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NEW OSCILLATION CRITERIA FOR THIRD ORDER NONLINEAR NEUTRAL DIFFERENCE EQUATIONS

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ABSTRACT. In this paper, we are concerned with oscillation of the third-order nonlinear neutral difference equation

$$\Delta(c_n\left[\Delta(d_n\Delta(x_n+p_nx_{n-\tau}))\right]^{\gamma})+q_nf(x_{g(n)})=0, \qquad n\geq n_0,$$

where $\gamma>0$ is the quotient of odd positive integers, c_n, d_n, p_n and q_n are positive sequences of real numbers, τ is a nonnegative integer, $g(n)$ is a sequence of nonnegative integers and $f\in C(\mathbb{R},\mathbb{R})$ such that $uf(u)>0$ for $u\neq 0$. Our results extend and improve some previously obtained ones. Some examples are considered to illustrate the main results.

1. Introduction

In recent years, the asymptotic properties and oscillation of difference equations and their applications have been and still are receiving intensive attention. In fact, in the last few years several monographs and hundreds of research papers have been written, see for example the monographs [1, 3, 6, 10]. Determination of oscillatory behavior for solutions of first and second order difference equations has occupied a great part of researchers' interest. Compared to the first and second order difference equations, the study of third order difference equations has received considerably less attention in the literature, even though such equations arise in the study of economics, mathematical biology, and other areas of mathematics which discrete models are used as well as their applications in the numerical solutions of third-order differential equations (see for example [4]).

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For contributions, for third order difference equations, we refer the reader to the papers [2, 5, 7, 8, 11, 12, 13, 15, 16, 17, 18, 19] and for neutral difference equations we refer the reader to the papers [14, 20] and the references cited therein. For completeness and comparison, we present below some of these results.

In [20], the authors considered the nonlinear neutral delay difference equation

$$\Delta(c_n\Delta\left(d_n\Delta(x_n+p_nx_{n-\tau})\right))+q_nf(x_{n-\sigma})=0, \qquad \text{for} \quad n\geq n_0, \tag{1.1}$$

and established some sufficient conditions for oscillation by employing the Riccati technique, when the following assumptions are satisfied:

(A_1) τ and σ are nonnegative integers such that $\tau\leq\sigma$,

(A_2) c_n, d_n, p_n, q_n are positive sequences of real numbers such $0\leq p_n<1$, and

$$\sum_{n=n_0}^{\infty}\left(\frac{1}{c_n}\right)=\sum_{n=n_0}^{\infty}\left(\frac{1}{d_n}\right)=\infty, \tag{1.2}$$

(A_3) $f\colon\mathbb{R}\to\mathbb{R}$ is a continuous function, $uf(u)>0$ for $u\neq 0$ and $f(u)/u^{\gamma}\geqslant K>0$.

In [14] the author considered the third order nonlinear neutral delay difference equation

$$\Delta\big(c_n\Delta\left[d_n\Delta(x_n+p_nx_{n-\tau})\right]^{\gamma}\big)+q_nf(x_{n-\sigma})=0, \qquad n\geq n_0, \tag{1.3}$$

where (A_1)–(A_3) are satisfied and $\gamma\geqslant 1$ is quotient of odd positive integers. In [14] the author established several sufficient conditions for oscillation which improved the results that has been established in [20]. To prove the main results in [20] and [14] and find effective oscillation criteria the authors used an additional sequence different from the coefficients in the equations. One of our aims in this paper is to delete this condition and find new oscillation criteria without any additional sequence.

In this paper, we are concerned with oscillation of the third-order nonlinear neutral difference equation

$$\Delta\left(c_n\left[\Delta(d_n\Delta(x_n+p_nx_{n-\tau}))\right]^{\gamma}\right)+q_nf(x_{g(n)})=0, \qquad n\geq n_0, \tag{1.4}$$

when the following assumptions are satisfied:

(h_1) $\gamma>0$ is quotient of odd positive integers,

(h_2) τ is a nonnegative integer and $g(n)$ is a sequence of nonnegative integers such that $\lim\limits_{n\to\infty}g(n)=\infty$,

(h_3) c_n, d_n, p_n and q_n are positive sequences of real numbers such $0\leq p_n<1$, and $\Delta p_n\geq 0$,

(h_4) $f\colon\mathbb{R}\to\mathbb{R}$ is a continuous function, $uf(u)>0$ for $u\neq 0$ and $f(u)/u^{\gamma}\geqslant K>0$.

We note that the results that has been established in [20] and [14] are obtained in the case when $\gamma \geqslant 1$ and (1.2) holds. The natural question now is: If one can find new oscillation criteria for the equation (1.4) when $0 < \gamma < 1$? The main aim in this paper is to give an affirmative answer to this question and establish some sufficient conditions which guarantee that the equation (1.4) has oscillatory solutions or the solutions tend to zero as $n \to \infty$.

The paper is organized as follows: In Section 2, we state and prove some useful lemmas. In Section 3, we will state and prove the main results and divided it into two subsections: In the Subsection 3.1, we consider the advanced case when $g(n) > n$ and in the Subsection 3.2, we consider the delay case when $g(n) < n$. The main investigation of the main oscillation results depends on the Riccati substitution and the analysis of the associated Riccati difference inequality. The results in this paper are different from the results established in [20] and [14] and can be applied on the case when $0 < \gamma < 1$. Some examples and applications are considered throughout the paper to illustrate the main results.

2. Some preliminary lemmas

In this section, we state and prove the fundamental lemmas which we will use in the proofs of the main results in Section 3. Let x_n is a solution of the equation (1.4), and

$$z_n := x_n + p_n x_{n-\tau}. \tag{2.1}$$

We define the quasi-differences of z_n by

$$z_n^{[0]} = z_n, \quad z_n^{[1]} = d_n \Delta z_n, \quad z_n^{[2]} = c_n \left[\Delta z_n^{[1]}\right]^{\gamma}, \quad \text{and} \quad z_n^{[3]} = \Delta\left(z_n^{[2]}\right). \tag{2.2}$$

We note that if x_n is a solution of (1.4) then $y_n = -x_n$ is also solution of (1.4), since from (h$_4$), $uf(u) > 0$ for $u \neq 0$. Thus, concerning nonoscillatory solutions of (1.4), we can restrict our attention only to the positive ones and from (2.1), since $p_n > 0$, we see that if x_n is positive and monotonic then z_n is also positive and monotonic. We start with the following Lemma which provides the sign of the quasi-differences of z_n.

