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ON PRIMITIVE FOR THE GAP-INTEGRAL

D. K. Ganguly* — Ranu Mukherjee**

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ABSTRACT. The concept of the GAP-integral was introduced by the authors. In this paper some results on primitive for the GAP-integral are presented.

1. Introduction

The Approximately Continuous Perron integral was defined by Burkill [2] and its Riemann-type definition was given by Bullen [3]. Schwabik [6] presented a generalized version of the Perron integral leading to the new approach to a generalized ordinary differential equation. The authors developed a generalization of Approximately Continuous Perron (GAP)-integral and defined the primitive for the GAP-integral [1]. In the present paper attempt has been made to study some properties of primitive related to GAP-integral and also it is possible to give a characterization of the GAP-integral by its primitive. A fundamental type theorem for the GAP-integral has been established.

2. Preliminaries and definitions

Definition 2.1. A collection Δ of closed subintervals of $[a, b]$ is called an approximate full cover (AFC) if for every $x \in [a, b]$ there exists a measurable set $D_x \subset [a, b]$ such that $x \in D_x$ and D_x has density 1 at x, with $[u, v] \in \Delta$ whenever $u, v \in D_x$ and $u \leq x \leq v$.

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A division of $[a, b]$ obtained by $a = x_0 < x_1 < \cdots < x_n = b$ and $\{\xi_1, \xi_2, \ldots, \xi_n\}$ is called a Δ-division if Δ is an approximate full cover with $[x_{i-1}, x_i]$ coming from Δ or more precisely, if we have $x_{i-1} \le \xi_i \le x_i$ and $x_{i-1}, x_i \in D_{\xi_i}$ for all i. We call ξ_i the associated point of $[x_{i-1}, x_i]$ and x_i $(i = 0, 1, \ldots, n)$ the division points.

A division of $[a, b]$ given by $a \le y_1 < z_1 < y_2 < z_2 \cdots < y_m < z_m \le b$ and $\{\zeta_1, \zeta_2, \ldots, \zeta_m\}$ is called a Δ-partial division if $\bigcup\limits_{i=1}^{m} [y_i, z_i] \subset [a, b]$ and Δ is an approximate full cover with $[y_i, z_i]$ coming from Δ or more precisely, $y_i \le \zeta_i \le z_i$ and $y_i, z_i \in D_{\zeta_i}$ for all i.

In [1], the GAP-integral is defined as follows:

Definition 2.2. A function $U\colon [a, b] \times [a, b] \to \mathbb{R}$ is said to be Generalized Approximate Perron (GAP)-integrable to a real number A if for every $\varepsilon > 0$ there is an AFC Δ of $[a, b]$ such that for every Δ-division $D = ([\alpha, \beta], \tau)$ of $[a, b]$ we have

$$\Big| (D) \sum \{U(\tau, \beta) - U(\tau, \alpha)\} - A \Big| < \varepsilon$$

and we write $A = (\mathrm{GAP}) \int\limits_a^b U$.

The set of all functions U which are GAP-integrable on $[a, b]$ is denoted by $\mathrm{GAP}[a, b]$. We use the notation

$$S(U, D) = (D) \sum \big\{U(\tau, \beta) - U(\tau, \alpha)\big\}$$

for the Riemann-type sum corresponding to the function U and the Δ-division $D = ([\alpha, \beta], \tau)$ of $[a, b]$.

Note that the integral is uniquely determined.

With the notion of partial division we have proved the following theorem in [1].

