous homomorphisms) when considering the forgetful functor $|\ |\colon$ TopGrp $\to$ Top (that forgets the group structure) and taking the (classical) Kuratowski closure operator on Top.

3. Neighborhoods

Definition 3.1. ([22]) Let K be a $\mathscr{K}$-object. A subobject $n \in \operatorname{sub} K$ is called a *c-neighborhood* of a given subobject $m \in \operatorname{sub} K$ if n is pseudocomplementable (in $\operatorname{sub} K$) and $m \wedge c_K(\overline{n}) = o_K$. We denote by $\mathscr{N}_{c_K}(m)$ the class of all c-neighborhoods of m. A subclass $\mathscr{B} \subseteq \mathscr{N}_{c_K}(m)$ is called a *base* of c-neighborhoods of m if, for every $n \in \mathscr{N}_{c_K}(m)$, there exists $p \in \mathscr{B}$ such that $p \leq n$.

We will write briefly $\mathscr{N}(m)$ instead of $\mathscr{N}_{c_K}(m)$ if only one $\mathscr{K}$-object K with $m \in \operatorname{sub} K$ is considered.

Example 3.2.

(1) Of course, if $m, n \in \operatorname{sub} K$ and both n and $c_K(\overline{n})$ are pseudocomplementable, then $n \in \mathscr{N}(m)$ if and only if $m \leq \overline{c_K(\overline{n})}$. Clearly, if $\mathscr{K} = \mathrm{Top}$, then c-neighborhoods coincide with the usual neighborhoods (of subsets) in topological spaces.

(2) Recall that a projection space is a pair $(X, (\alpha_n)_{n\in\mathbb{N}})$ where X is a set and $(\alpha_n)_{n\in\mathbb{N}} = (\alpha_n \colon X \to X)_{n\in\mathbb{N}}$ is a sequence of maps such that $\alpha_n \circ \alpha_m = \alpha_{\min(m,n)}$ — cf. [20]. Given projection spaces $(X, (\alpha_n)_{n\in\mathbb{N}})$ and $(Y, (\beta_n)_{n\in\mathbb{N}})$, a map $g \colon X \to Y$ is called a projection function of $(X, (\alpha_n)_{n\in\mathbb{N}})$ into $(Y, (\beta_n)_{n\in\mathbb{N}})$ provided that $\beta_n \circ g = g \circ \alpha_n$ for all $n \in \mathbb{N}$. Projection spaces $(X, (\alpha_n)_{n\in\mathbb{N}})$ with $\alpha_n = f$ for all $n \in \mathbb{N}$, where $f \colon X \to X$ is a map, coincide with idempotent mono-unary algebras. Let $\mathscr{K}$ be the category of projection spaces and projection functions.

(a) Let $\mathscr{X} = \mathscr{K}$, let $|\ | \colon \mathscr{K} \to \mathscr{K}$ be the identity functor, and consider the (surjections,injections)-factorization structure for morphisms in $\mathscr{K}$. With respect to this factorization structure, there is a closure operator $c = (c_K)_{K\in\mathscr{K}}$ on $\mathscr{K}$ given by $c_K(m) = \{x \in X : \ (\forall n \in \mathbb{N})(\alpha_n(x) \in m(M))\}$ whenever $K = (X, (\alpha_n)_{n\in\mathbb{N}})$ is a projection space and $m \colon M \to K$ is a subobject of K. This closure operator coincides with the closure operator c_∞ from [20]. So, by [20], c is idempotent, additive and hereditary. It can easily be seen that, given a $\mathscr{K}$-object K, $\operatorname{sub} K$ is pseudocomplemented but need not be a Boolean algebra (e.g., let $K = (X, f)$ be the idempotent mono-unary algebra with $X = \{0, 1\}$, $f(0) = 1$ and $f(1) = 1$).

(b) Let $\mathscr{X} = \mathrm{Set}$, let $|\ | \colon \mathscr{K} \to \mathscr{X}$ be the forgetful functor and consider the (surjections,injections)-factorization structure for morphisms in Set. With respect to this factorization structure, there is a closure operator $c = (c_K)_{K\in\mathscr{K}}$ on $\mathscr{K}$ given by $c_K(m) = m(M) \cup \{x \in X : \ (\forall n \in \mathbb{N})(\alpha_n(x) \in m(M))\}$ whenever $K = (X, (\alpha_n)_{n\in\mathbb{N}})$ is a projection space and $m \colon M \to X$ is a subobject

of K. Moreover, c is clearly idempotent and hereditary. It is a so-called non-standard closure operator — see [5]. Of course, for those subobjects m of K which coincide with (underlying sets of) subobjects of K in the sense of (a), $c_K(m)$ coincides with (the underlying set of) $c_K(m)$ from (a).
(c) Let the situation be the same as in (b). Then, with respect to the factorization structure considered, there is another closure operator $c = (c_K)_{K\in\mathscr{K}}$ on $\mathscr{K}$ defined as follows: $c_K(m) = m(M) \cup \{\alpha_n(x) : \ x \in m(M) \ \text{and} \ n \in \mathbb{N}\}$ whenever $K = (X, (\alpha_n)_{n\in\mathbb{N}})$ is a projection space and $m\colon M \to X$ is a subobject of K. Clearly, this closure operator is not only idempotent and hereditary, but also additive. Therefore, it is more appropriate than the closure operator c from (b). It is also obvious that c_K-closed subobjects of a $\mathscr{K}$-object K (i.e., subsets of K) coincide with the subobjects of K from (a). In other words, c is a so-called hull operator — see [5].

Now, let $K \in \mathscr{K}$ be the projection space $K = (\mathbb{N}, (\alpha_n)_{n\in\mathbb{N}})$ where, for each $n, p \in \mathbb{N}$, $\alpha_n(p) = \min(n, p)$.

If c is the closure operator on $\mathscr{K}$ given in part (a), then $n > o_K \Longrightarrow n \in \mathscr{N}(m)$ whenever m, n are subobjects of K (because the subobjects of K are c-closed and coincide with the subsets of $\mathbb{N}$ having the form $\{x \in \mathbb{N} : \ x < n\}$ where $n \in \mathbb{N} \cup \{\infty\}$, so that $n > o_K \Longrightarrow \overline{n} = o_k$ for each subobject n of K).