Lemma 2.1. *Assume that* (h$_1$)–(h$_4$) *hold. If x_n is a nonoscillatory solution of* (1.4)*, then there exists $N > n_0$ such that $z_n^{[i]} \neq 0$ for $i = 0, 1, 2,$ for $n \geq N$.*

Proof. Without loss of generality, we may assume that x_n be an eventually positive solution of (1.4) and there exists $n_1 \geq n_0$ such that $x_n > 0$, $x_{n-\tau} > 0$ and $x_{g(n)} > 0$ for $n \geq n_1$. Then from (2.1) and (h$_3$), we see that $z_n > 0$ and since $q_n > 0$, we have $z_n^{[3]} < 0$ and there exists $n_2 \geq n_1$ such that $z_n^{[2]}$ is either positive or negative for $n \geq n_2$. Thus $z_n^{[1]}$ is either increasing or decreasing for $n \geq n_2$ and so there exists $N \geq n_2$ such that $z_n^{[1]}$ is either positive or negative for $n \geq N$. The proof is complete. □

In view of Lemma 2.1, and (2.1), we see that if x_n is a nonoscillatory solution of (1.4) then quasi-differences of z_n belong to the following classes:

$$\begin{aligned}
C_0 &= \{z:\ (\exists N)(\forall n \geq N)\big(z_n z_n^{[1]} < 0,\ \ z_n z_n^{[2]} > 0\big)\},\\
C_1 &= \{z:\ (\exists N)(\forall n \geq N)\big(z_n z_n^{[1]} > 0,\ \ z_n z_n^{[2]} < 0\big)\},\\
C_2 &= \{z:\ (\exists N)(\forall n \geq N)\big(z_n z_n^{[1]} > 0,\ \ z_n z_n^{[2]} > 0\big)\},\\
C_3 &= \{z:\ (\exists N)(\forall n > N)\big(z_n z_n^{[1]} < 0,\ \ z_n z_n^{[2]} < 0\big)\}.
\end{aligned}$$

In the following we prove some lemmas which provide a classification of asymptotic behavior of the nonoscillatory solutions.

Lemma 2.2. *Assume that* (h$_1$)–(h$_4$) *hold, and*

$$\sum_{n=n_0}^{\infty}\left(\frac{1}{c_n}\right)^{\gamma} = \infty, \qquad \sum_{n=n_0}^{\infty}\left(\frac{1}{d_n}\right) = \infty. \tag{2.3}$$

If x_n is a nonoscillatory solution of (1.4), *then $z_n \in C_0 \cup C_2$.*

Proof. Without loss of generality, we may assume that x_n is an eventually positive solution of (1.4) and there exists $n_1 \geqslant n_0$ such that $x_n > 0$, $x_{n-\tau} > 0$ and $x_{g(n)} > 0$ for $n \geq n_1$. In view Lemma 2.1, we see that $z_n^{[0]}$, $z_n^{[1]}$ and $z_n^{[2]}$ are monotone and eventually of one sign. So to complete the proof, we prove that there are only the following two cases:

Case (I): $z_n^{[0]} > 0$, $z_n^{[1]} > 0$, $z_n^{[2]} > 0$, for $n \geqslant n_1$ sufficiently large.

Case (II): $z_n^{[0]} > 0$, $z_n^{[1]} < 0$, $z_n^{[2]} > 0$, for $n \geqslant n_1$ sufficiently large.

We claim that there exists $n_2 \geqslant n_1$ such that $z_n^{[2]} > 0$ for $n \geqslant n_2$. Suppose that $z_n^{[2]} \leq 0$ for $n \geqslant n_2$. From (1.4), we see that $z_n^{[3]} < 0$ for $n \geqslant n_1$ and then $z_n^{[2]}$ is decreasing. Therefore there exist a negative constant C and $n_3 \geqslant n_2$ such that $z_n^{[2]} \leq C$ for $n \geqslant n_3$. So that

$$z_n^{[1]} \leq z_{n_3}^{[1]} + C^{\frac{1}{\gamma}} \sum_{s=n_3}^{n-1} \frac{1}{(c_s)^{\frac{1}{\gamma}}},$$

which implies by (2.3) that $\lim\limits_{n\to\infty} z_n^{[1]} = -\infty$. Thus, there is an integer $n_4 \geqslant n_3$ such that for $n \geqslant n_4$, $d_n \Delta(z_n) \leq d_{n_4}\Delta(z_{n_4}) < 0$. This implies that after summing from n_4 to $n-1$, that

$$z_n - z_{n_4} \leq d_{n_4}\Delta(z_{n_4}) \sum_{s=n_3}^{n-1} \frac{1}{d_s},$$

which implies by (2.3) that $z_n \to -\infty$ as $n \to \infty$. This is a contradiction with $z_n > 0$. Then $z_n^{[2]} > 0$. The proof is complete. □

Lemma 2.3. *Assume that* (h_1)–(h_4) *hold and let* z_n *is defined as in* (2.1). *If* x_n *is a nonoscillatory solution of* (1.4) *and*

$$\sum_{n=n_1}^{\infty} \frac{1}{d_n} \sum_{s=n_1}^{n-1} \frac{1}{(c_s)^{\frac{1}{\gamma}}} = \infty, \qquad \textit{for} \quad n_1 \geq n_0, \tag{2.4}$$

then C_3 *is empty.*

P r o o f. Without loss of generality we assume that x_n is an eventually positive solution and there exists $n_1 > n_0$ such that $x_n > 0$, $x_{n-\tau} > 0$ and $x_{g(n)} > 0$ for $n \geq n_1$. This implies that $z_n > 0$ for $n \geq n_1$. To prove that C_3 is empty, we prove that the case $z_n z_n^{[1]} < 0$, and $z_n z_n^{[2]} < 0$ for $n \geq N > n_0$ is impossible. Assume for the sake of contradiction that there exists $n_2 > n_1$ such that $z_n^{[2]} < 0$ and $z_n^{[1]} < 0$ for $n \geq n_2$. Denote $a_0 = z_{n_2}^{[2]} < 0$. Then, since $z_n^{[2]}$ is deceasing we have $c_n(\Delta z_n^{[1]}) < a_0$ for $n \geq n_2$ and thus by summation from n_2 to $n-1$, we have

$$z_n^{[1]} < z_{n_2}^{[1]} + a_0^{\frac{1}{\gamma}} \sum_{s=n_2}^{n-1} \frac{1}{(c_s)^{\frac{1}{\gamma}}}.$$

