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WEAK RELATIVELY UNIFORM CONVERGENCES ON ABELIAN LATTICE ORDERED GROUPS

Štefan Černák — Ján Jakubík

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ABSTRACT. The notion of relatively uniform convergence has been applied in the theory of vector lattices and in the theory of archimedean lattice ordered groups. Let G be an abelian lattice ordered group. In the present paper we introduce the notion of weak relatively uniform convergence (wru-convergence, for short) on G generated by a system M of regulators. If G is archimedean and $M = G^+$, then this type of convergence coincides with the relative uniform convergence on G. The relation of wru-convergence to the o-convergence is examined. If G has the diagonal property, then the system of all convex ℓ-subgroups of G closed with respect to wru-limits is a complete Brouwerian lattice. The Cauchy completeness with respect to wru-convergence is dealt with. Further, there is established that the system of all wru-convergences on an abelian divisible lattice ordered group G is a complete Brouwerian lattice.

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1. Introduction

The notion of relatively uniform convergence (ru-convergence, for short) has been applied in the theory of vector lattices (cf. the monographs [2], [15], [17]) and in the theory of archimedean lattice ordered groups (cf. the papers [1], [3], [5], [6], [7], [14], [16]). Related notions for MV-algebras were studied in [4].

If H is an abelian lattice ordered group which fails to be archimedean, then the definition of ru-convergence can be used for H, but it has certain rather "pathological" properties, namely

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(i) there exists a sequence (x_n) in H such that the set of limits of this sequence is infinite;

(ii) there exists a sequence (y_n) in H such that $0 < y_1 < y_2 \dots$ and (y_n) converges to 0 under the ru-convergence.

To avoid these patological properties, we introduce the notion of weak relatively uniform convergence (wru-convergence, for short) on an abelian lattice ordered group G generated by a system M of regulators of convergence. We proceed as follows. The set of all positive integers will be denoted by $\mathbb{N}$.

An element $a \in G^+$ will be said to be *archimedean* if, whenever $0 \leqq b \in G$ and $nb \leqq a$ for each $n \in \mathbb{N}$, then $b = 0$.

Recall that a lattice ordered group G is called archimedean if all elements of G^+ are archimedean.

The set of all archimedean elements of G will be denoted by $\mathcal{A}$.

Let $A(G)$ be the ℓ-subgroup of G generated by the set $\mathcal{A}$. Then $A(G)$ is a convex ℓ-subgroup of G having the following properties (cf. [11]):

(i) $A(G)$ is an archimedean lattice ordered group;

(ii) if H is an archimedean convex ℓ-subgroup of G, then $H \subseteq A(G)$.

We say that $A(G)$ is the *archimedean kernel* of G.

Let $b \in \mathcal{A}$, (x_n) a sequence in G and $x \in G$. We say that (x_n) *b-converges* to x, written $x_n \xrightarrow{b} x$, if for each $k \in \mathbb{N}$ there exists $n_0(b,k) \in \mathbb{N}$ such that

$$k|x_n - x| \leqq b$$

holds whenever $n \in \mathbb{N}$, $n \geqq n_0(b,k)$.

Let M be a nonempty subset of $\mathcal{A}$. Assume that M is closed with respect to the addition.

Let (x_n) be a sequence in G and $x \in G$. We say that this sequence $\alpha(M)$*-converges* to x and we write

$$x_n \to_{\alpha(M)} x, \tag{1}$$

if $x_n \xrightarrow{b} x$ for some $b \in M$.

We denote this type of convergence as wru-convergence on G with the system M of regulators, or, shortly, as $\alpha(M)$-convergence.

If G is archimedean and if $M = G^+$, then the relation (1) is equivalent with the condition that (x_n) relatively uniformly converges to x.

Further, assume that G is archimedean and $0 < b \in G$. Consider the convergence in G dealt with in [3] applying the fixed regulator b. Let us denote this convergence as b-convergence on G. Put $M_b = \{nb\}_{n\in\mathbb{N}}$. It is easy to verify that the b-convergence on G coincides with the convergence $\alpha(M_b)$ on G.

A sequence (x_n) in G is a *Cauchy sequence* with respect to the convergence $\alpha(M)$ if for some $b \in M$ and each $k \in \mathbb{N}$ there exists $n_1(b,k) \in \mathbb{N}$ such that

$$k|x_n - x_m| \leqq b$$

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whenever n and m are positive integers with $n \geqq n_1(b, k)$, $m \geqq n_1(b, k)$.

A lattice ordered group is *Cauchy complete* with respect to the convergence $\alpha(M)$ if, whenever (x_n) is a Cauchy sequence with respect to $\alpha(M)$, then there exists $x \in G$ with $x_n \to_{\alpha(M)} x$.

In the present paper, the basic properties of the $\alpha(M)$-convergence in an abelian lattice ordered group G are deduced. The relations between $\alpha(M)$-convergence and o-convergence are examined. The Cauchy completeness of G with respect to $\alpha(M)$-convergence is investigated. Some results of the paper [14] are extended. We show that if G has the diagonal property, then the system of all $\alpha(M)$-closed convex ℓ-subgroups of G is a complete Browerian lattice. Further, there is proved that the system of all wru-convergences on an abelian divisible lattice ordered group G is also a complete Brouwerian lattice.

2. Basic properties of $\alpha(M)$-convergence and examples

In this section, basic properties of $\alpha(M)$-convergence and some examples will be given. As above, we apply the assumption that G is an abelian lattice ordered group and M is as in Section 1.

The fact that the set M of regulators of convergence is closed with respect to the addition, ensures that the results of Lemmas 2.1–2.5 can be proved in the same way as in [6].

The first lemma establishes that limits in $\alpha(M)$-convergence are uniquely determined.

Lemma 2.1. *Let (x_n) be a sequence in G and $x, y \in G$. If $x_n \to_{\alpha(M)} x$ and $x_n \to_{\alpha(M)} y$, then $x = y$.*

Lemma 2.2. *Let (x_n), (y_n) be sequences in G and $x, y \in G$. If $x_n \to_{\alpha(M)} x$ and $y_n \to_{\alpha(M)} y$, then*

(i) $x_n + y_n \to_{\alpha(M)} x + y$,

(ii) $x_n \vee y_n \to_{\alpha(M)} x \vee y$,

(iii) $x_n \wedge y_n \to_{\alpha(M)} x \wedge y$,

(iv) *$kx_n \to_{\alpha(M)} kx$ for any integer k,*

(v) *if $a, b \in G, a \leqq x_n \leqq b$ for all $n \in \mathbb{N}$, then $a \leqq x \leqq b$.*

We use the symbol F to denote the set of all sequences in G which are Cauchy with respect to $\alpha(M)$-convergence.