Theorem 2.1 (Saks-Henstock Lemma). *Let $U : [a, b] \times [a, b] \to \mathbb{R}$ be (GAP)-integrable over $[a, b]$. Then, given $\varepsilon > 0$, there is an approximate full cover Δ of $[a, b]$ such that for every Δ-division $D = \big\{([\alpha_{j-1}, \alpha_j], \tau_j) :\ j = 1, 2, \ldots, q\big\}$ of $[a, b]$ we have*

$$\Big| \sum_{j=1}^{q} \big\{U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})\big\} - (\mathrm{GAP}) \int_a^b U \Big| < \varepsilon.$$

Then, if $\big\{([\beta_j, \gamma_j], \zeta_j);\ j = 1, 2, \ldots, m\big\}$ represents a Δ-partial division of $[a, b]$, we have

$$\Big| \sum_{j=1}^{m} \big[\{U(\zeta_j, \gamma_j) - U(\zeta_j, \beta_j)\} - (\mathrm{GAP}) \int_{\beta_j}^{\gamma_j} U\big] \Big| < \varepsilon.$$

In [1], the indefinite GAP-integral is defined as follows:

Definition 2.3. Let $U \in \mathrm{GAP}[a,b]$. The function $\phi\colon [a,b] \to \mathbb{R}$ defined by $\phi(s) = (\mathrm{GAP}) \int_a^s U$, $a < s \le b$, $\phi(a) = 0$ is called the indefinite GAP-integral of U.

For $[\alpha,\beta] \subset [a,b]$, we put $\phi(\alpha,\beta) = \phi(\beta) - \phi(\alpha) = (\mathrm{GAP}) \int_\alpha^\beta U$.

We take this function ϕ as the primitive of U.

We need the following definitions to establish some results in the consequence:

Definition 2.4. ([5]) If E is a measurable set, then E^d represents the set of all points $x \in E$ such that x is a point of density of E.

Definition 2.5. ([4]) A number A is said to be the approximate limit of a function f at x_0 if for every $\varepsilon > 0$ there exists a set D of density 1 at x_0 such that

$$|f(x) - A| < \varepsilon \qquad \text{for every} \quad x \in D \backslash \{x_0\}.$$

We write $\lim\limits_{x \to x_0} \mathrm{ap}\, f(x) = A$.

Next we define a function f to be approximately continuous at x_0 if

$$\lim_{x \to x_0} \mathrm{ap}\, f(x) = f(x_0).$$

The approximate derivative of f at x_0, denoted by $\mathrm{AD}\, f(x_0)$, is

$$\mathrm{AD}\, f(x_0) = \lim_{x \to x_0} \mathrm{ap}\, \frac{f(x) - f(x_0)}{x - x_0}.$$

We state Vitali's covering theorem without proof for further use:

Theorem 2.2. ([4]) *If a family of closed intervals covers a set X in the Vitali sense, i.e., for every $x \in X$ and $\eta > 0$ there is a closed interval I in the family such that*

$$x \in I \quad \text{and} \quad |I| < \eta,$$

then for every $\varepsilon > 0$ there is a finite number of disjoint closed intervals $I_1, I_2, \dots$ $\dots, I_n$ in the family such that

$$|X| < \sum_{i=1}^{n} |I_i| + \varepsilon$$

where $|X|$ denotes the outer measure of X.

We recall that the outer measure of X is defined to be

$$|X| = \inf\Big\{ \sum_i |I_i| : \bigcup_i I_i \supset X \Big\}$$

in which I_i denote intervals.

3. Some results on primitives

Theorem 3.1. *If $U \in \mathrm{GAP}[a,b]$ and ϕ is the primitive of U, then ϕ is measurable if the function $U(s,\cdot)\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ for $s \in [a,b]$.*

Proof. In order to prove that the function ϕ is measurable, let α be a real number.

Let $E = \big\{s \in (a,b) :\ \phi(s) > \alpha\big\}$. We shall show that the set E is measurable.

Let $s \in E$ and choose $\eta > 0$ such that $3\eta = \phi(s) - \alpha$.

Since $U \in \mathrm{GAP}[a,b]$, given $\eta > 0$, there is an approximate full cover Δ of $[a,b]$ such that for every Δ-division D of $[a,b]$ we have

$$|S(U,D) - \phi(a,b)| < \eta.$$

Again, since the function $U(s,\cdot)$ is continuous on $[a,b]$, for $\eta > 0$ there exists a $\delta_s > 0$ such that

$$|U(s,t) - U(s,s)| < \eta \qquad \text{whenever} \quad |t-s| < \delta_s.$$

Let $\big\{D_s :\ s \in [a,b]\big\}$ be the collection of sets generated by Δ. We will assume that each point of D_s is a point of density of D_s.