Now, since $z_{n_2}^{[1]} < 0$, we see after summation from n_2 to $n-1$, that

$$z_n < a_0^{\frac{1}{\gamma}} \sum_{n=n_2}^{n-1} \frac{1}{d_n} \sum_{s=n_2}^{n-1} \frac{1}{(c_s)^{\frac{1}{\gamma}}}.$$

Letting $n \to \infty$, we get by (2.4) that $\lim_{n\to\infty} z_n = -\infty$, which contradicts the positivity of z_n. The proof is complete. □

Lemma 2.4. *Assume that* (h_1)–(h_4) *hold and let* x_n *is a nonoscillatory solution of* (1.4). *Let* z_n *is defined as in* (2.1) *and assume that* $z_n \in C_2$. *Then* z_n *is solution of the inequality*

$$\Delta\left(c_n \left[\Delta z_n^{[1]}\right]^{\gamma}\right) + P_n z_{g(n)}^{\gamma} \leq 0, \qquad \textit{for} \quad n \geq n_1, \tag{2.5}$$

where $P_n = K q_n (1 - p_{g(n)})^{\gamma}$.

P r o o f. Without loss of generality, we assume that x_n is an eventually positive solution and there exists $n_1 > n_0$ such that $x_n > 0$, $x_{n-2\tau} > 0$ and $x_{g(n)} > 0$ for $n \geq n_1$ sufficiently large. This implies from (2.1) that $z_n > 0$ $n \geq n_1$. Since $z_n \in C_2$, we see that

$$z_n = x_n + p_n x_{n-\tau} = x_n + p_n[z_{n-\tau} - p_{n-\tau} x_{n-2\tau}] \leq x_n + p_n z_n, \qquad \text{for} \quad n \geq n_1.$$

Thus we have $x_n \geqslant (1 - p_n) z_n$ for $n \geqslant n_1$. Then there exists $n_2 \geqslant n_1$ such that $x_{g(n)} \geqslant (1 - p_{g(n)}) z_{g(n)}$. This and (h_4) imply that (2.5) holds. The proof is complete. □

Lemma 2.5. *Assume that* (h_1)–(h_4) *hold and let* x_n *is a nonoscillatory solution of* (1.4). *Let* z_n *is defined as in* (2.1) *and assume that* $z_n \in C_0$. *Then there exists* $n_1 \geqslant n_0$ *such that*

$$x_{n-\tau} \geqslant \frac{z_n}{1+p_n} > (1-p_n)z_n \qquad \textit{for} \quad n \geqslant n_1. \tag{2.6}$$

P r o o f. Without loss of generality we may assume that x_n is an eventually positive solution and there exists $n_1 > n_0$ such that $x_n > 0$, $x_{n-\tau} > 0$ and $x_{g(n)} > 0$ for $n \geq n_1$. This implies from (2.1) that $z_n > 0$ for $n \geq n_1$. Now, since $z_n \in C_0$, then z_n is decreasing, we may assume without loss of generality that x_n is also decreasing. If this is not the case, i.e., if x_n and $x_{n-\tau}$ are eventually nondecreasing for large $n \geq n_1$, we see (note from (h_3) that $p_n \geq 0$ and $\Delta p_n \geq 0$) that

$$\Delta z_n = \Delta x_n + \Delta p_n x_{n-\tau} + p_{n+1}(\Delta x_{n-\tau}) > \Delta x_n \geq 0,$$

which is a contradiction with $\Delta z_n < 0$, for $n \geq n_1$. Hence

$$z_n = x_n + p_n x_{n-\tau} \leq x_{n-\tau} + p_n x_{n-\tau}.$$

From which we obtain that $x_{n-\tau} \geqslant z_n/(1+p_n)$, which is the first part in (2.6). Since $0 \leq p_n < 1$, then $1 \geq 1 - p_n^2$, and this implies that $1/(1+p_n) \geq (1-p_n)$. Therefore $z_n/\left(1+p_n\right) \geq (1-p_n)z_n$ and then the second part of the inequality in (2.6) holds. The proof is complete. $\square$

Lemma 2.6. *Assume that* (h_1)–(h_4) *hold. Let* x_n *be a nonoscillatory solution of* (1.4) *and* z_n *is defined as in* (2.1) *such that* $z_n \in C_0$. *Furthermore assume that*

(h_5) $\displaystyle\sum_{n_0}^{\infty} \frac{1}{d_n}\left[\sum_{n_0}^{n-1} \frac{1}{c_t} \sum_{n_0}^{t-1} \left(q_s/\left(1+p_{g(s)+\tau}\right)^{\gamma}\right)\right]^{1/\gamma} = \infty.$

Then

$$\lim_{n\to\infty} z_n = 0, \tag{2.7}$$

and if $\lim\limits_{n\to\infty} p_n = p^* \in [0,1)$ *then* $\lim\limits_{n\to\infty} x_n = 0$.

P r o o f. Without loss of generality we may assume that $x_n > 0$, $x_{n-\tau} > 0$ and $x_{g(n)} > 0$ for $n \geq n_1$ where n_1 is chosen so large that Lemma 2.1 holds. (The proof when x_n is eventually negative is similar, since the substitution $y_n = -x_n$ transforms (1.4) into an equation of the same form). From Lemma 2.5, (2.1) implies that there exists $n_2 \geqslant n_1$ such that

$$x_{g(n)} \geqslant \frac{z_{g(n)+\tau}}{1+p_{g(n)+\tau}} \qquad \text{for} \quad n \geqslant n_2. \tag{2.8}$$

From (h_3), (1.4) and (2.8) we obtain

$$\Delta\left(c_n\left[\Delta z_n^{[1]}\right]^{\gamma}\right) + \frac{Kq_n}{(1+p_{g(n)+\tau})^{\gamma}} z^{\gamma}_{g(n)+\tau} \leq 0, \qquad n \geq n_2. \tag{2.9}$$

Since $z_n > 0$, decreasing and $\lim_{n\to\infty} g(n) = \infty$, it follows that $\lim_{n\to\infty} z_{g(n)+\tau} = b \geqslant 0$. Now we claim that $b = 0$. If not then $z^{\gamma}_{g(n)+\tau} \to b^{\gamma} > 0$ as $n \to \infty$, and hence there exists $n_3 \geq n_2$ such that $z^{\gamma}_{g(n)+\tau} \geq b^{\gamma}$. Therefore from (2.9) we have

$$\Delta\left(c_n\left[\Delta z_n^{[1]}\right]^{\gamma}\right) + \frac{Kq_n}{(1+p_{g(n)+\tau})^{\gamma}} b^{\gamma} \leq 0, \qquad n \geq n_2, \tag{2.10}$$