On the other hand, if c is the closure operator on $\mathscr{K}$ given in part (b), then c_K coincides with the discrete topology on $\mathbb{N}$. Therefore, we have $n \in \mathscr{N}(m) \Longleftrightarrow m \leq n$ whenever m, n are subobjects of K.

Finally, let c be the closure operator on $\mathscr{K}$ from part (c). Let $m\colon M \to \mathbb{N}$ be an arbitrary subobject of K with $m > o_K$. Then one can easily see that $c_K(m) = \mathbb{N}$ if $m(M)$ is infinite, and $c_K(m) = \{1, 2, \ldots, \max m(M)\}$ if $m(M)$ is finite. Consequently, we have $\mathscr{N}(m) = \{N \subseteq \mathbb{N} : \ x \in N \text{ for each } x \in \mathbb{N} \text{ with } x \geq \min m(M)\}$. Thus, if $x \in \mathbb{N}$ is a point, then $\{y \in \mathbb{N} : \ y \geq x\}$ is the smallest neighborhood of x. It follows that $(\mathbb{N}, c_K)$ is nothing but the so-called right topology on (the linearly ordered set) $\mathbb{N}$.

(3) Let $\mathrm{Alg}_{(2)}$ be the construct of algebras of type (2) (and the usual algebraic homomorphisms). Let $|\ |\colon \mathrm{Alg}_{(2)} \to \mathrm{Alg}_{(2)}$ be the identity functor. One can easily show (cf. [18, Exercise 2.D(a)]) that, with respect to the (surjections, injections)-factorization system for morphisms in $\mathrm{Alg}_{(2)}$, there is a closure operator c on $\mathrm{Alg}_{(2)}$ given as follows: For every algebra X of type (2) and every subalgebra M of X,

$$c_X(M) = \begin{cases} \emptyset & \text{if } M = \emptyset, \\ \bigcap\{N : \ M \leq N \trianglelefteq X\} & \text{if } M \neq \emptyset \end{cases}$$

where $M \leq N$ stands for "M is a subalgebra of N" and $N \trianglelefteq X$ for "N is a left ideal of X". Recall that, when using the multiplicative denotation for the binary operation of X, a left ideal of X is a nonempty subalgebra N of X such that $xa \in N$ whenever $x \in X$ and $a \in N$. Let $G = \{a, b, c\}$ be the three-element commutative and idempotent algebra of type (2) with $ab = c$, $ac = b$ and $bc = a$. Then the subobject lattice of G is a diamond (see [24]), hence only the least and the greatest elements have pseudocomplements. Clearly, $c_G(M) = G$ for each nonempty subalgebra M of G and $c_G(\emptyset) = \emptyset$. Further, we clearly have $\mathscr{N}(\emptyset) = \{\emptyset, G\}$ while $\mathscr{N}(M) = \{G\}$ for each nonempty subalgebra M of G.

Lemma 3.3. ([22]) *Let K be a $\mathscr{K}$-object and $m, p \in \operatorname{sub} K$. Then*

(1) $\mathrm{id}_K \in \mathscr{N}(m)$ *if c is grounded,*

(2) $\mathscr{N}(o_K) = \{n \in \operatorname{sub} K : \ n$ *pseudocomplementable*$\}$,

(3) *if $m > o_K$, then $n > o_K$ for each $n \in \mathscr{N}(m)$,*

(4) *$n \in \mathscr{N}(m)$ implies $m \le n$ provided that*
   (a) *m is an atom or*
   (b) *$\operatorname{sub} K$ is pseudocomplemented and $\overline{\overline{n}} = n$,*

(5) *if $n \in \mathscr{N}(m)$ and $p \in \operatorname{sub} K$ is pseudocomplementable with $p \ge n$, then $p \in \mathscr{N}(m)$,*

(6) $p \le m \Longrightarrow \mathscr{N}(m) \subseteq \mathscr{N}(p)$,

(7) *if $m > o_K$ and $n_1, n_2, \ldots, n_k \in \mathscr{N}(m)$ $(k \in \mathbb{N})$, then $m \wedge n_1 \wedge n_2 \wedge \ldots \cdots \wedge n_k > o_K$,*

(8) *if $m > o_K$ and $n_1, n_2, \ldots, n_k \in \mathscr{N}(m)$ $(k \in \mathbb{N})$, then $n_1 \wedge n_2 \wedge \cdots \wedge n_k > o_K$,*

(9) *if $n_1, n_2 \in \mathscr{N}(m)$, then $n_1 \wedge n_2 \in \mathscr{N}(m)$ provided that c is additive and $\operatorname{sub} K$ is a Boolean algebra.*

(10) *If $f\colon K \to L$ is a $\mathscr{K}$-morphism and $q \in \mathscr{N}\big(f(m)\big)$, then $f^{-1}(q) \in \mathscr{N}(m)$.*

Lemma 3.4. ([22]) *Let K be a $\mathscr{K}$-object and $m, p \in \operatorname{sub} K$, $m > o_K$, and let $\mathscr{B} \subseteq \mathscr{N}(m)$ be a base of c-neighborhoods of m. If $m \le c_K(p)$, then $n \wedge p > o_K$ for each $n \in \mathscr{B}$, and vice versa provided that m is an atom of $\operatorname{sub} K$ and p, $\overline{p}$, $c_K(p)$ are pseudocomplementable with $\overline{\overline{p}} = p$.*

4. Separation and compactness

If $\mathscr{K}$ is a (large) complete lattice with the smallest element 0, then a subclass $\mathscr{T} \subseteq \mathscr{K}$ is said to be *centered* if $\bigwedge \mathscr{S} > 0$ for every finite subclass $\mathscr{S} \subseteq \mathscr{T}$.

Definition 4.1. ([22]) A $\mathscr{K}$-object K is said to be

(a) *separated* (with respect to c) provided that, whenever $m, p \in \operatorname{sub} K$ are different atoms, there are $n \in \mathscr{N}(m)$ and $q \in \mathscr{N}(p)$ with $n \wedge q = o_K$,

(b) *compact* (with respect to c) if $\bigwedge \mathscr{T} > o_K$ for every centered class $\mathscr{T} \subseteq \operatorname{sub} K$ of c-closed subobjects of K.

Example 4.2.

(1) If $\mathscr{K} = \mathrm{Top}$, then the above defined concepts of separation and compactness coincide with the well-known separation and compactness of topological spaces. For separation, this is true also in the case when $\mathscr{K}$ is the construct of Čech closure spaces [8].