Suppose that $t \in (s, s+\delta_s) \cap D_s$. Let D be a Δ-division of $[a,s]$ and let $D' = D \cup ([s,t],s)$. Then D' is a Δ-division of $[a,t]$.

Hence

$$\begin{aligned} \phi(t) = \phi(a,t) &> S(U,D') - \eta \\ &= S(U,D) + \{U(s,t) - U(s,s)\} - \eta \\ &> \phi(a,s) - 3\eta = \phi(s) - 3\eta = \alpha. \end{aligned}$$

In a similar way, we can show that $\phi(t) > \alpha$ for all $t \in (s-\delta_s, s) \cap D_s$.

For each s, let $A_s = (s-\delta_s, s+\delta_s) \cap D_s$.

Then $A_s = A_s^d$ and $A_s \subseteq E$.

Hence the set $E = \bigcup\limits_{s \in E} A_s$ is measurable, ([5]).

Therefore, ϕ is a measurable function. □

Theorem 3.2. *If $U \in \mathrm{GAP}[a,b]$, then its primitive ϕ is approximately differentiable almost everywhere and the approximate derivative*

$$\mathrm{AD}\,\phi(\tau) = \lim_{t\to\tau} \mathrm{ap}\, \frac{U(\tau,t) - U(\tau,\tau)}{t-\tau}$$

for almost all $\tau \in [a,b]$, if the limit exists.

P r o o f. Since $U \in \mathrm{GAP}[a,b]$, given $\varepsilon > 0$, there exists an approximate full cover Δ of $[a,b]$ such that for every Δ-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$ we have

$$\sum |\{\phi(\beta) - \phi(\alpha)\} - \{U(\tau,\beta) - U(\tau,\alpha)\}| < \varepsilon.$$

Let

$$X = \Big\{\tau \in [a,b] : \text{either } \ \mathrm{AD}\,\phi(\tau) \ \text{ does not exist,} \qquad \text{or, if it does, it is not equal to } \ \lim_{t\to\tau}\mathrm{ap}\, \tfrac{U(\tau,t)-U(\tau,\tau)}{t-\tau}\Big\}.$$

We shall show that X is of measure zero.

From the definition of X we see that for each $\tau \in [a,b] - X$, there exists a measurable set $D_\tau \subset [a,b]$ of density 1 at τ and

$$\mathrm{AD}\,\phi(\tau) = \lim_{\substack{\beta\to\tau\\ \beta\in D_\tau}} \frac{\phi(\beta)-\phi(\tau)}{\beta-\tau}.$$

Then for each $\tau \in X$ there is a $\eta(\tau) > 0$ such that for every approximate neighbourhood D_τ of τ either there is a point $\alpha \in D_\tau$ with $0 < \tau - \alpha < \delta$ $(\delta > 0)$ and

$$\left|\frac{\phi(\tau)-\phi(\alpha)}{\tau-\alpha} - \left\{\frac{U(\tau,\tau)-U(\tau,\alpha)}{\tau-\alpha}\right\}\right| > \eta(\tau)$$

i.e.

$$|\{\phi(\tau) - \phi(\alpha)\} - \{U(\tau,\tau) - U(\tau,\alpha)\}| > \eta(\tau)|\tau-\alpha|.$$

or there is a point $\beta \in D_\tau$ with $0 < \beta - \tau < \delta$ and

$$|\{\phi(\beta) - \phi(\tau)\} - \{U(\tau,\beta) - U(\tau,\tau)\}| > \eta(\tau)|\beta-\tau|.$$

Let $X_n = \big\{\tau \in X :\ \eta(\tau) \geq \frac{1}{n}\big\}$, $n = 1,2,\ldots$ and so $X = \bigcup\limits_{n=1}^{\infty} X_n$.