Define the sequence $u_n = c_n\Delta(d_n\Delta z_n)^{\gamma}$ for $n \geq n_3$. Then, we have

$$\Delta u_n \leq -\frac{Kq_n}{(1+p_{g(n)+\tau})^{\gamma}} b^{\gamma}.$$

Summing the last inequality from n_3 to $n-1$, we have

$$u_n \leq u_{n_3} - b^{\gamma}K\sum_{s=n_3}^{n-1}\frac{q_s}{(1+p_{g(s)+\tau})^{\gamma}}.$$

In view of (h$_5$) it is possible to choose an integer n_4 sufficiently large such that for all $n \geq n_4$

$$u_n \leq -A\sum_{s=n_3}^{n-1}\frac{q_s}{(1+p_{g(s)+\tau})^{\gamma}},$$

where $A = \frac{b^{\gamma}K}{2} > 0$. Hence

$$\Delta z_n^{[1]} \leq -(A)^{\frac{1}{\gamma}}\left(\frac{1}{c_n}\sum_{s=n_3}^{n-1}\frac{q_s}{(1+p_{g(s)+\tau})^{\gamma}}\right)^{\frac{1}{\gamma}},$$

Summing the last inequality from n_4 to $n-1$ we obtain

$$z_n^{[1]} \leq z_{n_4}^{[1]} - (A)^{\frac{1}{\gamma}}\sum_{t=n_4}^{n-1}\left(\frac{1}{c_t}\sum_{s=n_3}^{t-1}\frac{q_s}{(1+p_{g(s)+\tau})^{\gamma}}\right)^{\frac{1}{\gamma}}.$$

Since $z_n^{[1]} < 0$ for $n \geqslant n_0$, the last inequality implies that

$$d_n\Delta z_n \leq -(A)^{\frac{1}{\gamma}}\sum_{t=n_3}^{n-1}\left(\frac{1}{c_t}\sum_{s=n_2}^{t-1}\frac{q_s}{(1+p_{s-\sigma+\tau})^{\gamma}}\right)^{\frac{1}{\gamma}},$$

or

$$\Delta z_n \leq -(A)^{\frac{1}{\gamma}}\frac{1}{d_n}\sum_{t=n_4}^{n-1}\left(\frac{1}{c_t}\sum_{s=n_3}^{t-1}\frac{q_s}{(1+p_{g(s)+\tau})^{\gamma}}\right)^{\frac{1}{\gamma}}.$$

Summing from n_4 to $n-1$ we have

$$z_n \leq z_{n_4} - (A)^{\frac{1}{\gamma}}\sum_{l=n_4}^{n-1}\frac{1}{d_l}\sum_{t=n_4}^{l-1}\left(\frac{1}{c_t}\sum_{s=n_3}^{t-1}\frac{q_s}{(1+p_{g(s)+\tau})^{\gamma}}\right)^{\frac{1}{\gamma}}.$$

Condition (h_5) implies that $z_n \to -\infty$ as $n \to \infty$ which is a contradiction with the fact that z_n is positive. Then $b = 0$ and then (2.7) holds. From this since $\lim_{n\to\infty} p_n = p^*$, we see that

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + p^* \lim_{n\to\infty} x_{n-\tau} = (1 + p^*) \lim_{n\to\infty} x_n.$$

This and (2.7) implies that $\lim_{n\to\infty} x_n = 0$. This completes the proof. □

To prove the next lemma which plays an important role in the proof of the main results in the delay case, we need the following functions which are define by

$$h_k(n, s) := \frac{(n-s)^{(k)}}{k!}, \qquad k = 0, 1, 2, \ldots, \tag{2.11}$$

where $t^{(k)} = t(t-1)\cdots(t-k+1)$ is the so-called falling function (see [10]). The summation and the difference of the functions $h_k(n, s)$ are defined by

$$h_{k+1}(n, s) := \sum_{\tau=s}^{n-1} h_k(\tau, s),$$
$$\Delta_1 h_k(n, s) := h_{k-1}(n, s), \quad \Delta_2 h_k(n, s) := -h_{k-1}(n, s),$$

where Δ_1 denotes the difference with respect to n and Δ_2 denotes the difference with respect to s. As a special case when $n = 2$, we see that $n^{(2)} = n(n-1)$ and one can easily prove that $\Delta n^{(2)} = 2n$. Also since $\Delta(1/n^{(2)}) = -2/(n+1)^{(3)}$ then $\sum_{s=n}^{\infty} \left(-2/(n+1)^{(3)}\right) = 1/n^{(2)}$.

Lemma 2.7. *Assume that $g(n) \le n$, and*

$$z_n > 0, \quad \Delta z_n > 0, \quad \Delta^2 z_n > 0, \quad \text{and} \quad \Delta^3 z_n < 0, \qquad \text{for} \quad n \ge n_0. \tag{2.12}$$

Then

$$\liminf_{n\to\infty} \frac{n z_n}{h_2(n, n_0)\Delta z_n} \ge 1, \tag{2.13}$$

and there exists $N > n_0$ such that

$$\frac{\Delta z_{g(n)}}{\Delta z_{n+1}} \ge \frac{(g(n) - N)}{(n + 1 - N)}. \tag{2.14}$$

Proof. First, we prove that (2.13) holds. Let

$$G_n := (n - N)z_n - \frac{(n-N)^{(2)}}{2}\Delta z_n.$$

Then $G_N = 0$, and

$$\begin{aligned}\Delta G_n &= (n+1-N)\Delta z_n + z_n - \frac{(n+1-N)^{(2)}}{2}\Delta^2 z_n - (n-N)\Delta z_n \\ &= \Delta z_n + z_n - \frac{(n+1-N)^{(2)}}{2}\Delta^2 z_n \\ &= z_{n+1} - \frac{(n+1-N)^{(2)}}{2}\Delta^2 z_n \\ &= z_{n+1} - \sum_{\tau=N}^{n}(\tau - N)\Delta^2 z_n.\end{aligned}$$