(2) Let Alg_τ be the construct of algebras of a given type τ (with the usual homomorphisms as morphisms) and let $|\ |\colon \mathrm{Alg}_\tau \to \mathrm{Set}$ be the forgetful functor. Then, with respect to the (surjections,injections)-factorization structure for morphisms in Set, there is an idempotent closure operator $c = (c_A)_{A\in \mathrm{Alg}_\tau}$ on Alg_τ given by $c_A(X) = \langle X\rangle_A$ for every algebra A of type τ and every subset $X \subseteq A$ where $\langle X\rangle_A$ denotes the subalgebra of A generated by X. It is evident that every object of Alg_τ is compact. An object $A \in \mathrm{Alg}_\tau$ is separated if, for instance, A is a projection algebra (i.e., all operations of A are projections).

The concepts of separation and compactness are studied in [22]. In this section, we complete the study by discussing relationships of the concepts to the well known c-separation and c-compactness.

Theorem 4.3. *Let K be a $\mathscr{K}$-object such that $\mathrm{sub}(K\times K)$ is atomistic and both δ_K and $c_{K\times K}(\delta_K)$ are pseudocomplementable with $\overline{\overline{\delta_K}} = \delta_K$ and $\overline{\overline{c_{K\times K}(\delta_K)}} = c_{K\times K}(\delta_K)$. Let, for every atom $m \in \mathrm{sub}(K \times K)$, both the projections $\mathrm{pr}_i\colon |K|\times|K| \to |K|$, $i = 1,2$, fulfill $\mathrm{pr}_i \circ m \in \mathscr{M}$ and let from $p \in \mathscr{N}(\mathrm{pr}_1 \circ m)$ and $q \in \mathscr{N}(\mathrm{pr}_2 \circ m)$ it follows that $p \times q \in \mathscr{N}(m)$. If K is separated, then it is c-separated.*

P r o o f. Let $m \in \mathrm{sub}(K \times K)$ be an atom with $m \le \overline{\delta_K}$. Then $m \not\le \delta_K$, hence $\mathrm{pr}_1 \circ m = \mathrm{pr}_2 \circ m$ (because δ_K is an equalizer of pr_1 and pr_2). Therefore, there are $p \in \mathscr{N}(\mathrm{pr}_1 \circ m)$ and $q \in \mathscr{N}(\mathrm{pr}_2 \circ m)$ such that $p \wedge q = o_K$. Suppose that $(p\times q)\wedge\delta_K > o_{K\times K}$. Then there is an atom $s \in \mathrm{sub}(K\times K)$ with $s \le (p\times q)\wedge\delta_K$. Since $s \le p \times q$, we clearly have $\mathrm{pr}_1 \circ s \le p$ and $\mathrm{pr}_2 \circ s \le q$. From $s < \delta_K$ it follows that $\mathrm{pr}_1 \circ s = \mathrm{pr}_2 \circ s$. Consequently, $p\wedge q > o_K$, which is a contradiction. Thus, there holds $(p \times q) \wedge \delta_K = o_{K\times K}$. Further, we have $p \times q \in \mathscr{N}(m)$ by the assumptions of the statement. Suppose that $m \le c_{K\times K}(\delta_K)$. Then $n \wedge \delta_K > o_{K\times K}$ for every $n \in \mathscr{N}(m)$ by Lemma 3.4. Therefore, $(p \times q) \wedge \delta_K > o_{K\times K}$, which is a contradiction. Hence, we get $m \le \overline{c_{K\times K}(\delta_K)}$. We have shown that $\overline{\delta_K} \le \overline{c_{K\times K}(\delta_K)}$, which yields $c_{K\times K}(\delta_K) \le \delta_K$. Thus, δ_K is c-closed. □

Theorem 4.4. *Let K be a $\mathscr{K}$-object such that $\mathrm{sub}\, K$ is pseudocomplemented, all atoms of $\mathrm{sub}\, K$ have the same domain (up to isomorphisms) and $\overline{\overline{\delta_K}} = \delta_K$. Let, for any pair of atoms $p, q \in \mathrm{sub}\, K$, from $n \in \mathscr{N}(\langle p, q\rangle)$ it follows that $\mathrm{pr}_1 \circ n \in \mathscr{N}(p)$ and $\mathrm{pr}_2 \circ n \in \mathscr{N}(q)$. If K is c-separated, then it is separated.*

P r o o f. Let δ_K be c-closed and let $p, q \in \operatorname{sub} K$ be different atoms. Then $\langle p, q\rangle \in \operatorname{sub}(K \times K)$ is an atom by Lemma 2.1 and we have $\langle p, q\rangle \not\leq \delta_K = c_{K\times K}(\delta_K)$ because δ_K is an equalizer of pr_1 and pr_2. Therefore, by Lemma 3.4, there exists $n \in \mathscr{N}(\langle p, q\rangle)$ such that $n \wedge \delta_K = o_{K\times K}$. We have $\mathrm{pr}_1 \circ n \in \mathscr{N}(p)$ and $\mathrm{pr}_2 \circ n \in \mathscr{N}(q)$ by the assumptions of the statement. Suppose that $\mathrm{pr}_1 \circ n \wedge \mathrm{pr}_2 \circ n > o_K$ and let $r \in \operatorname{sub} K$ be an atom with $r \leq \mathrm{pr}_1 \circ n \wedge \mathrm{pr}_2 \circ n$. Then there are $s, t \in \mathscr{M}$ with the same domain (up to isomorphisms) as r and such that $r = \mathrm{pr}_1 \circ n \circ s = \mathrm{pr}_2 \circ n \circ t$. It follows that $\mathrm{pr}_1 \circ \langle r, r\rangle = \mathrm{pr}_1 \circ n \circ s$ and $\mathrm{pr}_2 \circ \langle r, r\rangle = \mathrm{pr}_2 \circ n \circ t$. Hence, $\langle r, r\rangle = n \circ s$ (and $\langle r, r\rangle = n \circ t$, so that $s = t$), which yields $\langle r, r\rangle \leq n$. Thus, since $\langle r, r\rangle \leq \delta_K$, we have $\langle r, r\rangle \leq n \wedge \delta_K$. Consequently, $n \wedge \delta_K > o_{K\times K}$, which is a contradiction. Therefore, $\mathrm{pr}_1 \circ n \wedge \mathrm{pr}_2 \circ n = o_K$ and the proof is complete. □

THEOREM 4.5. *Let c be additive and $\operatorname{sub} L$ be an atomic Boolean algebra for each $\mathscr{K}$-object L. Let K be a $\mathscr{K}$-object satisfying the following condition:*

> *Given a $\mathscr{K}$ object L, an atom $y \in \operatorname{sub} L$ and a subobject $m \in \operatorname{sub}(K \times L)$ with $\mathrm{pr}_L(c_{K\times L}(m)) \wedge y = o_L$, for each atom $x \in \operatorname{sub} K$ there are subobjects $u_x \in \operatorname{sub} K$ and $v_x \in \operatorname{sub} L$, u_x c-closed, such that $u_x \wedge x = o_K$, $c_L(v_x) \wedge y = o_L$, and $c_{K\times L}(m) \leq \mathrm{pr}_K^{-1}(u_x) \vee \mathrm{pr}_L^{-1}(v_x)$.*

If K is compact, then it is c-compact.