Then the above family of closed intervals $[\alpha,\tau]$ or $[\tau,\beta]$ covers X_n in the Vitali sense.

By the Vitali's covering theorem, for every $\varepsilon > 0$, we can find a finite number of disjoint closed intervals $[\alpha_k,\beta_k]$ for $k = 1,2,\ldots,m$ with $\alpha_k = \tau_k$ or $\beta_k = \tau_k$ such that

$$\begin{aligned}|X_n| &< \sum_{k=1}^{m} |\beta_k - \alpha_k| + \varepsilon\\ &< \sum_{k=1}^{m} \frac{|\{\phi(\beta_k) - \phi(\alpha_k)\} - \{U(\tau_k,\beta_k) - U(\tau_k,\alpha_k)\}|}{\eta(\tau_k)} + \varepsilon\\ &< n\varepsilon + \varepsilon = \varepsilon(n+1)\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the outer measure of X_n is 0 and so is X. $\square$

Theorem 3.3. *Let $U\colon [a,b]\times[a,b]\to\mathbb{R}$ be a function such that $U(\tau,\cdot)\colon [a,b]\to\mathbb{R}$ is a continuous function on $[a,b]$ for $\tau\in[a,b]$. Also let $\phi\colon [a,b]\to\mathbb{R}$ be a function which is approximately continuous on $[a,b]$ and approximately differentiable nearly everywhere i.e. everywhere except perhaps for a countable number of points in $[a,b]$ such that*

$$\mathrm{AD}\,\phi(\tau)=\lim_{t\to\tau}\mathrm{ap}\,\frac{U(\tau,t)-U(\tau,\tau)}{t-\tau}\qquad \text{for}\quad \tau\in[a,b].$$

Then $U\in\mathrm{GAP}[a,b]$ and $(\mathrm{GAP})\int_a^b U=\phi(b)-\phi(a)$.

Proof. Let $X=\{\tau_n:\ n\in Z^+\}$ be the set of all points $\tau\in[a,b]$ for which $\mathrm{AD}\,\phi(\tau)$ does not exist.

For each $\tau\in[a,b]-X$, there exists a measurable set $D_\tau\subseteq[a,b]$ of density 1 at τ and

$$\mathrm{AD}\,\phi(\tau)=\lim_{\substack{\beta\to\tau\\ \beta\in D_\tau}}\frac{\phi(\beta)-\phi(\tau)}{\beta-\tau}.$$

Again, by the given condition, for given $\varepsilon>0$ we have

$$|\,\mathrm{AD}\,\phi(\tau)-\frac{U(\tau,\beta)-U(\tau,\tau)}{\beta-\tau}|<\varepsilon\qquad\text{whenever}\quad \beta\in D_\tau.$$

For each n, there exists a measurable set $D_{\tau_n}\subseteq[a,b]$ of density 1 at τ_n and

$$\phi(\tau_n)=\lim_{\substack{\tau\to\tau_n\\ \tau\in D_{\tau_n}}}\phi(\tau).$$

For each $\tau\in[a,b]-X$, there exists $\delta_\tau>0$ such that $\beta\in D_\tau\cap(\tau-\delta_\tau,\tau+\delta_\tau)=S_\tau$ implies

$$|\{\phi(\beta)-\phi(\tau)\}-\mathrm{AD}\,\phi(\tau)(\beta-\tau)|\le\varepsilon|\beta-\tau|$$

and

$$|\,\mathrm{AD}\,\phi(\tau)(\beta-\tau)-\{U(\tau,\beta)-U(\tau,\tau)\}|<\varepsilon|\beta-\tau|$$

Therefore,

$$\begin{aligned}&|\{\phi(\beta)-\phi(\tau)\}-\{U(\tau,\beta)-U(\tau,\tau)\}|\\ &\le|\{\phi(\beta)-\phi(\tau)\}-\mathrm{AD}\,\phi(\tau)(\beta-\tau)|+|\,\mathrm{AD}\,\phi(\tau)(\beta-\tau)-\{U(\tau,\beta)-U(\tau,\tau)\}|\\ &<\varepsilon|\beta-\tau|+\varepsilon|\beta-\tau|<2\varepsilon|\beta-\tau|.\end{aligned}$$

The set S_τ is measurable and $\tau\in S_\tau^d$.