Lemma 2.3. *Let (x_n) be a sequence in G and $x \in G$. If $x_n \to_{\alpha(M)} x$, then $(x_n) \in F$.*

Lemma 2.4. *If $(x_n) \in F$ then (x_n) is bounded.*

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Lemma 2.5. *Let $(x_n), (y_n) \in F$. Then*

(i) $(x_n + y_n) \in F$,

(ii) $(x_n \vee y_n) \in F$,

(iii) $(x_n \wedge y_n) \in F$.

Example 2.6. Let us modify the previous definition of the convergence $\alpha(M)$ in such a way that we do not assume all elements of M to be archimedean. Let $b \in M$ and suppose that b fails to be archimedean. Hence there exists $x \in G$ such that $0 < nx < b$ for each $n \in \mathbb{N}$. Let $x_n = nx$ for each $n \in \mathbb{N}$. Then we have

$$x_n < x_{n+1} \qquad \text{and} \qquad k|x_n - 0| \leqq b$$

for each $n \in \mathbb{N}$ and each $k \in \mathbb{N}$. Hence $x_n \to_{\alpha(M)} 0$.

Further, let m be a positive integer. If $n > m$, then for each $k \in \mathbb{N}$ we get

$$0 < k|x_n - x_m| < k|x_n| < b,$$

whence $x_n \to_{\alpha(M)} x_m$. Thus the number of limits of the sequence (x_n) is infinite.

We verified that we arrived at the pathological properties mentioned in Section 1 above. This is the reason to suppose all elements of M to be archimedean.

We remark that the possibility $M = \{0\}$ is not excluded; in this case, the only $\alpha(M)$-convergent sequences are those which are eventually constant.

The following example shows that there exists a lattice ordered group $G \neq \{0\}$ with 0 being the only archimedean element; in such a case we have $M = \{0\}$.

Example 2.7. Consider the lexicographic product $G = \Gamma G_i$ $(i \in \mathbb{N})$, $G_i = Z$ for each $i \in \mathbb{N}$ where Z is the additive group of all integers with the natural linear order (for the notion of the lexicographic product of partially ordered groups cf., e.g., Fuchs [8]). The component of an element $g \in G$ in G_i will be denoted by $g(i)$.

Let $0 < x \in G$. Then with respect to the order of G, there exists $i_0 \in \mathbb{N}$ with $x(i_0) > 0$ and $x(i) = 0$ for each $i \in \mathbb{N}$, $i < i_0$. Let $i_1 \in \mathbb{N}$, $i_1 > i_0$. There exists $y \in G$ such that $y(i_1) = 1$ and $y(i) = 0$ for each $i \in \mathbb{N}$, $i \neq i_1$. Hence $y > 0$. We get $(ny)(i_1) = n$ and $(ny)(i) = 0$ for each $i \in \mathbb{N}$, $i \neq i_1$. Therefore $ny < x$ for each $n \in \mathbb{N}$. Consequently, x cannot be archimedean, so $\mathcal{A} = \{0\}$.

Example 2.8. Let G_1 be any abelian linearly ordered group, G_2 any abelian lattice ordered group, and let G be their lexicographic product. It is easy to verify that for every set M of regulators, the relation $M \subseteq \{0\} \times G_2^+$ holds. Assume that $M \neq \{0\}$.

Let (x_n) be a sequence in G and let $x_n \to_{\alpha(M)} 0$.

Let $k \in \mathbb{N}$. There exist $b \in M$ and $n_0 \in \mathbb{N}$ such that

$$k|x_n| \leqq b \qquad \text{for each} \quad n \in \mathbb{N}, \;\; n \geqq n_0.$$

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Since $b(1) = 0$, $x_n(1) = 0$ for each $n \in \mathbb{N}$, $n \geqq n_0$ and $k|x_n(2)| = k|x_n|(2) \leqq b(2) = b$. Hence $x_n(2) \to_{\alpha(M)} 0$.

Let (y_n) be a sequence in G, $y \in G$ and let $y_n \to_{\alpha(M)} y$. The above considerations entail that there exists $n_0 \in \mathbb{N}$ with $y_n(1) = y(1)$ for each $n \in \mathbb{N}$, $n \geqq n_0$ and $y_n(2) \to_{\alpha(M)} y(2)$.

We intend now to generalize the results established in the foregoing two examples.

Let $I \neq \emptyset$ be a linearly ordered set and let $A_i \neq \{0\}$ be a partially ordered group for each $i \in I$. Suppose that G is the lexicographic product of A_i, $G = \Gamma A_i$ $(i \in I)$. We distinguish two cases:

(a) The set I has no greatest element. Then all A_i are linearly ordered groups. Zero element 0 is the only archimedean element of G. Hence $M = \{0\}$. For the proof of this, a similar procedure to that in Example 2.7 can be applied.

(b) The set I possesses the greatest element i_0. Then A_{i_0} is a lattice ordered group and A_i is a linearly ordered group for each $i \in I \backslash \{i_0\}$. We have $G = A \circ A_{i_0}$ where $A = \Gamma A_i$ $(i \in I \setminus \{i_0\})$. Assume that $M \neq \{0\}$. Concerning $\alpha(M)$-convergence in G, we obtain an analogous result to that in Example 2.8.

The direct product of lattice ordered groups is defined in the usual way. Let $G = \prod\limits_{i \in I} G_i$ be the direct product of the system $\{G_i\}_{i \in I}$ and let H be a subset of all elements $g \in G$ such that the set $\big\{i \in I : \ g(i) \neq 0\big\}$ is finite. Then H is an ℓ-subgroup of G; it is said to be a direct sum of the system $\{G_i\}_{i \in I}$; we express this fact by writing $H = \sum\limits_{i \in I} G_i$. If the set I is finite then $\prod\limits_{i \in I} G_i = \sum\limits_{i \in I} G_i$. We apply the notion of direct sum to investigate the $\alpha(M)$-convergence.

Let I be a nonempty set and let G_i be an abelian lattice ordered group for each $i \in I$. In the following two lemmas we assume that G is the direct sum of G_i, $G = \sum\limits_{i \in I} G_i$, M is as before. It is easy to see that $M_i = \big\{b_i \in G_i : \ b \in M\big\}$ is a set of archimedean elements in G_i, which is closed with respect to the addition. We consider M_i as the set of convergence regulators in G_i. The set of all Cauchy sequences in G_i with respect to $\alpha(M_i)$-convergence is denoted by F_i.