P r o o f. Let K be compact, L be a $\mathscr{K}$-object and $m \in \operatorname{sub}(K \times L)$. If $\mathrm{pr}_L(c_{K\times L}(m)) = \mathrm{id}_L$, then we clearly have $c_L(\mathrm{pr}_L(m)) \leq \mathrm{pr}_L(c_{K\times L}(m))$. Let $\mathrm{pr}_L(c_{K\times L}(m)) < \mathrm{id}_L$. Then $\overline{\mathrm{pr}_L(c_{K\times L}(m))} > o_L$. Let $y \in \operatorname{sub} L$ be an atom with $y \leq \overline{\mathrm{pr}_L(c_{K\times L}(m))}$, i.e., with $\mathrm{pr}_L(c_{K\times L}(m)) \wedge y = o_L$. For each atom $x \in \operatorname{sub} K$, let $u_x \in \operatorname{sub} K$ and $v_x \in \operatorname{sub} L$ be the subobjects from the condition of the statement. Then $\bigwedge\{u_x : x \in \operatorname{sub} K \text{ is an atom}\} = o_K$ (because otherwise there is an atom $x_0 \in \operatorname{sub} K$ with $x_0 \leq u_x$ for each atom $x \in \operatorname{sub} K$, which is a contradiction with $u_{x_0} \wedge x_0 = o_K$). Thus, there is a finite set $\{x_1, \ldots, x_k\}$ of atoms of $\operatorname{sub} K$ such that $\bigwedge\limits_{i=1}^{k} u_{x_i} = o_K$. Put $v = \bigvee\limits_{i=1}^{k} v_{x_i}$. Then $c_L(v) \wedge y = c_L\Big(\bigvee\limits_{i=1}^{k} v_{x_i}\Big) \wedge y = \bigvee\limits_{i=1}^{k} (c_L(v_{x_i}) \wedge y) = o_L$. Consequently, $\overline{v} \in \mathscr{N}(y)$. Further, we have $c_{K\times L}(m) \leq \bigwedge\limits_{i=1}^{k} (\mathrm{pr}_K^{-1}(u_{x_i}) \vee \mathrm{pr}_L^{-1}(v_{x_i})) \leq \bigwedge\limits_{i=1}^{k} \mathrm{pr}_K^{-1}(u_{x_i}) \vee \bigvee\limits_{i=1}^{k} \mathrm{pr}_L^{-1}(v_{x_i}) \leq \mathrm{pr}_L^{-1}(\bigvee\limits_{i=1}^{k} v_{x_i}) = \mathrm{pr}_L^{-1}(v)$, hence $\mathrm{pr}_L(c_{K\times L}(m)) \leq v$. This yields $\overline{v} \leq \overline{\mathrm{pr}_L(c_{K\times L}(m))}$, i.e., $\overline{v} \wedge \mathrm{pr}_L(c_{K\times L}(m)) = o_L$. It follows that $\overline{v} \wedge \mathrm{pr}_L(m) = o_L$. By Lemma 3.4, $y \wedge c_L(\mathrm{pr}_L(m)) = o_L$. Consequently, $y \leq \overline{c_L(\mathrm{pr}_L(m))}$. We have shown that $\overline{c_L(\mathrm{pr}_L(m))} \geq \overline{\mathrm{pr}_L(c_{K\times L}(m))}$. Therefore, $c_L(\mathrm{pr}_L(m)) \leq \mathrm{pr}_L(c_{K\times L}(m))$ and the proof is complete. □

Theorem 4.6. *Let c be idempotent and K be a $\mathscr{K}$-object with the properties that $\operatorname{sub} K$ is a Boolean algebra and for any centered subclass $\mathscr{F} \subseteq \operatorname{sub} K$ of c-closed subobjects of K there exist a $\mathscr{K}$-object L and a c-dense subobject $m\colon |K| \to |L|$ of L such that the following conditions are satisfied:*

(1) $\operatorname{sub}(K \times L)$ *is atomic.*

(2) *For any atom $z \in \operatorname{sub}(K \times L)$, from $p \in \mathscr{N}(\operatorname{pr}_K(z))$ and $q \in \mathscr{N}(\operatorname{pr}_L(z))$ it follows that $p \times q \in \mathscr{N}(z)$.*

(3) *There exists a subobject $y \in \operatorname{sub} L$ with $y > o_L$, $y \wedge m = o_L$, and $y \vee m(s) \in \mathscr{N}(y)$ for each $s \in \mathscr{F}$.*

If K is c-compact, then it is compact.