For each n, there exists $\delta_n>0$ such that $\alpha,\beta\in D_{\tau_n}\cap(\tau_n-\delta_n,\tau_n+\delta_n)=S_{\tau_n}$ implies

$$|\phi(\beta)-\phi(\alpha)|<\varepsilon 2^{-n}.$$

Again, since $U(\tau,\cdot)$ is continuous on $[a,b]$, we have

$$|U(\tau_n,\beta)-U(\tau_n,\alpha)|<\varepsilon 2^{-n}\qquad\text{for}\quad \alpha,\beta\in S_{\tau_n}.$$

The set S_{τ_n} is measurable and $\tau_n \in S^d_{\tau_n}$.

The sets $\{S_\tau : \ \tau \in [a,b] - X\}$ and $\{S_{\tau_n} : \ n \in Z^+\}$ define an approximate full cover Δ of $[a,b]$.

Let $D = ([\alpha,\beta],\tau)$ be any Δ-division of $[a,b]$.

We split up the sum $\sum$ over the Δ-division into two partial sums $\sum\limits_1$ and $\sum\limits_2$ in which $\tau \in [a,b] - X$ and $\tau \in X$ respectively.

Then we obtain

$$\begin{aligned}
&\Big|\sum\{U(\tau,\beta) - U(\tau,\alpha)\} - \phi(a,b)\Big| \\
&\le \sum_1 |\{U(\tau,\beta) - U(\tau,\alpha)\} - \phi(\alpha,\beta)| + \sum_2 |\phi(\alpha,\beta)| + \sum_2 |U(\tau,\beta) - U(\tau,\alpha)| \\
&< \sum_1 \big[|\{U(\tau,\beta) - U(\tau,\tau)\} - \{\phi(\beta) - \phi(\tau)\}| \\
&\qquad + |\{U(\tau,\tau) - U(\tau,\alpha)\} - \{\phi(\tau) - \phi(\alpha)\}|\big] + 2\sum_2 \varepsilon 2^{-n} \\
&< \sum_1 \{2\varepsilon(\beta - \tau) + 2\varepsilon(\tau - \alpha)\} + 2\varepsilon \\
&< 2\varepsilon(b - a) + 2\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$U \in \mathrm{GAP}[a,b] \qquad \text{and} \qquad (\mathrm{GAP})\int_a^b U = \phi(b) - \phi(a).$$

$\square$

Theorem 3.4. *A function $U\colon [a,b]\times[a,b] \to \mathbb{R}$ is* GAP*-integrable on $[a,b]$ with primitive ϕ if and only if for every $\varepsilon > 0$ there exists an approximate full cover Δ of $[a,b]$ such that*

$$(D)\sum |\phi(\alpha,\beta)| < \varepsilon \qquad \textit{and} \qquad (D)\sum |U(\tau,\beta) - U(\tau,\alpha)| < \varepsilon$$

whenever $D = ([\alpha,\beta],\tau)$ is a partial division in $\Delta \cap \Gamma_\varepsilon$ where Γ_ε is defined as

$$\Gamma_\varepsilon = \Big\{([\alpha,\beta],\tau) : \ \big|\phi(\alpha,\beta) - \{U(\tau,\beta) - U(\tau,\alpha)\}\big| \ge \varepsilon|\beta - \alpha|\Big\}.$$

P r o o f. Let there be an approximate full cover Δ of $[a,b]$ such that the two inequalities are satisfied.