Lemma 2.9. *Let (x_n) be a sequence in G. If $(x_n) \in F$ then $(x_n(i)) \in F_i$ for each $i \in I$.*

Lemma 2.10. *Let (x_n) be a sequence in G and $x \in G$. If $x_n \to_{\alpha(M)} x$ then $x_n(i) \to_{\alpha(M_i)} x(i)$ for each $i \in I$.*

Lemmas 2.9 and 2.10 are easy to prove. If I is a finite set then also the converse assertions are satisfied. However, if I is infinite, the converses fail to hold in general.

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Example 2.11. Let $G = \sum\limits_{i\in\mathbb{N}} G_i$ where $G_i = R$ for each $i \in \mathbb{N}$. Consider the sequence (x_n) in G, $x_1 = (1, 0, 0, \dots)$, $x_2 = (0, 2, 0, 0, \dots)$, $x_3 = (0, 0, 3, 0, 0, \dots)$, $x_4 = (0, 0, 0, 4, 0, \dots)$, $\dots$. For each $i \in I$ and any $M_i \subseteq G_i^+$ we have $x_n(i) \to_{\alpha(M_i)} 0$, but $(x_n) \notin F$ for an arbitrary $M \subseteq G^+$.

Let $M_1, M_2 \subseteq \mathcal{A}$. Assuming that the sets M_1 and M_2 are nonempty and closed under the addition, we put $\alpha(M_1) \leqq \alpha(M_2)$ if and only if $x_n \to_{\alpha(M_1)} x$ implies $x_n \to_{\alpha(M_2)} x$. If $M_1 \subseteq M_2$ then $\alpha(M_1) \leqq \alpha(M_2)$, but not conversely. We will show that the converse implication is valid for a particular type of sets M_1 and M_2.

Let M be a nonempty subset of $\mathcal{A}$ closed under the addition. We form the set

$$\widetilde{M} = \big\{b \in \mathcal{A}:\ \big(\forall \boldsymbol{x} = (x_n) \in G^{\mathbb{N}}\big)\big(\forall x \in G\big)\big(x_n \xrightarrow{b} x \implies x_n \to_{\alpha(M)} x\big)\big\}.$$

In 2.12–2.14, G is assumed to be an abelian divisible lattice ordered group (for a construction of a divisible lattice ordered group cf., e.g. [10]).

THEOREM 2.12. *The set $\widetilde{M}$ is closed with respect to the addition.*

P r o o f. Assume that $b_1, b_2 \in \widetilde{M}$. We have to show that $b = b_1 + b_2 \in \widetilde{M}$. For this purpose, suppose that (x_n) is a sequence in G and $x \in G$ with the property $x_n \xrightarrow{b} x$. Our aim is to prove that $x_n \to_{\alpha(M)} x$.

The assumption $x_n \xrightarrow{b} x$ is equivalent to $|x_n - x| \xrightarrow{b} 0$. Denoting $y_n = |x_n - x|$, we have $y_n \geqq 0$ for all $n \in \mathbb{N}$ and $y_n \xrightarrow{b} 0$. Thus for each $k \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ with the property

$$k y_n \leqq b \qquad \text{for each} \quad n \in \mathbb{N},\ \ n \geqq n_0.$$

Hence

$$y_n \leqq \frac{1}{k} b = \frac{1}{k} b_1 + \frac{1}{k} b_2 \qquad \text{for each} \quad n \in \mathbb{N},\ \ n \geqq n_0.$$

Applying the Riesz decomposition property, we get

$$y_n = y_n^1 + y_n^2$$

where

$$0 \leqq y_n^1 \leqq \frac{1}{k} b_1,\ \ 0 \leqq y_n^2 \leqq \frac{1}{k} b_2 \qquad \text{for each} \quad n \in \mathbb{N},\ \ n \geqq n_0.$$

Whence $y_n^1 \xrightarrow{b_1} 0$ and $y_n^2 \xrightarrow{b_2} 0$. The assumption yields $y_n^1 \to_{\alpha(M)} 0$ and $y_n^2 \to_{\alpha(M)} 0$. By 2.2, $|x_n - x| = y_n = y_n^1 + y_n^2 \to_{\alpha(M)} 0$, i.e., $x_n \to_{\alpha(M)} x$. □

It is easy to see that $M \subseteq \widetilde{M}$, $0 \in \widetilde{M}$, $\widetilde{M}$ is a convex subset of $\mathcal{A}$, $\alpha(M) = \alpha(\widetilde{M})$ and that $\widetilde{M}$ is the greatest of all $M \subseteq \mathcal{A}$, closed under addition with $\alpha(M) = \alpha(\widetilde{M})$.

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Lemma 2.13. *Let M_1 and M_2 be nonempty subsets of $\mathcal{A}$ closed with respect to the addition. Then $\alpha(M_1) \leqq \alpha(M_2)$ if and only if $\widetilde{M_1} \subseteq \widetilde{M_2}$.*

Proof. If $\widetilde{M_1} \subseteq \widetilde{M_2}$ then obviously $\alpha(M_1) = \alpha(\widetilde{M_1}) \leqq \alpha(\widetilde{M_2}) = \alpha(M_2)$. In order to prove the converse implication, we assume that $b_1 \in \widetilde{M_1}$, (x_n) is a sequence in G and $x \in G$ such that $x_n \xrightarrow{b_1} x$. Then $x_n \to_{\alpha(M_1)} x$. The assumption implies $x_n \to_{\alpha(M_2)} x$. This shows that $b_1 \in \widetilde{M_2}$. □

Let $b \in \mathcal{A}$. The convergence $\alpha(\widetilde{M_b})$ is said to be the *principal convergence* generated by the element b.

As observed earlier in Section 1, b-convergence and $\alpha(M_b)$-convergence coincide. Then we get:

Lemma 2.14. *Let $b \in \mathcal{A}$. The following conditions are equivalent:*

(i) $x_n \xrightarrow{b} x$.

(ii) $x_n \to_{\alpha(\widetilde{M_b})} x$.

Let H be a convex ℓ-subgroup of G, (x_n) a sequence in H and $x \in H$. It can happen that $x_n \to_{\alpha(\widetilde{M_g})} x$ for some $g \in \mathcal{A}$, but there is no archimedean element $h \in H$ such that $x_n \to_{\alpha(\widetilde{M_h})} x$.