Proof. Let K be c-compact and $\mathscr{F} \subseteq \operatorname{sub} K$ be a centered class of c-closed subobjects of K. Put $d = \langle \mathrm{id}_K, m\rangle$. Then $m = \operatorname{pr}_L(d) \leq \operatorname{pr}_L(c_{K\times L}(d))$. As m is dense, we have $y \leq c_L(m)$. Consequently, $y \leq c_L(\operatorname{pr}_L(c_{K\times L}(d))) = \operatorname{pr}_L(c_{K\times L}(d))$ because $\operatorname{pr}_L\colon K \times L \to L$ is c-preserving and c is idempotent. Thus, since $\operatorname{pr}_L(c_{K\times L}(d)) \wedge y > o_L$, we have $c_{K\times L}(d) \wedge \operatorname{pr}_L^{-1}(y) > o_{K\times L}$. Let $z \in \operatorname{sub}(K\times L)$ be an atom with $z \leq c_{K\times L}(d)\wedge \operatorname{pr}_L^{-1}(y)$. Then $z \leq c_{K\times L}(d)$ and $\operatorname{pr}_L(z) \leq y$. Let $a \in \operatorname{sub} K$ be the atom with $a = \operatorname{pr}_K(z)$ and put $q_s = y \vee m(s)$ for each $s \in \mathscr{F}$. By Lemma 2.1, $a \in \operatorname{sub} K$ is an atom. Let $p \in \mathscr{N}(a)$. Since $q_s \in \mathscr{N}(y)$, we have $p \times q_s \in \mathscr{N}(z)$ for each $s \in \mathscr{F}$. By Lemma 3.4, $w \wedge d > o_{K\times L}$ for each $w \in \mathscr{N}(z)$. Thus, $(p \times (y \vee m(s))) \wedge d > o_{K\times L}$ for each $s \in \mathscr{F}$. Hence, there is an atom $v_s \in \operatorname{sub}(K \times L)$ with $v_s \leq (p \times (y \vee m(s))) \wedge d$ for each $s \in \mathscr{F}$. As $v_s \leq d$, there is an element $u_s \in \operatorname{sub} K$, $u_s > o_K$, with $v_s = d \circ u_s$. We have $\operatorname{pr}_K \circ v_s = \operatorname{pr}_K \circ\langle \mathrm{id}_K, m\rangle \circ u_s = u_s$ and $\operatorname{pr}_L \circ v_s = \operatorname{pr}_L \circ \langle \mathrm{id}_K, m\rangle \circ u_s = m \circ u_s$. From $v_s \leq p \times (y \vee m(s))$ it follows that $u_s \leq \operatorname{pr}_K(p\times(y\vee m(s)))$ and $m(u_s) \leq \operatorname{pr}_L(p\times(y\vee m(s)))$ (for each $s \in \mathscr{F}$). Now, using the $(\mathscr{E},\mathscr{M})$-diagonalization property, we get $u_s \leq p$ and $m(u_s) \leq y\vee m(s)$, i.e., $u_s \leq m^{-1}(y)\vee m^{-1}(s) = m^{-1}((y\vee m(s))\wedge m) = m^{-1}((y\wedge m)\vee(m(s)\wedge m)) = m^{-1}(m(s) \wedge m) = m^{-1}(m(s)) = s$. Consequently, $p \wedge s \geq u_s > o_K$ for each $s \in \mathscr{F}$. Therefore, by Lemma 3.4, $a \leq c_K(s) = s$ for each $s \in \mathscr{F}$. Hence $\bigwedge \mathscr{F} > o_K$, so that K is compact. □

Example 4.7. If $\mathscr{K} = \mathrm{Top}$, then the assumptions of each of the Theorems 4.3–4.6 are satisfied and the Theorems give the well-known results that c-separation and c-compactness coincide with separation and compactness of topological spaces. Theorems 4.3 and 4.5 are also valid if $\mathscr{K}$ is the larger category of Čech closure spaces [8]. The assumptions of Theorem 4.5 are satisfied whenever $\mathscr{K}$ is a full subcategory of Top (the subobjects u_x and v_x are then obtained as complements of certain open neighborhoods of x and y, respectively — see [19]). As for Theorem 4.6, its assumptions are satisfied, for example, whenever $\mathscr{K}$ is the category of T_1-spaces or the category of normal spaces (the topological space L is then defined to be the space with $|L| = |K|\cup\{y\}$ where $y \notin |K|$ is a point and the open sets in L are just the open sets in K and the sets of the form $\{y\}\cup T\cup X$

where T is a finite intersection of elements of $\mathscr{F}$ and $X \subseteq |K|$ is a subset — see [19] again). On the other hand, there hardly exist topological categories which are not subcategories of Top and fulfill the conditions of Theorem 4.5 or 4.6.

Remark 4.8. The assumptions of Theorems 4.3 and 4.4 are quite natural (especially if $\mathscr{K}$ is a construct), thus there is a strong relationship between separation and c-separation. But this is not true for compactness and c-compactness in general (if $\mathscr{K}$ differs from Top with the Kuratowski closure operator c). Nevertheless, the two concepts of compactness behave still quite analogously. For example, the Tychonoff's theorem for each of them is based on using a certain finiteness property of products — cf. [22, Theorem 4.11] and [10].

5. Convergence

The well-known concepts of *filter, ultrafilter, filter base* and *filter subbase* defined for lattices may be naturally extended to possibly large lattices. Similarly, we may extend the concept of a stack from ordered sets to ordered classes (recall that a *stack* on an ordered class G is a subclass $S \subseteq G$ such that $x \leq y$ implies $y \in S$ whenever $x \in S$ and $y \in G$). Of course, filters are just the filter bases that are stacks and, for (large) lattices with a least element, filter subbases not containing the least element coincide with centered subclasses (i.e., nonempty subclasses $\mathscr{C}$ such that $\bigwedge \mathscr{B}$ is different from the least element for every finite subset $\mathscr{B} \subseteq \mathscr{C}$). Let $\mathscr{G}$ be a possibly large lattice with a smallest element. If $\mathscr{R}$, $\mathscr{S}$ are centered subclasses of $\mathscr{G}$, then $\mathscr{S}$ is said to be *finer* than $\mathscr{R}$, and $\mathscr{R}$ is said to be *coarser* than $\mathscr{S}$, provided that $\mathscr{R} \subseteq \mathscr{S}$. It is evident that maximal centered subclasses of $\mathscr{G}$ (and maximal filter bases on $\mathscr{G}$) coincide with ultrafilters on $\mathscr{G}$. Thus, as the Axiom of Choice for conglomerates is assumed, each centered subclass of $\mathscr{G}$ (and each filter base on $\mathscr{G}$) is coarser than an ultrafilter on $\mathscr{G}$.

For each $\mathscr{X}$-object X we denote by $\mathbf{R}_X$ the conglomerate of all centered subclasses of $\operatorname{sub} X$. Thus, $\mathbf{R}_X = \emptyset$ if and only if X is a trivial object (because otherwise $\{\mathrm{id}_X\} \in \mathbf{R}_X$). Given a $\mathscr{K}$-object K, we write briefly $\mathbf{R}_K$ instead of $\mathbf{R}_{|K|}$.