To complete the proof we will apply the discrete Taylor's Theorem [1, Theorem 1.113] of the sequence f_n, which is defined by

$$f_n := \sum_{k=0}^{m-1} h_k(n,\alpha)\Delta^k f(\alpha) + \frac{1}{(m-1)!}\sum_{\tau=\alpha}^{n-m} h_{m-1}(n,\tau+1)\Delta^m f(\tau), \qquad (2.15)$$

where $h_n(t,s)$ be defined as in (2.11). Replacing f_n by z_{n+1} and putting $m = 2$ in (2.15), we have (noting from (2.12) that $\Delta^2 z_n$ is decreasing)

$$\begin{aligned}z_{n+1} &= \sum_{k=0}^{2-1} h_k(n+1,N)\Delta^k z_N + \frac{1}{(2-1)!}\sum_{\tau=N}^{n+1-2} h_{2-1}(n+1,\tau+1)\Delta^2 z_\tau \\ &= z_N + (n+1-N)\Delta z_N + \sum_{\tau=N}^{n-1} h_1(n+1,\tau+1)\Delta^2 z_\tau \\ &\geq z_N + (n+1-N)\Delta z_N + \Delta^2 z_n \sum_{\tau=N}^{n-1} h_1(n+1,\tau+1).\end{aligned}$$

It would follows that $\Delta G_n > 0$ on $[N,\infty)$ provided, we can prove that

$$\sum_{\tau=N}^{n-1} h_1(n+1,\tau+1) = \sum_{\tau=N}^{n}(\tau - N).$$

To see this, we use the summation by parts formula [1, Theorem 1.77],

$$\sum_{\tau=a}^{b} f(\tau+1)\Delta g(\tau) = f(\tau)g(\tau)_a^{b+1} - \sum_{\tau=a}^{b}\Delta f(\tau)g(\tau),$$

to get

$$\sum_{\tau=N}^{n} h_1(n+1,\tau+1) = h_1(n+1,\tau)(\tau-N)_{\tau=N}^{\tau=n+1} - \sum_{\tau=N}^{n}(-1)(\tau-N) = \sum_{\tau=N}^{n}(\tau-N),$$

which is the desired result. Hence $\Delta G_n > 0$ for $n \geq N$. Since $G_N = 0$, we get that $G_n > 0$ for $n \geq N$. This implies that

$$\frac{(n-N)z_n}{h_2(n,N)\Delta z_n} \geq 1, \qquad \text{for} \quad n \geq N. \tag{2.16}$$

Therefore, since

$$\frac{nz_n}{h_2(n,n_0)\Delta z_n} = \frac{(n-N)z_n}{h_2(n,N)\Delta z_n}\frac{n}{n-N}\frac{h_2(n,N)}{h_2(n,n_0)},$$

and since

$$\lim_{n\to\infty}\frac{n}{n-N} = 1 = \lim_{n\to\infty}\frac{h_2(n,N)}{h_2(n,n_0)},$$

we get that

$$\liminf_{n\to\infty}\frac{nz(n)}{h_2(n,n_0)\Delta z_n} \geq 1,$$

which proves (2.13). Next, we prove that (2.14) holds. From (2.12), since $\Delta^2 z_n$ is decreasing, we have

$$\Delta z_n - \Delta z_N \geq \Delta^2 z_n(n-N).$$

Dividing by $\Delta z_n \Delta z_{n+1}$, we get that

$$\frac{\Delta z_n - \Delta z_N - \Delta^2 z_n(n-N)}{\Delta z_n \Delta z_{n+1}} \geq 0.$$

Thus

$$\Delta\left(\frac{n-N}{\Delta z_n}\right) \geq 0.$$

This implies that

$$\frac{(n+1-N)}{\Delta z_{n+1}} \geq \frac{(g(n)-N)}{\Delta z_{\sigma(n)}}.$$

where $g(n) \leq n < n+1$. Hence

$$\left(\Delta z_{g(n)}\right)^{\gamma} \geq \left(\frac{g(n)-N}{n+1-N}\right)^{\gamma}\left(\Delta z_{n+1}\right)^{\gamma},$$

and this proves (2.14). The proof is complete. □

3. Main results

In this section, we establish some sufficient conditions which guarantee that the solution x_n of (1.4) oscillates or satisfies $\lim_{n\to\infty} x_n = 0$. If (2.3) holds then in view of Lemma 2.2 if x_n is a solution of (1.4), and z_n be as defined by (2.1), then $z_n \in C_0 \cup C_2$.

3.1. The case when $g(n) > n$

In this subsection, we consider the case when $g(n) > n$ and establish some sufficient conditions which guarantee that the solution x_n of (1.4) is either oscillates or satisfies $\lim_{n\to\infty} z_n = \infty$ where z_n is defined as in (2.1). To simplify the presentation of the results, we introduce the following notations:

$$r_* := \liminf_{n\to\infty} \frac{n^\gamma w_{n+1}}{c_n}, \qquad R := \limsup_{n\to\infty} \frac{n^\gamma w_{n+1}}{c_n},$$

$$q_* := \liminf_{n\to\infty} \frac{1}{n} \sum_{s=N}^{n-1} \frac{s^{\gamma+1}}{c_s} Q_s, \qquad p_* := \liminf_{n\to\infty} \frac{n^\gamma}{c_n} \sum_{s=n+1}^{\infty} Q_s,$$

$$Q_n := P_n \left(\frac{D_n (c_n)^{\frac{1}{\gamma}} C_n}{(c_n)^{\frac{1}{\gamma}} C_n + 1} \right)^\gamma, \qquad P_n = K q_n (1 - p_{g(n)})^\gamma,$$

$$C_n := \sum_{s=N}^{n-1} \frac{1}{(c_s)^{\frac{1}{\gamma}}} \quad \text{and} \quad D_n := \sum_{s=n}^{g(n)-1} \frac{1}{d_s}.$$

Lemma 3.1. *Assume that* (h$_1$)–(h$_4$) *and* (2.3) *hold. Furthermore assume that* $g(n) > n$. *Let* x_n *is a solution of* (1.4) *and let* z_n *is defined as in* (2.1) *such that* $z_n \in C_2$. *Define* w_n *by the Riccati substitution*

$$w_n := \frac{z_n^{[2]}}{(z_n^{[1]})^\gamma}. \tag{3.1}$$

Then $w_n > 0$, *and*

$$\Delta w_n + Q_n + \gamma \frac{1}{(c_n)^{\frac{1}{\gamma}}} (w_{n+1})^{1+\frac{1}{\gamma}} \le 0, \qquad \text{for} \quad n \in [N, \infty). \tag{3.2}$$