Example 2.15. Let G be the set of all real functions f defined on the interval $[0,1]$ such that $f(1) = 0$. Then G is an archimedean lattice ordered group with respect to the operation $+$ and the partial order performed componentwise. Let H be the set of all functions f from G such that there exists $0 < x_f < 1$ with the property $f(x) = 0$ whenever $x \in [0,1]$, $x > x_f$. Then H is a convex ℓ-subgroup of G. Consider the sequence $(f_n(x))$ with

$$f_n(x) = \begin{cases} \frac{1}{n}, & \text{if } 0 \leqq x \leqq 1 - \frac{1}{n}, \\ 0, & \text{if } 1 - \frac{1}{n} < x \leqq 1 \end{cases}$$

and the function

$$g(x) = \begin{cases} 1, & \text{if } 0 \leqq x < 1, \\ 0, & \text{if } x = 1. \end{cases}$$

We see that $(f_n(x))$ is a sequence in H and $0 < g(x) \in G \backslash H$. It is easy to check that $f_n(x) \xrightarrow{g(x)} 0$, but there is no element $0 < h(x)$ in H with $f_n(x) \xrightarrow{h(x)} 0$. Finally, we apply Lemma 2.14.

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3. Cauchy completeness of G and $A(G)$

Again, let G be an abelian lattice ordered group, $A(G)$ as in Section 1 and let M be as above. If the role of G is to be emphasized, then we write $\alpha(M;G)$ rather than $\alpha(M)$.

We obviously have $M \subseteq A(G)$; then from the definition of $\alpha(M;G)$ we immediately obtain:

Lemma 3.1. *Let (x_n) be a sequence in $A(G)$ and $x \in A(G)$. Then the following conditions are equivalent:*

(i) $x_n \to_{\alpha(M;G)} x$;

(ii) $x_n \to_{\alpha(M;A(G))} x$.

Order convergence (o-convergence) of a sequence (x_n) in G to an element $x \in G$ will be denoted by $x_n \to_o x$.

Also, the following assertion is easy to verify.

Lemma 3.2. *Let (x_n) be a sequence in G and $x \in G$. Then the following conditions are equivalent:*

(i) $x_n \to_o x$ *in* G;

(ii) *there exists $m \in \mathbb{N}$ such that the sequence (x_{m+n}) o-converges to x in G.*

Assume that (x_n) is a sequence in G, $x \in G$ and that

$$x_n \to_{\alpha(M;G)} x.$$

Thus there exists $b \in M$ and $m \in \mathbb{N}$ such that

$$|x_n - x| \leqq b$$

for each $n \in \mathbb{N}$ with $n \geqq m$.

Denote $|x_n - x| = y_n$. Hence $y_n \in A(G)$ for $n \geqq m$. Also,

$$y_{m+n} \to_{\alpha(M;G)} 0.$$

Then according to Lemma 3.1, we have

$$y_{m+n} \to_{\alpha(M;A(G))} 0.$$

Assume that the lattice ordered group $A(G)$ is either σ-complete or divisible. Then according to [14, Proposition 3.4, Proposition 3.5], we obtain that the sequence (y_{m+n}) o-converges to 0 in $A(G)$. From this and from the fact that $A(G)$ is a convex ℓ-subgroup of G we conclude that the sequence (y_{m+n}) o-converges to 0 in G. Then Lemma 3.2 yields that the sequence (y_n) o-converges to 0 in G. Due to the definition of y_n we get that the sequence (x_n) o-converges to x in G.

Therefore we obtain:

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Proposition 3.3. *Let (x_n) be a sequence in G, $x \in G$ and $x_n \to_{\alpha(M;G)} x$. Assume that the archimedean kernel $A(G)$ of G is either σ-complete or divisible. Then $x_n \to_o x$ in the lattice ordered group G.*

This extends [14, Proposition 3.4, Proposition 3.5].

If G is divisible, then, clearly, $A(G)$ is divisible as well. If G is σ-complete, then it is archimedean and hence $A(G) = G$. Thus from Proposition 3.3 we infer:

Corollary 3.4. *Let G, (x_n) and x be as in* Proposition 3.3. *Assume that G is either σ-complete or divisible. Then $x_n \to_o x$ is valid in G.*

We remark that when speaking about Cauchy completeness in this section, we always consider this notion with respect to a convergence generated by a fixed system M of regulators.

Proposition 3.5. *Let M be as above. The following conditions are equivalent:*

(i) *G is Cauchy complete with respect to $\alpha(M)$;*

(ii) *$A(G)$ is Cauchy complete with respect to $\alpha(M)$.*

Proof. Let (i) be valid. From the fact that $A(G)$ is a convex ℓ-subgroup of G we infer that (ii) holds.

Conversely, assume that (ii) is satisfied. Let (x_n) be a Cauchy sequence (with respect to $\alpha(M)$) in G. Hence there exists $b \in M$ such that for each $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ with

$$k|x_n - x_m| \leqq b$$

whenever $n, m \geqq n_k$.

For each $n \in \mathbb{N}$, let us put

$$y_n = x_n - x_{n_1}.$$

Let us notice that

$$|y_n| = |x_n - x_{n_1}| \leqq b$$

for all $n \geqq n_1$. Thus (y_{n+n_1}) is a sequence in $A(G)$. Further, as

$$k|y_n - y_m| = k|x_n - x_{n_1} - x_m + x_{n_1}| = k|x_n - x_m| \leqq b.$$

for all $n, m \geqq n_k$, we infer that (y_n), just like (y_{n+n_1}) is a Cauchy sequence in $A(G)$. By (ii), there exists $y \in A(G)$ such that

$$y_{n+n_1} \to_{\alpha(M;A(G))} y.$$

Then $y_{n+n_1} \to_{\alpha(M;G)} y$ and this implies that

$$x_{n+n_1} = y_{n+n_1} + x_{n_1} \to_{\alpha(M)} y + x_{n_1}.$$

Consequently

$$x_n \to_{\alpha(M)} y + x_{n_1}.$$

Therefore (x_n) is convergent with respect to $\alpha(M)$. Thus the condition (i) is satisfied. □

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4. Dedekind completion

In the present section we deduce some results concerning Dedekind completions of lattice ordered groups. We apply the notation as in [8, Chapter §10] with the distinction that the group operation is written additively.

We recall some relevant notions. Let G be a lattice ordered group. For each nonempty upper bounded subset X of G we denote by $U(X)$ the set of all upper bounds of X; further, let $X^{\#} = L(U(X))$ be the set of all lower bounds of $U(X)$. The system of all such sets $X^{\#}$ will be denoted by $D_0(G)$; this system is partially ordered by the set-theoretical inclusion. For $X^{\#}$ and $Y^{\#}$ from $D_0(G)$ we put

$$X^{\#} +_1 Y^{\#} = (X+Y)^{\#}.$$

Further, let $D(G)$ be the set of all sets $X^{\#}$ having the property that there exists $Y^{\#} \in D_0(G)$ with

$$X^{\#} +_1 Y^{\#} = \{0\}^{\#}.$$

Then (cf. [8]) $D(G)$ is closed with respect to the operation $+_1$. If we consider the mapping $G \to D(G)$ defined by

$$x \to \{x\}^{\#},$$

then we obtain an embedding of G into $D(G)$. In fact, we will identify x and $\{x\}^{\#}$; in this way, G turns out to be an ℓ-subgroup of $D(G)$. We say that $D(G)$ is the Dedekind completion of G.