Let X, Y be $\mathscr{X}$-objects and $\mathscr{B} \subseteq \operatorname{sub} X$ a subclass. As usual, if $f\colon X \to Y$ is an $\mathscr{X}$-morphism, we put $f(\mathscr{B}) = \{f(r) :\ r \in \mathscr{B}\}$. Clearly, if $\mathscr{B}$ is a centered subclass of $\operatorname{sub} X$ (or a filter base on $\operatorname{sub} X$ respectively), then $f(\mathscr{B})$ is a centered subclass of $\operatorname{sub} Y$ (or a filter base on $\operatorname{sub} Y$ respectively).

Let $X = \prod\limits_{i \in I} X_i$ be a product in $\mathscr{X}$ and let $\mathscr{B}_i \subseteq \operatorname{sub} X_i$ for each $i \in I$. Then we put $\prod\limits_{i \in I} \mathscr{B}_i = \Big\{ \prod\limits_{i \in I} m_i :\ m_i \in \mathscr{B}_i \text{ for each } i \in I \Big\}$. If in $\mathscr{X}$ the nontrivial objects are stable under products and if $\mathscr{B}_i$ is centered for each $i \in I$, then $\prod_{i\in I} \mathscr{B}_i \in \operatorname{sub} X$ is centered, too (because $\bigwedge_{j\in J} \prod_{i\in I} m_i^j \geq \prod_{i\in I} \bigwedge_{j\in J} m_i^j$ whenever $m_i^j \in \mathscr{B}_i$ for each $j \in J$ and each $i \in I$, and the domain of $\bigwedge_{j\in J} m_i^j$ is non-trivial for each $i \in I$).

Let K be a $\mathscr{K}$-object and $m \in \operatorname{sub} K$, $m > o_K$. By Lemma 3.3, $\mathscr{N}(m)$ is a centered subclass of $\operatorname{sub} K$ and each base $\mathscr{B} \subseteq \mathscr{N}(m)$ is a centered subclass of $\operatorname{sub} K$ too. But $\mathscr{N}(m)$ need not be a filter in general (by Lemma 3.3, $\mathscr{N}(m)$ is a filter on $\operatorname{sub} K$ provided that c is additive and $\operatorname{sub} K$ is a Boolean algebra). For this reason, centered classes will be used as tools for defining convergence:

Definition 5.1. Let K be a $\mathscr{K}$-object, $m \in \operatorname{sub} K$ and $\mathscr{R} \in \mathbf{R}_K$.

(a) We say that $\mathscr{R}$ *converges* to m, in symbols $\mathscr{R} \to m$, if, for each $p \in \operatorname{sub} K$ with $o_K < p \leq m$ and each $n \in \mathscr{N}(p)$ there exists $r \in \mathscr{R}$ such that $r \leq n$.

(b) m is called a *clustering* of $\mathscr{R}$ provided that $m \leq c_K(r)$ for each $r \in \mathscr{R}$ (i.e., provided that $m \leq \bigwedge_{r\in\mathscr{R}} c_K(r)$).

Example 5.2. (1) Let $\mathscr{K} = \mathrm{Top}$, let K be a $\mathscr{K}$-object, $\mathscr{R} \in \mathbf{R}_K$ be a filter, and $m\colon M \to |K|$ be an inclusion (in Set). Then $\mathscr{R} \to m$ (respectively, m is a clustering of $\mathscr{R}$) if and only if $\mathscr{R}$ converges to x (respectively, x is a cluster point of $\mathscr{R}$) — in the usual topological sense — for each $x \in M$. But some authors do not require $\mathscr{R}$ to be a filter when defining convergence in a topological space and work with convergence of centered systems — see e.g. [2].

(2) If $\mathscr{K}$ is the construct of Čech closure spaces [8], $K \in \mathscr{K}$, $\mathscr{R} \in \mathbf{R}_K$ is a filter base and $m \in K$ is a point, then Definition 5.1 is equivalent to the definitions of convergence and a cluster point from [8].

The following statement is obvious:

Proposition 5.3. *Let K be a $\mathscr{K}$-object. Then*

(1) *$\mathscr{R} \to o_K$ for each $\mathscr{R} \in \mathbf{R}_K$.*

(2) *$\mathscr{N}(m) \to m$ whenever m is an atom of* $\operatorname{sub} K$*.*

(3) *For any $\mathscr{R} \in \mathbf{R}_K$ and any $m \in \operatorname{sub} K$, from $\mathscr{R} \to m$ it follows that $\mathscr{R} \to p$ for each $p \in \operatorname{sub} K$, $p \leq m$.*

(4) *Let the lattice* $\operatorname{sub} K$ *be atomic, let $\mathscr{R} \in \mathbf{R}_K$ and let $m \in \operatorname{sub} K$. If $\mathscr{R} \to a$ for each atom $a \in \operatorname{sub} K$ with $a \leq m$, then $\mathscr{R} \to m$.*

(5) *For any $\mathscr{R} \in \mathbf{R}_K$ and any $m \in \operatorname{sub} K$, from $\mathscr{R} \to m$ it follows that $\mathscr{S} \to m$ whenever $\mathscr{S} \in \mathbf{R}_K$ is finer than $\mathscr{R}$.*

(6) *If $\mathscr{R} \in \mathbf{R}_K$ is a stack on* $\operatorname{sub} K$ *and $m \in \operatorname{sub} K$, then $\mathscr{R} \to m$ if and only if $\mathscr{N}(p) \subseteq \mathscr{R}$ for each $p \in \operatorname{sub} K$ with $o_K < p \leq m$.*

(7) *o_K is a clustering of every $\mathscr{R} \in \mathbf{R}_K$.*

(8) *Let $\mathscr{R} \in \mathbf{R}_K$ and $m, n \in \operatorname{sub} K$. If m is a clustering of $\mathscr{R}$ and $n \leq m$, then n is a clustering of $\mathscr{R}$, too.*

As an immediate consequence of Lemma 3.4, Definition 5.1 and Proposition 5.3 we get:

Proposition 5.4. *Let K be a $\mathscr{K}$-object, $m, p \in \operatorname{sub} K$, and let m be an atom of $\operatorname{sub} K$. If $m \leq c_K(p)$, then there exists $\mathscr{R} \in \mathbf{R}_K$ such that $\mathscr{R} \to m$ and $n \wedge p > o_K$ for each $n \in \mathscr{R}$, and vice versa provided that $\operatorname{sub} K$ is a Boolean algebra.*