P r o o f. Let x_n be as in the statement of this Theorem and without loss of generality, we may assume that there exists $n_1 > n_0$ such that $x_n > 0$, $x_{n-\tau} > 0$ and $x_{g(n)} > 0$ for $n \ge n_1$. Then from Lemma 2.2, there exists $N > n_1$ such that $z_n^{[0]} > 0$, $z_n^{[1]} > 0$, $z_n^{[2]} > 0$, $z_n^{[3]} \le 0$. By the difference quotient rule, we have

$$\begin{aligned}
\Delta w_n &= \frac{\left(z_n^{[1]}\right)^\gamma z_n^{[3]} - \Delta \left(z_n^{[1]}\right)^\gamma z_n^{[2]}}{\left(z_n^{[1]}\right)^\gamma \left(z_{n+1}^{[1]}\right)^\gamma} \\
&= \frac{z_n^{[3]} \left(z_{g(n)}^{[0]}\right)^\gamma}{\left(z_{g(n)}^{[0]}\right)^\gamma \left(z_{n+1}^{[1]}\right)^\gamma} - \frac{\Delta \left(z_n^{[1]}\right)^\gamma z_n^{[2]}}{\left(z_n^{[1]}\right)^\gamma \left(z_{n+1}^{[1]}\right)^\gamma}.
\end{aligned}$$

From Lemma 2.4, we see that

$$\Delta w_n \leq -P_n \frac{\left(z_{g(n)}^{[0]}\right)^{\gamma}}{\left(z_{n+1}^{[1]}\right)^{\gamma}} - \frac{\Delta \left(z_n^{[1]}\right)^{\gamma} z_n^{[2]}}{\left(z_n^{[1]}\right)^{\gamma} \left(z_{n+1}^{[1]}\right)^{\gamma}}. \tag{3.3}$$

Using the inequality ([9, p. 39]),

$$x^{\gamma} - y^{\gamma} \geq \gamma y^{\gamma-1}(x-y), \qquad \text{for all} \quad x \neq y \;\text{ and }\; \gamma \geq 1, \tag{3.4}$$

we have

$$\Delta \left(z_n^{[1]}\right)^{\gamma} = \left(z_{n+1}^{[1]}\right)^{\gamma} - \left(z_n^{[1]}\right)^{\gamma} \geq \gamma \left(z_n^{[1]}\right)^{\gamma} \left(\Delta z_n^{[1]}\right), \qquad \text{when} \quad \gamma \geq 1. \tag{3.5}$$

From the definition of $z_n^{[2]}$ we see that $\Delta z_n^{[1]} = \left(z_n^{[2]}/c_n\right)^{\frac{1}{\gamma}}$. This and (3.5) imply that

$$\Delta \left(z_n^{[1]}\right)^{\gamma} \geq \gamma \left(z_n^{[1]}\right)^{\gamma} \left(\frac{z_n^{[2]}}{c_n}\right)^{\frac{1}{\gamma}}. \tag{3.6}$$

Using the inequality ([9, p. 39]),

$$x^{\gamma} - y^{\gamma} \geq \gamma x^{\gamma-1}(x-y), \qquad \text{for all} \quad x \neq y \;\text{ and }\; 0 < \gamma \leq 1, \tag{3.7}$$

we see that

$$\Delta \left(z_n^{[1]}\right)^{\gamma} = \left(z_{n+1}^{[1]}\right)^{\gamma} - \left(z_n^{[1]}\right)^{\gamma} \geq \gamma \left(z_{n+1}^{[1]}\right)^{\gamma} \left(\Delta z_n^{[1]}\right) \geq \gamma \left(z_{n+1}^{[1]}\right)^{\gamma} \left(\frac{z_n^{[2]}}{c_n}\right)^{\frac{1}{\gamma}}. \tag{3.8}$$

Combining (3.6) and (3.8), since $z_n^{[1]}$ is increasing and $z^{[2]}$ is decreasing, we obtain

$$\begin{aligned}
\frac{\Delta \left(z_n^{[1]}\right)^{\gamma} z_n^{[2]}}{\left(z_n^{[1]}\right)^{\gamma} \left(z_{n+1}^{[1]}\right)^{\gamma}} &\geq \frac{\gamma z_n^{[2]} (z_n^{[2]})^{\frac{1}{\gamma}}}{c_n^{\frac{1}{\gamma}} \left(z_n^{[1]}\right) \left(z_{n+1}^{[1]}\right)^{\gamma}} \\
&\geq \frac{\gamma \left(z_{n+1}^{[2]}\right) \left(z_{n+1}^{[2]}\right)^{\frac{1}{\gamma}}}{(c_n)^{\frac{1}{\gamma}} \left(z_{n+1}^{[1]}\right) \left(z_{n+1}^{[1]}\right)^{\gamma}} \\
&= \frac{\gamma}{(c_n)^{\frac{1}{\gamma}}} \left(w_{n+1}\right)^{\frac{1}{\gamma}+1}, \qquad \text{for} \quad \gamma > 0.
\end{aligned}$$

Substituting in (3.3), we have

$$\Delta w_n \leq -P_n \left(\frac{z_{g(n)}}{z_{n+1}^{[1]}}\right)^{\gamma} - \frac{\gamma}{(c_n)^{\frac{1}{\gamma}}} \left(w_{n+1}\right)^{1+\frac{1}{\gamma}}. \tag{3.9}$$

Next, we consider the coefficient of P_n in (3.9). Since $z_{n+1}^{[1]} = z_n^{[1]} + \Delta(z_n^{[1]})$, we have

$$\frac{z_{n+1}^{[1]}}{z_n^{[1]}} = 1 + \frac{\Delta(z_n^{[1]})}{z_n^{[1]}} = 1 + \frac{\left(z_n^{[2]}\right)^{\frac{1}{\gamma}}}{(c_n)^{\frac{1}{\gamma}} z_n^{[1]}}.$$

Also since $z_n^{[2]}$ is decreasing, we get

$$z_n^{[1]} = z_N^{[1]} + \sum_{s=N}^{n-1} \left(z_s^{[2]}\right)^{\frac{1}{\gamma}} \frac{1}{(c_s)^{\frac{1}{\gamma}}} \geq z_N^{[1]} + \left(z_n^{[2]}\right)^{\frac{1}{\gamma}} \sum_{s=N}^{n-1} \frac{1}{(c_s)^{\frac{1}{\gamma}}} > \left(z_n^{[2]}\right)^{\frac{1}{\gamma}} \sum_{s=N}^{n-1} \frac{1}{(c_s)^{\frac{1}{\gamma}}}. \tag{3.10}$$