We denote by $\mathcal{D}$ the class of all lattice ordered groups G such that $G = D(G)$. Obviously, each complete lattice ordered group belongs to $\mathcal{D}$. On the other hand, a lattice ordered group belonging to $\mathcal{D}$ need not be complete. A necessary and sufficient condition for a lattice ordered group G to belong to $\mathcal{D}$ is given in [9].

The notion of a generalized Dedekind completion $D_1(G)$ of a lattice ordered group G has been introduced and studied in [11]; cf. also [12] and [13]; we recall the relevant basic facts.

Let G be a lattice ordered group. There exists a lattice ordered group $D_1(G)$ such that the following conditions are fulfilled:

(i) G is an ℓ-subgroup of $D_1(G)$.

(ii) $D(A(G))$ is an ℓ-ideal of $D_1(G)$.

(iii) If $x \in G$ and X is a nonempty subset of $x + A(G)$ such that X is upper-bounded in $x + A(G)$, then there is $x_0 \in D_1(G)$ with $\sup X = x_0$.

(iv) For each $x_0 \in D_1(G)$ there exists $x \in G$ and a nonempty subset $X \subseteq x + A(G)$ such that X is upper-bounded in $x + A(G)$ and $x_0 = \sup X$.

The lattice ordered group $D_1(G)$ is said to be the generalized Dedekind completion of G. A constructive description of $D_1(G)$ was presented in [11].

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In fact, $D_1(G)$ is an amalgam of lattice ordered groups G and $D(A(G))$ with the common ℓ-subgroup $A(G)$. The generalized Dedekind completion $D_1(G)$ is uniquely determined, up to isomorphisms leaving all elements of G fixed.

If G is archimedean, then $D_1(G) = D(G)$. There exists an abelian lattice ordered group G such that $D_1(G)$ fails to be isomorphic to $D(G)$ (cf. [13]).

Proposition 4.1. (Cf. [11, Proposition 2.14].) *For each lattice ordered group G, the relation*

$$A(D_1(G)) = D(A(G))$$

is valid.

We remark that since the lattice ordered group $A(G)$ is archimedean, $D(A(G)) = D_1(A(G))$. Thus the relation given in Proposition 4.1 can be written in the form

$$A(D_1(G)) = D_1(A(G)).$$

It is well-known that the Dedekind completion of an archimedean lattice ordered group is a complete lattice ordered group. Hence applying Proposition 4.1 we get:

Lemma 4.2. *For each lattice ordered group G, the lattice ordered group $A(D_1(G))$ is complete.*

Similarly as above, when speaking about Cauchy completeness, we have in mind the convergence $\alpha(M)$, where M is a fixed system of regulators of convergence with $M \subseteq (A(G))^+$ (cf. Section 1).

Proposition 4.3. *Let G be an abelian lattice ordered group. Then $D_1(G)$ is Cauchy complete.*

Proof. In view of [6, Corollary 4.5], each complete lattice ordered group is Cauchy complete. Thus according to Lemma 4.2, $A(D_1(G))$ is Cauchy complete. Now, it suffices to apply Proposition 3.5. $\square$

Our aim is to verify that a result analogous to Proposition 4.3 is valid for the Dedekind completion of an abelian lattice ordered group.

Lemma 4.4. (Cf. [11, Corollary 2.19].) *Let $\emptyset \neq \{a_i\}_{i\in I}$ be a set of archimedean elements of a lattice ordered group G. Assume that the relation $\bigvee_{I\in I} a_i = b$ is valid in G. Then b is an archimedean element of G.*

Proposition 4.5. (Cf. [13, Proposition 3.1].) *The archimedean kernel of $D(G)$ is the set of all elements $h \in D(G)$ with the property that $|h| = \sup Z$ for a subset $Z \subseteq A(G)$.*

From Lemma 4.4 and Proposition 4.5, we obtain by a simple calculation:

Proposition 4.6. *Let G be an abelian lattice ordered group. Then the relation*

$$A(D(G)) = D(A(G))$$

is valid.

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Proposition 4.7. *Let G be an abelian lattice ordered group. Then $D(G)$ is Cauchy complete.*

Proof. Since $A(G)$ is archimedean, from Proposition 4.6 we get the assertion

$$A(D(G)) \text{ is a complete lattice ordered group.} \tag{$*$}$$

Now, it suffices to apply the same argument as in the proof of Proposition 4.3 with the distinction that $(*)$ is used instead of Lemma 4.2. □

5. $\alpha(M)$-closed convex ℓ-subgroups

Again, let $M \subseteq \mathcal{A}, M \neq \emptyset$. We assume that M is closed under the addition.

The set $c(G)$ of all convex ℓ-subgroups of G is a complete lattice under the set inclusion. The lattice operations in $c(G)$ will be denoted by $\wedge$ and $\vee$. Let $\{G_i :\ i \in I\} \subseteq c(G)$. Then $\bigwedge\limits_{i\in I} G_i = \bigcap\limits_{i\in I} G_i$ and $\bigvee\limits_{i\in I} G_i$ coincides with the lattice operation of join in the lattice of all subgroups of G, i.e., it is the subgroup of G generated by the subgroups G_i $(i \in I)$ of G.

Let A be a convex ℓ-subgroup of G. Then A is called $\alpha(M)$-*closed* if for every sequence (x_n) in A with $x_n \to_{\alpha(M)} x$ in G, the limit x belongs to A.

The set of all $\alpha(M)$-closed convex ℓ-subgroups of G will be denoted by $\mathrm{cl}(G)$. Let $\{G_i :\ i \in I\} \subseteq \mathrm{cl}(G)$. It is easy to check that $\bigcap\limits_{i\in I} G_i \in \mathrm{cl}(G)$. As $G \in \mathrm{cl}(G)$, the set $\mathrm{cl}(G)$ is a complete lattice under the set inclusion. The lattice operations in $\mathrm{cl}(G)$ will be denoted by $\sqcap$ and $\sqcup$; thus $\bigsqcap\limits_{i\in I} G_i = \bigcap\limits_{i\in I} G_i$ and $\bigsqcup\limits_{i\in I} G_i$ is the set intersection of all convex $\alpha(M)$-closed ℓ-subgroups of G including the set $\bigcup\limits_{i\in I} G_i$.