Corollary 5.5. *Let K be a $\mathscr{K}$-object such that $\operatorname{sub} K$ is a Boolean algebra and let $p \in \operatorname{sub} K$. If $c_K(p)$ equals a join of atoms of $\operatorname{sub} K$, then $c_K(p) = \bigvee\{m \in \operatorname{sub} K : \ m$ is an atom such that there exists $\mathscr{R} \in \mathbf{R}_K$ with $\mathscr{R} \to m$ and $n \wedge p > o_K$ for each $n \in \mathscr{R}\}$.*

Proposition 5.6. *Let K be a $\mathscr{K}$-object such that $\operatorname{sub} K$ is a Boolean algebra, let $\mathscr{R} \in \mathbf{R}_K$ be a stack on $\operatorname{sub} K$ and let $m \in \operatorname{sub} K$ be a join of atoms. If there exists $\mathscr{S} \in \mathbf{R}_K$ with $\mathscr{R} \subseteq \mathscr{S}$ and $\mathscr{S} \to m$, then m is a clustering of $\mathscr{R}$, and vice versa provided that $\operatorname{sub} K$ is atomic, c is additive and $\mathscr{R}$ is a filter.*

Proof. For $m = o_K$ the statement is trivial. Let $m > o_K$ and let there exist $\mathscr{S} \in \mathbf{R}_K$ with $\mathscr{R} \subseteq \mathscr{S}$ and $\mathscr{S} \to m$. Then, for an arbitrary atom $p \in \operatorname{sub} K$ with $p \leq m$, we have $\mathscr{S} \to p$. As $\mathscr{N}(p) \subseteq \mathscr{S}$ by Proposition 5.3(6), it follows that $r \wedge n > o_K$ whenever $r \in \mathscr{R}$ and $n \in \mathscr{N}(p)$. By Lemma 3.4, $p \leq c_K(r)$ for each $r \in \mathscr{R}$. Hence p is a clustering of $\mathscr{R}$, i.e., $p \leq \bigwedge\{c_K(r) : \ r \in \mathscr{R}\}$. Consequently, m is a clustering of $\mathscr{R}$.

Conversely, let $\operatorname{sub} K$ be atomic, c be additive and $\mathscr{R}$ be a filter. Suppose that m is a clustering of $\mathscr{R}$ and let $p \leq m$ be an arbitrary atom of $\operatorname{sub} K$. Put $\mathscr{B} = \{r \wedge n : \ r \in \mathscr{R}, \ n \in \mathscr{N}(p)\}$. By Lemma 3.4, $r \wedge n > o_K$ whenever $r \in \mathscr{R}$ and $n \in \mathscr{N}(p)$. As $\mathscr{R}$ is a filter and, by Lemma 3.3(9), $\mathscr{N}(p)$ is a filter, too, $\mathscr{B}$ is a filter base. Let $\mathscr{S}$ be the filter generated by $\mathscr{B}$, i.e., $\mathscr{S} = \{s \in \operatorname{sub} K : \ (\exists q \in \mathscr{B})(q \leq s)\}$. We have $\mathscr{N}(p) \subseteq \mathscr{S}$, hence $\mathscr{S} \to p$. But we also have $\mathscr{R} \subseteq \mathscr{S}$ and, by Proposition 5.3(4), $\mathscr{S} \to m$. The proof is complete. $\square$

Corollary 5.7. *Let $K \in \mathscr{K}$ be an object such that $\operatorname{sub} K$ is a Boolean algebra, let $\mathscr{R} \in \mathbf{R}_K$ be a stack and let $m \in \operatorname{sub} K$ be a join of atoms. If $\mathscr{R} \to m$, then m is a clustering of $\mathscr{R}$.*

Corollary 5.8. *Let c be additive, K be a $\mathscr{K}$-object such that $\operatorname{sub} K$ is an atomic Boolean algebra, and let $\mathscr{R} \in \mathbf{R}_K$ be an ultrafilter. Then $\mathscr{R} \to m$ if and only if m is a clustering of $\mathscr{R}$.*

Theorem 5.9. *Let $f \colon K \to L$ be a $\mathscr{K}$-morphism, $m \in \operatorname{sub} K$ and $\mathscr{R} \in \mathbf{R}_K$. If $\mathscr{R} \to m$, then $f(\mathscr{R}) \to f(m)$.*

Proof. Let $\mathscr{R} \to m$, $p \in \operatorname{sub} L$, $o_L < p \leq f(m)$, and let $n \in \mathscr{N}(p)$. Since $f(f^{-1}(p)) = p$, we have $n \in \mathscr{N}(f(f^{-1}(p)))$. Thus, by Lemma 3.3, $f^{-1}(n) \in \mathscr{N}(f^{-1}(p))$. From $f^{-1}(p) \wedge m \leq f^{-1}(p)$ it follows that $f^{-1}(n) \in \mathscr{N}\big(f^{-1}(p) \wedge m\big)$.

Further, $f(m) \wedge p = p > o_L$ implies $o_K < f^{-1}(p) \wedge m \leq m$. Thus, there exists $r \in \mathscr{R}$ such that $r \leq f^{-1}(n)$ because $\mathscr{R} \to m$. Hence, $f(r) \leq f(f^{-1}(n)) \leq n$. Since $f(r) \in f(\mathscr{R})$, we have $f(\mathscr{R}) \to f(m)$. $\square$

Let $K = \prod_{i\in I} K_i$ be a product in $\mathscr{K}$ and let $\mathscr{R} \in \mathbf{R}_K$. By Theorem 5.9, given $m \in \operatorname{sub} K$, $\mathscr{R} \to m$ implies $\mathrm{pr}_i(\mathscr{R}) \to \mathrm{pr}_i(m)$ for each $i \in I$. If the converse implication is also valid, we say that the centered class $\mathscr{R}$ is *convergence-compatible* with the product K. For example, it is well known that in the case $\mathscr{K} = \mathbf{Top}$ filters are convergence-compatible with products.