It follows that

$$\frac{z_n^{[1]}}{\left(z_n^{[2]}\right)^{\frac{1}{\gamma}}} \geq \sum_{s=N}^{n-1} \frac{1}{(c_s)^{\frac{1}{\gamma}}} = C_n. \tag{3.11}$$

Hence

$$\frac{z_{n+1}^{[1]}}{z_n^{[1]}} = 1 + \frac{\Delta(z^{[1]})}{z_n^{[1]}} \leq \left(\frac{(c_n)^{\frac{1}{\gamma}} C_n + 1}{(c_n)^{\frac{1}{\gamma}} C_n}\right).$$

Thus

$$\frac{z_n^{[1]}}{z_{n+1}^{[1]}} \geq \frac{(c_n)^{\frac{1}{\gamma}} C_n}{(c_n)^{\frac{1}{\gamma}} C_n + 1}.$$

This gives that

$$\frac{z_{g(n)}}{z_{n+1}^{[1]}} = \frac{z_{g(n)}}{z_n^{[1]}} \frac{z_n^{[1]}}{z_{n+1}^{[1]}} \geq \frac{z_{g(n)}}{z_n^{[1]}} \frac{(c_n)^{\frac{1}{\gamma}} C_n}{(c_n)^{\frac{1}{\gamma}} C_n + 1}. \tag{3.12}$$

Now, since $g(n) > n$ and $z_n^{[1]}$ is increasing, we have

$$z_{g(n)} > z_{g(n)} - z_n = \sum_{s=n}^{g(n)-1} \Delta z_s = \sum_{s=n}^{g(n)-1} \frac{z_s^{[1]}}{d_s} \geq z_n^{[1]} \sum_{s=n}^{g(n)-1} \frac{1}{d_s} = z_n^{[1]} D_n.$$

This and (3.12) show that

$$\frac{z_{g(n)}}{z_{n+1}^{[1]}} \geq \frac{D_n (c_n)^{\frac{1}{\gamma}} C_n}{(c_n)^{\frac{1}{\gamma}} C_n + 1}. \tag{3.13}$$

Substituting from (3.13) into (3.9), we have the inequality (3.2) and this completes the proof. $\square$

In the following, we assume that

$$\sum_{s=n_0}^{\infty} Q_s < \infty, \tag{3.14}$$

which is different from the assumption that has been posed in all the above mentioned results in the introduction.

Now, we are ready to state and prove the main oscillation theorem in the advanced case.

Theorem 3.1. *Assume that* (h_1)–(h_5), *and* (2.3) *hold. Furthermore assume that* $g(n) > n$, *and* $\Delta c_n \geq 0$. *Let* x_n *be a solution of* (1.4). *If*

$$p_* > \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}, \tag{3.15}$$

or

$$p_* + q_* > 2^{\gamma(\gamma+1)}. \tag{3.16}$$

Then either x_n *oscillates or* $\lim_{n\to\infty} z_n = 0$.

P r o o f. Suppose the contrary and assume that x_n is a nonoscillatory solution of equation (1.4). Without loss of generality, we may assume that $x_n > 0$, $x_{n-\tau} > 0$, $x_{g(n)} > 0$, for $n \geq n_1$ where n_1 is chosen so large. We consider only this case, because the proof when $x_n < 0$ is similar, since $uf(u) > 0$. Then from (2.1) and in view of Lemma 2.2, since (2.3) holds, $z_n \in C_0 \cup C_2$. If $z_n \in C_0$ and (h_5) holds, we are back to the proof of Lemma 2.6 to show that $\lim_{n\to\infty} z_n = 0$. Next, we consider the case when $z_n \in C_2$ and w_n is defined as in (3.1). Then from Lemma 3.1, there exists $n_2 > n_1$ such that $w_n > 0$ and satisfies the difference inequality

$$\Delta w_n \leq -Q_n - \frac{\gamma}{(c_n)^{\frac{1}{\gamma}}} (w_{n+1})^{1+\frac{1}{\gamma}}, \qquad \text{for} \quad n \geq n_2. \tag{3.17}$$

Also from Lemma 3.1, since

$$z_n^{[1]} > \left(z_n^{[2]}\right)^{\frac{1}{\gamma}} \sum_{s=N}^{n-1} \frac{1}{(c_s)^{\frac{1}{\gamma}}},$$

we see that

$$w_n := \frac{z_n^{[2]}}{(z_n^{[1]})^\gamma} < \left(\sum_{s=N}^{n-1} \frac{1}{(c_s)^{\frac{1}{\gamma}}}\right)^{-\gamma}.$$

This and (2.3) imply that $\lim_{n\to\infty} w_n = 0$. First, we give a contradiction to (3.15). Summing (3.17) from $n+1$ to ∞ and using that $\lim_{n\to\infty} w_n = 0$, we get

$$w_{n+1} \geq \sum_{n+1}^{\infty} Q_s + \gamma \sum_{n+1}^{\infty} \frac{1}{(c_s)^{\frac{1}{\gamma}}} (w_{s+1})^{\frac{1}{\gamma}} w_{s+1}. \tag{3.18}$$

It follows from (3.18) that

$$\frac{n^\gamma w_{n+1}}{c_n} \geq \frac{n^\gamma}{c_n} \sum_{n+1}^{\infty} Q_s + \gamma \frac{n^\gamma}{c_n} \sum_{n+1}^{\infty} \frac{1}{(c_s)^{\frac{1}{\gamma}}} (w_{s+1})^{\frac{1}{\gamma}} w_{s+1}. \tag{3.19}$$

Let $\varepsilon > 0$, then by the definition of p_* and r_*, we may assume that there exists $N \geq n_2$, sufficiently large, so that

$$\frac{n^\gamma}{c_n}\sum_{n+1}^{\infty} Q_s \geq p_* - \varepsilon, \quad \text{and} \quad \frac{n^\gamma w_{n+1}}{c_n} \geq r_* - \varepsilon, \qquad \text{for} \quad N \geq n_2. \tag{3.20}$$