Assume that A is a convex ℓ-subgroup of G. Let $\{A_i :\ i \in I\}$ be the system of all elements of $\mathrm{cl}(G)$ with $A \subseteq A_i$. Then $\overline{A} = \bigcap\limits_{i\in I} A_i$ is the least convex $\alpha(M)$-closed ℓ-subgroup of G containing A as an ℓ-subgroup.

The following lemma is easy to verify.

Lemma 5.1. *Let $\{G_i :\ i \in I\} \subseteq \mathrm{cl}(G)$. Then $\bigsqcup\limits_{i\in I} G_i = \overline{\bigvee\limits_{i\in I} G_i}$.*

Let $\{G_i :\ i \in I\} \subseteq \mathrm{cl}(G)$. It can happen that $\bigvee\limits_{i\in I} G_i$ is different from $\bigsqcup\limits_{i\in I} G_i$ in $\alpha(M)$-convergence for some $M \subseteq \mathcal{A}$.

Example 5.2. Let $G = \prod\limits_{i\in\mathbb{N}} G_i$ where $G_i = R$ for each $i \in \mathbb{N}$ and $G_i^0 = \big\{x \in G :\ x(j) = 0 \text{ for each } j \in \mathbb{N},\ j \neq i\big\}$. Then G_i^0 is a convex $\alpha(M)$-closed ℓ-subgroup of G for each $M \subseteq \mathcal{A}$ and for each $i \in \mathbb{N}$. Since $H = \bigvee\limits_{i\in\mathbb{N}} G_i^0$ is the subgroup of G generated by all subgroups G_i^0 of G, H consists of all elements x from G

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such that the set $\{i \in \mathbb{N} : \ x(i) \neq 0\}$ is finite. It suffices to prove that H is not $\alpha(M)$-closed for some $M \subseteq \mathcal{A}$. Consider $\alpha(M)$-convergence such that the element $b = (1, 1, \dots) \in M$. The sequence (x_n) with $x_n = \left(1, \frac{1}{2}, \dots, \frac{1}{n}, 0, 0, \dots\right)$ for each $n \in \mathbb{N}$ is a sequence in H and $x_n \to_{\alpha(M)} x = \left(1, \frac{1}{2}, \frac{1}{3}, \dots\right)$, because $x_n \xrightarrow{b} x$. Then $x \in G$, but $x \notin H$ and so H fails to be $\alpha(M)$-closed.

Let A be a convex ℓ-subgroup of G. We denote by A' the set of all elements x of G such that there exists a sequence (x_n) in A such that $x_n \to_{\alpha(M)} x$.

Lemma 5.3. *A' is a convex ℓ-subgroup of G.*

Proof. Evidently, A' is an ℓ-subgroup of G. To prove that A' is convex, assume that $x \in A'$, $y \in G$, $0 \leqq y \leqq x$. There is a sequence (x_n) in A such that $x_n \to_{\alpha(M)} x$. It is easy to verify that without loss of generality we can suppose that $x_n \geqq 0$ for each $n \in \mathbb{N}$. Hence, $(x_n \wedge y)$ is a sequence in A and $x_n \wedge y \to_{\alpha(M)} x \wedge y = y$, so $y \in A'$ and A' is convex. $\square$

In [15] there is defined a diagonal property for relatively uniform convergence in a vector lattice. This notion can be defined analogously for $\alpha(M)$-convergence in G. The definition is as follows.

We say that the lattice ordered group G has the *diagonal property* if the following condition is satisfied:

Let (x_{nk}) be a double sequence in G, (x_n) a sequence in G and $x_0 \in G$ such that $x_{nk} \to_{\alpha(M)} x_n$ for each $n \in \mathbb{N}$ (if $k \to \infty$) and $x_n \to_{\alpha(M)} x_0$. Then for each $n \in \mathbb{N}$ there exists $k(n) \in \mathbb{N}$ such that $x_{n,k(n)} \to_{\alpha(M)} x_0$.

Lemma 5.4. *If G has the diagonal property for $\alpha(M)$-convergence then $A' = \overline{A}$ for each convex ℓ-subgroup A of G.*

The proof is analogous to that in vector lattices for relatively uniform convergence [15].

Lemma 5.5. *Let G possess the diagonal property. If $A, B \in c(G)$, then $\overline{A \cap B} = \overline{A} \cap \overline{B}$.*

Proof. From $A \cap B \subseteq A, B$ we infer that $\overline{A \cap B} \subseteq \overline{A}, \overline{B}$, so $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. Conversely, we will show that $\overline{A} \cap \overline{B} \subseteq \overline{A \cap B}$. With respect to hypothesis and Lemma 5.4, we have to verify that $A' \cap B' \subseteq (A \cap B)'$. Let $0 \leqq x \in A' \cap B'$. Then there are sequences (x_n) in A and (y_n) in B such that $x_n \to_{\alpha(M)} x$ and $y_n \to_{\alpha(M)} x$ in G. Similarly as in the proof of Lemma 5.3, we can suppose that $x_n, y_n \geqq 0$ for any $n \in \mathbb{N}$. The sequence $(x_n \wedge y_n)$ is in $A \cap B$ and $x_n \wedge y_n \to_{\alpha(M)} x$ in G. This yields that $x \in (A \cap B)'$. Therefore, we obtain the desired result. $\square$

Theorem 5.6. *Let G have the diagonal property. Then the lattice* $\mathrm{cl}(G)$ *is Brouwerian.*

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P r o o f. Let $A \in \mathrm{cl}(G)$ and $B_i \in \mathrm{cl}(G)$ for each $i \in I$. We have to verify that

$$A \sqcap \left(\bigsqcup_{i\in I} B_i \right) = \bigsqcup_{i\in I} (A \sqcap B_i)$$

is valid. It is well-known that $c(G)$ is a Brouwerian lattice. Using this fact together with Lemmas 5.1 and 5.5 we obtain

$$\begin{aligned} A \sqcap \left(\bigsqcup_{i\in I} B_i \right) &= A \cap \overline{\bigvee_{i\in I} B_i} = \overline{A} \cap \overline{\bigvee_{i\in I} B_i} = \overline{A \cap \left(\bigvee_{i\in I} B_i \right)} = \overline{\bigvee_{i\in I} (A \cap B_i)} \\ &= \bigsqcup_{i\in I} (A \cap B_i) = \bigsqcup_{i\in I} (A \sqcap B_i). \end{aligned}$$

□

6. The system $s(G)$

Let $s(G)$ be the system of all wru-convergences on G (for all possible $\emptyset \neq M \subseteq \mathcal{A}$ closed under the addition). It will be established that $s(G)$ is a complete Brouwerian lattice.