Proposition 5.10. *Let in $\mathscr{X}$ the non-trivial objects be stable under products and let all projections in $\mathscr{K}$ belong to $\mathscr{E}$. Let $K = \prod_{i\in I} K_i$ be a product in $\mathscr{K}$ and, for each $i \in I$, let $\mathscr{R}_i \in \mathbf{R}_{K_i}$, $m_i \in \operatorname{sub} K_i$ and $\mathscr{R}_i \to m_i$. If $\prod_{i\in I} \mathscr{R}_i \in \mathbf{R}_K$ is convergence-compatible with K, then $\prod_{i\in I} \mathscr{R}_i \to \prod_{i\in I} m_i$.*

P r o o f. By the assumptions, $\prod_{i\in I} \mathscr{R}_i \in \mathbf{R}_K$. Let $r_i \in \mathscr{R}_i$, $r_i \colon R_i \to K_i$ for each $i \in I$. Then, for each $i \in I$, $\mathrm{pr}_i \circ \prod_{i\in I} r_i = r_i \circ p_i$ where $\mathrm{pr}_i \colon \prod_{i\in I} K_i \to K$ and $p_i \colon \prod_{i\in I} R_i \to R_i$ are the projections. Thus, $\mathrm{pr}_i\Big(\prod_{i\in I} r_i\Big)$ is the $\mathscr{M}$-part of the $(\mathscr{E}, \mathscr{M})$-factorization of $r_i \circ p_i$. Now, for each $i \in I$, the diagonalization property results in $\mathrm{pr}_i\Big(\prod_{i\in I} r_i\Big) \leq r_i$ and, since $p_i \in \mathscr{E}$, also in $r_i \leq \mathrm{pr}_i\Big(\prod_{i\in I} r_i\Big)$. Therefore, we have $\mathrm{pr}_i\Big(\prod_{i\in I} r_i\Big) = r_i$ for each $i \in I$ and, analogously, we get $\mathrm{pr}_i\Big(\prod_{i\in I} m_i\Big) = m_i$ for each $i \in I$. Hence $\mathrm{pr}_i\Big(\prod_{i\in I} \mathscr{R}_i\Big) = \mathscr{R}_i$, which yields $\mathrm{pr}_i\Big(\prod_{i\in I} \mathscr{R}_i\Big) \to m_i$ for each $i \in I$. Consequently, $\mathrm{pr}_i\Big(\prod_{i\in I} \mathscr{R}_i\Big) \to \mathrm{pr}_i\Big(\prod_{i\in I} (m_i)\Big)$ for each $i \in I$. Since $\prod_{i\in I} \mathscr{R}_i$ is convergence-compatible with K, we have $\prod_{i\in I} \mathscr{R}_i \to \prod_{i\in I} m_i$. $\square$

Theorem 5.11. *Let K be a $\mathscr{K}$-object. If K is separated, then from $\mathscr{R} \to m$ and $\mathscr{R} \to p$ it follows that $m = p$ whenever $m, p \in \operatorname{sub} K$ are atoms and $\mathscr{R} \in \mathbf{R}_K$, and vice versa provided that c is additive and $\operatorname{sub} K$ is a Boolean algebra.*

P r o o f. Let K be separated. Then, for any pair $p, q \in \operatorname{sub} K$ of different atoms, there exist $m \in \mathscr{N}(p)$ and $n \in \mathscr{N}(q)$ such that $m \wedge n = o_K$. Let $\mathscr{R} \in \mathbf{R}_K$ be a centered class with $\mathscr{R} \to r$ and $\mathscr{R} \to s$ where $r, s \in \operatorname{sub} K$ are atoms. Let $m \in \mathscr{N}(r)$ and $n \in \mathscr{N}(s)$ be arbitrary neighborhoods. Then there are $t, u \in \mathscr{R}$ such that $t \leq m$ and $u \leq n$. Since $t \wedge u > o_K$, we have $m \wedge n > o_K$. Therefore, $r = s$.

Conversely, let c be additive and $\operatorname{sub} K$ be a Boolean algebra. Suppose there is a pair $p, q \in \operatorname{sub} K$ of different atoms such that $m \wedge n > o_K$ whenever $m \in \mathscr{N}(p)$ and $n \in \mathscr{N}(q)$. Put $\mathscr{B} = \mathscr{N}(p) \cup \mathscr{N}(q)$. Then $\mathscr{B} \subseteq \operatorname{sub} K$ is centered because $\mathscr{N}(p)$ and $\mathscr{N}(q)$ are filters by Lemma 3.3. We have both $\mathscr{B} \to p$ and $\mathscr{B} \to q$. □

Theorem 5.12. *Let K be a $\mathscr{K}$-object. If every $\mathscr{R} \in \mathbf{R}_K$ has a clustering different from o_K, then K is compact, and vice versa provided that c is idempotent.*

Proof. Suppose that K is not compact. Then there exists a centered class $\mathscr{T} \subseteq \operatorname{sub} K$ of c-closed subobjects of K such that $\bigwedge \mathscr{T} = o_K$. Hence, $\mathscr{T} \in \mathbf{R}_K$ and $\bigwedge \{c_K(p) : p \in \mathscr{T}\} = \bigwedge \mathscr{T} = o_K$. Thus, the only clustering of $\mathscr{T}$ is o_K.

Conversely, let c be idempotent and let K be compact. Let $\mathscr{R} \in \mathbf{R}_K$ and put $\mathscr{S} = \{c_K(r) : r \in \mathscr{R}\}$. Then $\mathscr{S}$ is a clustering of $\mathscr{R}$ and, since $\mathscr{S}$ is a centered class of c-closed subobjects of K, we have $\bigwedge \mathscr{S} > o_K$. □

Remark 5.13.

(a) The introduced concept of convergence may be strengthened by saying that $\mathscr{R} \in \mathbf{R}_K$ converges to $m \in \operatorname{sub} K$ if $\mathscr{N}(p) \subseteq \mathscr{R}$ for each $p \in \operatorname{sub} K$ with $o_K < p \leq m$. Then all statements of this section remain valid and, in the case of Proposition 5.6 and Corollary 5.7, this is true even if the assumption that $\mathscr{R}$ is a stack is omitted.

(b) A concept of convergence with respect to a closure operator on a category was introduced and investigated in [21]. But it is supposed in [21] that all $\operatorname{sub} K$, $K \in \mathscr{K}$, are pseudocomplemented, so that the centered class of all neighborhoods of a given subobject of K forms a stack. Therefore, centered stacks are used in [21] as a tool for expressing the convergence. The convergence introduced in Definition 5.1 generalizes the convergence from [21].

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Department of Mathematics
Brno University of Technology
Technická 2
CZ–616 69 Brno
CZECH REPUBLIC
E-mail: slapal@fme.vutbr.cz