Then for any Δ-division $D = ([\alpha, \beta], \tau)$ of $[a, b]$ we have

$$\begin{aligned}
&(D) \sum |\phi(\alpha, \beta) - \{U(\tau, \beta) - U(\tau, \alpha)\}| \\
&< (D \cap \Gamma_\varepsilon) \sum |\phi(\alpha, \beta)| + (D \cap \Gamma_\varepsilon) \sum |U(\tau, \beta) - U(\tau, \alpha)| \\
&\quad + (D \setminus \Gamma_\varepsilon) \sum |\phi(\alpha, \beta) - \{U(\tau, \beta) - U(\tau, \alpha)\}| \\
&< \varepsilon + \varepsilon + \sum \varepsilon |\beta - \alpha| \\
&< 2\varepsilon + \varepsilon(b - a) = \varepsilon(2 + b - a).
\end{aligned}$$

Therefore, $U \in \mathrm{GAP}[a, b]$ with primitive ϕ.

Conversely, suppose that $U \in \mathrm{GAP}[a, b]$ with primitive ϕ.

Then for every $\varepsilon \in (0, 1)$ and for every $k \in N$ there exists an approximate full cover Δ_k of $[a, b]$ such that for any Δ_k-division $D = ([\alpha, \beta], \tau)$ of $[a, b]$ we have

$$(D) \sum |\phi(\alpha, \beta) - \{U(\tau, \beta) - U(\tau, \alpha)\}| < \frac{\varepsilon^2}{k 2^{k+1}}$$

Let

$$E_k = \left\{([\alpha, \beta], \tau) : \; k - 1 \leq \left| \tfrac{U(\tau, \beta) - U(\tau, \alpha)|}{\beta - \alpha} \right| < k, \;\; \tau \in [a, b] \right\}, \qquad k = 1, 2, \ldots.$$

Consider

$$\Delta = \bigcup_{k=1}^{\infty} \{([\alpha, \beta], \tau) : \; ([\alpha, \beta], \tau) \in \Delta_k \cap E_k\}.$$

Then Δ is an approximate full cover of $[a, b]$.
Then for any partial division $D = ([\alpha, \beta], \tau)$ of $[a, b]$ in $\Delta \cap \Gamma_\varepsilon$ we have

$$\begin{aligned}
&(D) \sum |U(\tau, \beta) - U(\tau, \alpha)| \\
&< \sum k |\beta - \alpha| \\
&< \sum_{k=1}^{\infty} \frac{k}{\varepsilon} \quad (D) \sum |\phi(\alpha, \beta) - \{U(\tau, \beta) - U(\tau, \alpha)\}| \\
&< \sum_{k=1}^{\infty} \frac{k}{\varepsilon} \frac{\varepsilon^2}{k 2^{k+1}} \\
&< \varepsilon/2 \; < \varepsilon.
\end{aligned}$$

Again, since $U \in \mathrm{GAP}[a, b]$ with primitive ϕ, by Saks-Henstock Lemma, given $\varepsilon > 0$, there exists an approximate full cover Δ of $[a, b]$ such that for any partial division $D = ([\alpha, \beta], \tau)$ of $[a, b]$ in $\Delta \cap \Gamma_\varepsilon$ we have

$$(D) \sum |\phi(\alpha, \beta) - \{U(\tau, \beta) - U(\tau, \alpha)\}| < \varepsilon.$$

Hence

$$\begin{aligned}&(D)\sum|\phi(\alpha,\beta)|\\ &\leq (D)\sum|\phi(\alpha,\beta)-\{U(\tau,\beta)-U(\tau,\alpha)\}|+(D)\sum|U(\tau,\beta)-U(\tau,\alpha)|\\ &<\varepsilon+\varepsilon=2\varepsilon.\end{aligned}$$

□

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* *Department of Pure Mathematics*
University of Calcutta
35,Ballygunge Circular Road
Kolkata–700019
INDIA
E-mail: gangulydk@yahoo.co.in

** *Department of Pure Mathematics*
University of Calcutta
35,Ballygunge Circular Road
Kolkata–700019
INDIA
E-mail: ranumukherjee05@yahoo.co.in