Lemma 6.1. (Cf. [11].) *Let $b_1, b_2 \in \mathcal{A}$. Then $b_1 \vee b_2 \in \mathcal{A}$.*

Lemma 6.2. *Let $b_1, b_2 \in \mathcal{A}$. Then $b_1 + b_2 \in \mathcal{A}$.*

P r o o f. Let $b_1, b_2 \in \mathcal{A}$. By Lemma 6.1, $b_1 \vee b_2 \in \mathcal{A}$. From $b_1, b_2 \leqq b_1 \vee b_2$ we get $b_1 + b_2 \leqq 2(b_1 \vee b_2)$. Then $2(b_1 \vee b_2) \in \mathcal{A}$ implies $b_1 + b_2 \in \mathcal{A}$. □

Let $b_1, \dots, b_n \in \mathcal{A}$. Applying Lemma 6.2 and by induction we obtain $b_1 + \dots + b_n \in \mathcal{A}$.

When dealing with regulators of a relative uniform convergence, in some situations, it seems to be more convenient to proceed without the assumption that the set M under consideration is closed with respect to the addition.

Thus, we introduce the following definition.

Let M be a nonempty subset of $\mathcal{A}$, (x_n) a sequence in G and $x \in G$. We say that this sequence $\alpha_0(M)$-*converges* to x, written $x_n \to_{\alpha_0(M)} x$, if $x_n \xrightarrow{b} x$ for some $b = b_1 + \dots + b_m$ with $b_i \in M$ $(i = 1, \dots, m)$.

Remark that $\alpha_0(M) = \alpha(M)$ whenever M is closed with respect to the addition.

Given $\emptyset \neq M \subseteq \mathcal{A}$, the symbol M^0 will denote the set consisting of all elements $b \in G$ which can be expressed in the form $b = b_1 + \dots + b_m$ for some $b_1, \dots, b_m \in M$. The set M^0 is closed with respect to the addition and $\alpha_0(M) = \alpha(M^0)$ is valid.

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Analogously to definition of $\widetilde{M}$ in Section 2, we define the set

$$\overline{M} = \big\{b \in \mathcal{A}:\ \big(\forall \mathbf{x} = (x_n) \in G^{\mathbb{N}}\big)\big(\forall x \in G\big)\big(x_n \xrightarrow{b} x \implies x_n \to_{\alpha_0(M)} x\big)\big\}.$$

In 6.3–6.7, we assume that G is an abelian divisible lattice ordered group.

Lemma 6.3. *Let $\emptyset \neq M \subseteq \mathcal{A}$. Then the set $\overline{M}$ is closed with respect to the addition.*

P r o o f. It is easy to verify that the relations

$$\overline{M} = \widetilde{M^0} = \overline{M^0} \tag{2}$$

are fulfilled. By 2.12, $\widetilde{M^0}$ is closed under the addition. Then (2) completes the proof. □

Lemma 6.4. *Let $\emptyset \neq M \subseteq \mathcal{A}$. Then $\alpha_0(M) = \alpha_0(\overline{M})$.*

P r o o f. In view of (2) and 2.12 we have

$$\alpha_0(\overline{M}) = \alpha_0(\widetilde{M^0}) = \alpha(\widetilde{M^0}) = \alpha(M^0) = \alpha_0(M).$$

□

It is easy to see that $M \subseteq \overline{M}$, $0 \in \overline{M}$, $\overline{M}$ is a convex subset of $\mathcal{A}$ and that $\overline{M}$ is the greatest of all $\emptyset \neq M \subseteq \mathcal{A}$, with $\alpha_0(M) = \alpha_0(\overline{M})$.

The proof of the following lemma is analogous to that of Lemma 2.13.

Lemma 6.5. *Let $M_1 and M_2$ be nonempty subsets of $\mathcal{A}$. Then $\alpha_0(M_1) \leqq \alpha_0(M_2)$ if and only if $\overline{M}_1 \subseteq \overline{M}_2$.*

Theorem 6.6. *The set $s(G)$ is a complete lattice. If I is a nonempty set and for each $i \in I$, M_i is a nonempty subset of $\mathcal{A}$ closed with respect to the addition, then*

(i) $\bigwedge\limits_{i\in I} \alpha(M_i) = \alpha\Big(\bigcap\limits_{i\in I} \overline{M}_i\Big)$,

(ii) $\bigvee\limits_{i\in I} \alpha(M_i) = \alpha\Big(\overline{\bigcup\limits_{i\in I} M_i}\Big)$.

P r o o f.

(i) The relation $\bigwedge\limits_{i\in I} \alpha(M_i) = \bigwedge\limits_{i\in I} \alpha_0(M_i)$ is valid, since all sets M_i are closed with respect to the addition. According to Lemma 6.3, all $\overline{M}_i$ are closed under the addition, thus so does $\bigcap\limits_{i\in I} \overline{M}_i$. Hence $\alpha\Big(\bigcap\limits_{i\in I} \overline{M}_i\Big) = \alpha_0\Big(\bigcap\limits_{i\in I} \overline{M}_i\Big)$. Thus we have to prove that the relation

$$\bigwedge_{i\in I} \alpha_0(M_i) = \alpha_0\Big(\bigcap_{i\in I} \overline{M}_i\Big) \tag{3}$$

is valid.

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From $\bigcap\limits_{i\in I}\overline{M}_i \subseteq \overline{M}_{i_1}$ for each $i_1 \in I$ and from Lemma 6.4 we deduce that $\alpha_0\Bigl(\bigcap\limits_{i\in I}\overline{M}_i\Bigr) \leqq \alpha_0(\overline{M}_{i_1}) = \alpha_0(M_{i_1})$ for each $i_1 \in I$.

Suppose that $M \subseteq \mathcal{A}$ and that $\alpha_0(M) \leqq \alpha_0(\overline{M}_i)$ for each $i \in I$. Then by Lemma 6.5, $\overline{M} \subseteq \overline{\overline{M}}_i = \overline{M}_i$ for each $i \in I$ and so $\alpha_0(M) = \alpha_0(\overline{M}) \leqq \alpha_0\Bigl(\bigcap\limits_{i\in I}\overline{M}_i\Bigr)$ because $\overline{M} \subseteq \bigcap\limits_{i\in I}\overline{M}_i$. Consequently, (3) is valid and hence (i) is satisfied.

The convergence $\alpha(\mathcal{A})$ is the greatest element of $s(G)$. Hence $s(G)$ is a complete lattice; $\alpha(\{0\})$ is the least element of $s(G)$.

(ii) By using the same argument as above we get $\bigvee\limits_{i\in I}\alpha(M_i) = \bigvee\limits_{i\in I}\alpha_0(M_i)$ and $\alpha\Bigl(\overline{\bigcup\limits_{i\in I}M_i}\Bigr) = \alpha_0\Bigl(\overline{\bigcup\limits_{i\in I}M_i}\Bigr) = \alpha_0\Bigl(\bigcup\limits_{i\in I}M_i\Bigr)$. Hence we want to show that

$$\bigvee_{i\in I}\alpha_0(M_i) = \alpha_0\Bigl(\bigcup_{i\in I}M_i\Bigr) \tag{4}$$

holds.

For each $i_1 \in I$, $M_{i_1} \subseteq \bigcup\limits_{i\in I}M_i$ is valid, so $\alpha_0(M_{i_1}) \leqq \alpha_0\Bigl(\bigcup\limits_{i\in I}M_i\Bigr)$ for each $i_1 \in I$.

Assume that $M \subseteq \mathcal{A}$ and $\alpha_0(M_i) \leqq \alpha_0(M)$ for each $i \in I$. Lemma 6.5 yields $\overline{M}_i \subseteq \overline{M}$ for each $i \in I$. Hence $\bigcup\limits_{i\in I}\overline{M}_i \subseteq \overline{M}$. This and Lemma 6.4 imply $\alpha_0\Bigl(\bigcup\limits_{i\in I}\overline{M}_i\Bigr) \leqq \alpha_0(\overline{M}) = \alpha_0(M)$.

Now we show that

$$\alpha_0\Bigl(\bigcup_{i\in I}\overline{M}_i\Bigr) = \alpha_0\Bigl(\bigcup_{i\in I}M_i\Bigr). \tag{5}$$

On account of $\bigcup\limits_{i\in I}M_i \subseteq \bigcup\limits_{i\in I}\overline{M}_i$ we have $\alpha_0\Bigl(\bigcup\limits_{i\in I}M_i\Bigr) \leqq \alpha_0\Bigl(\bigcup\limits_{i\in I}\overline{M}_i\Bigr)$. The inclusion $M_{i_1} \subseteq \bigcup\limits_{i\in I}M_i$ for each $i_1 \in I$ yields $\overline{M}_{i_1} \subseteq \overline{\bigcup\limits_{i\in I}M_i}$ for each $i_1 \in I$, so $\bigcup\limits_{i\in I}\overline{M}_i \subseteq \overline{\bigcup\limits_{i\in I}M_i}$. Therefore $\alpha_0\Bigl(\bigcup\limits_{i\in I}\overline{M}_i\Bigr) \leqq \alpha_0\Bigl(\overline{\bigcup\limits_{i\in I}M_i}\Bigr) = \alpha_0\Bigl(\bigcup\limits_{i\in I}M_i\Bigr)$ by Lemma 6.4, whence (5) is satisfied which completes the proof of the part (ii). □

Theorem 6.7. *The lattice $s(G)$ is Brouwerian.*

Proof. Let I be a nonempty set, $M_i \subseteq \mathcal{A}$ for each $i \in I$ and $M \subseteq \mathcal{A}$. Suppose that all M_i and M are nonempty and closed under the addition. We have to

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prove that the relation

$$\alpha(M) \wedge \left(\bigvee_{i\in I} \alpha(M_i)\right) = \bigvee_{i\in I} (\alpha(M) \wedge \alpha(M_i)) \tag{6}$$

holds.

In view of (3) and (4) we get

$$\begin{aligned}
\alpha(M) \wedge \left(\bigvee_{i\in I} \alpha(M_i)\right) &= \alpha_0(M) \wedge \left(\bigvee_{i\in I} \alpha_0(M_i)\right)\\
&= \alpha_0(M) \wedge \alpha_0\left(\bigcup_{i\in I} M_i\right) = \alpha_0\left(\overline{M} \cap \overline{\bigcup_{i\in I} M_i}\right),\\
\bigvee_{i\in I}\big(\alpha(M) \wedge \alpha(M_i)\big) &= \bigvee_{i\in I}\big(\alpha_0(M) \wedge \alpha_0(M_i)\big)\\
&= \bigvee_{i\in I} \alpha_0(\overline{M} \cap \overline{M}_i) = \alpha_0\left(\bigcup_{i\in I}(\overline{M} \cap \overline{M}_i)\right).
\end{aligned}$$

To prove that the relation (6) is valid, it suffices to verify that $\alpha(M) \wedge \left(\bigvee_{i\in I} \alpha(M_i)\right) \leqq \bigvee_{i\in I} (\alpha(M) \wedge \alpha(M_i))$, i.e., that

$$\alpha_0\left(\overline{M} \cap \overline{\bigcup_{i\in I} M_i}\right) \leqq \alpha_0\left(\bigcup_{i\in I}(\overline{M} \cap \overline{M}_i)\right).$$

Assume that $x_n \to_{\alpha_0\left(\overline{M}\cap\overline{\bigcup_{i\in I} M_i}\right)} x$. Then $x_n \to_{\alpha_0(\overline{M})} x$ and $x_n \to_{\alpha_0\left(\overline{\bigcup_{i\in I} M_i}\right)} x$. By Lemma 6.4 and (5), $x_n \to_{\alpha_0\left(\bigcup_{i\in I} \overline{M_i}\right)} x$. Therefore there are $u_1, \dots, u_m \in \overline{M}$ and $v_1, \dots, v_k \in \bigcup_{i\in I} \overline{M}_i$ such that for each $p \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ with the property

$$p|x_n - x| \leqq u_1 + \dots + u_m \qquad \text{and} \qquad p|x_n - x| \leqq v_1 + \dots + v_k$$

for each $n \in \mathbb{N}$, $n \geqq n_0$. Hence

$$\begin{aligned}
p|x_n - x| &\leqq (u_1 + \dots + u_m) \wedge (v_1 + \dots + v_k)\\
&\leqq u_1 \wedge v_1 + \dots + u_1 \wedge v_k + \dots + u_m \wedge v_1 + \dots + u_m \wedge v_k
\end{aligned}$$

for each $n \in \mathbb{N}$, $n \geqq n_0$. We have $u_j \wedge v_\ell \leqq u_j, v_\ell$ $(j = 1, \dots, m;\ \ell = 1, \dots, k)$, so $u_j \wedge v_\ell \in \overline{M} \cap \left(\bigcup_{i\in I} \overline{M}_i\right) = \bigcup_{i\in I} (\overline{M} \cap \overline{M}_i)$ $(j = 1, \dots, m;\ \ell = 1, \dots, k)$. Consequently, $x_n \to_{\alpha_0\left(\bigcup_{i\in I}(\overline{M}\cap\overline{M}_i)\right)} x$ and the proof is finished. $\square$

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Mathematical Institute
Slovak Academy of Sciences
Grešákova 6
Košice
SLOVAKIA
E-mail: stefan.cernak@tuke.sk
kstefan@saske.sk

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