From (3.19), (3.20) and using the fact $\Delta c_n \geq 0$, we get

$$\begin{aligned}
\frac{n^\gamma w_{n+1}}{c_n} &\geq (p_* - \varepsilon) + \gamma \frac{n^\gamma}{c_n}\sum_{n+1}^{\infty} \frac{c_s}{s^{\gamma+1}} \frac{s\,(w_{s+1})^{\frac{1}{\gamma}}}{(c_s)^{\frac{1}{\gamma}}} \frac{s^\gamma w_{s+1}}{c_s} \\
&\geq (p_* - \varepsilon) + (r_* - \varepsilon)^{1+\frac{1}{\gamma}} \frac{n^\gamma}{c_n}\sum_{n+1}^{\infty} \frac{\gamma c_s}{s^{\gamma+1}} \\
&\geq (p_* - \varepsilon) + (r_* - \varepsilon)^{1+\frac{1}{\gamma}}\, n^\gamma \sum_{n+1}^{\infty} \frac{\gamma}{s^{\gamma+1}}.
\end{aligned} \tag{3.21}$$

Using the inequality (3.4), we have

$$\Delta\left(\frac{-1}{s^\gamma}\right) = \frac{(s+1)^\gamma - s^\gamma}{s^\gamma (s+1)^\gamma} \leq \frac{\gamma (s+1)^{\gamma-1}}{s^\gamma (s+1)^\gamma} = \frac{\gamma}{s^\gamma (s+1)} < \frac{\gamma}{s^{\gamma+1}}, \qquad \text{for} \quad \gamma \geq 1.$$

Using the inequality (3.7), we have

$$\Delta\left(\frac{-1}{s^\gamma}\right) = \frac{(s+1)^\gamma - s^\gamma}{s^\gamma (s+1)^\gamma} \leq \frac{\gamma (s)^{\gamma-1}}{s^\gamma (s+1)^\gamma} = \frac{\gamma}{s (s+1)^\gamma} < \frac{\gamma}{s^{\gamma+1}}, \qquad \text{for} \quad 0<\gamma<1.$$

So that for $\gamma > 0$, we have

$$\sum_{n+1}^{\infty} \frac{\gamma}{s^{\gamma+1}} > \sum_{n+1}^{\infty} \Delta\left(\frac{-1}{s^\gamma}\right) = \frac{1}{(n+1)^\gamma}. \tag{3.22}$$

Then from (3.21) and (3.22), we obtain

$$\frac{n^\gamma w_{n+1}}{c_n} \geq (p_* - \varepsilon) + (r_* - \varepsilon)^{1+\frac{1}{\gamma}} \left(\frac{n}{n+1}\right)^\gamma, \qquad \text{for} \quad \gamma > 0.$$

Taking the $\liminf$ of both sides as $n \to \infty$, we get that

$$r_* \geq p_* - \varepsilon + (r_* - \varepsilon)^{1+\frac{1}{\gamma}}.$$

Since $\varepsilon > 0$ is arbitrary, we get

$$p_* \leq r_* - r_*^{1+\frac{1}{\gamma}}. \tag{3.23}$$

Using the fact that

$$u - u^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}},$$

we have

$$p_* \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}},$$

which contradicts (3.15). Next, we give a contradiction to (3.16). Multiplying both sides of (3.17) by $n^{\gamma+1}/c_n$, and summing from N to $n-1$ $(n-1 \geq N)$, we get

$$\sum_{s=N}^{n-1} \frac{s^{\gamma+1}}{c_s} \Delta w_s \leq -\sum_{s=N}^{n-1} \frac{s^{\gamma+1}}{c_s} Q_s - \gamma \sum_{s=N}^{n-1} \left(\frac{s^\gamma w_{s+1}}{c_s} \right)^{\frac{\gamma+1}{\gamma}}.$$

Using summation by parts, we obtain

$$\frac{n^{\gamma+1} w_n}{c_n} \leq \frac{N^{\gamma+1} w_N}{c_N} + \sum_{s=N}^{n-1} \Delta \left(\frac{s^{\gamma+1}}{c_s} \right) w_{s+1} - \sum_{s=N}^{n} \frac{s^{\gamma+1}}{c_s} Q_s - \gamma \sum_{s=N}^{n-1} \left(\frac{s^\gamma w_{s+1}}{c_s} \right)^{\frac{\gamma+1}{\gamma}}.$$

By the quotient rule, we have

$$\Delta \left(\frac{s^{\gamma+1}}{c_s} \right) = \frac{\Delta(s^{\gamma+1})}{c_{s+1}} - \frac{s^{\gamma+1} \Delta c_s}{c_s c_{s+1}} \leq \frac{(\gamma+1)(s+1)^\gamma}{c_{s+1}} \leq \frac{(\gamma+1)(s+1)^\gamma}{c_s}. \quad (3.24)$$

Hence

$$\begin{aligned} \frac{n^{\gamma+1} w_n}{c_n} \quad \leq \quad & \frac{N^{\gamma+1} w_N}{c_N} - \sum_{s=N}^{n-1} \frac{s^{\gamma+1}}{c_s} Q_s + \sum_{s=N}^{n-1} (\gamma+1) \left(\frac{(s+1)^\gamma w_{s+1}}{c_s} \right) \\ & - \gamma \sum_{s=N}^{n-1} \left(\frac{s^\gamma w_{s+1}}{c_s} \right)^{\frac{\gamma+1}{\gamma}}. \end{aligned}$$

Now, since $s > n_0 > 0$ we can assume for s sufficiently large that $(s+1) \leq Ls$ $< 2s$. Using this and the last inequality, we obtain

$$\frac{n^{\gamma+1} w_n}{c_n} \leq \frac{N^{\gamma+1} w_N}{c_N} - \sum_{s=N}^{n-1} \frac{s^{\gamma+1}}{c_s} Q_s + \sum_{s=N}^{n-1} \left\{ (\gamma+1) L^\gamma W_{s+1} - \gamma W_{s+1}^{\frac{\gamma+1}{\gamma}} \right\}.$$

where $W_{s+1} := (s^\gamma w_{s+1}/c_s)$. Using the inequality

$$BW - AWu^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma},$$

we have

$$\begin{aligned} \frac{n^{\gamma+1} w_n}{c_n} \quad & \leq \quad \frac{N^{\gamma+1} w_N}{c_N} - \sum_{s=N}^{n-1} \frac{s^{\gamma+1}}{c_s} Q_s + \sum_{s=N}^{n-1} \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{[(\gamma+1) L^\gamma]^{\gamma+1}}{\gamma^\gamma} \\ & = \quad \frac{N^{\gamma+1} w_N}{c_N} - \sum_{s=N}^{n-1} \frac{s^{\gamma+1}}{c_s} Q_s + L^{\gamma(\gamma+1)} (n-N). \end{aligned}$$

It follows that

$$\frac{n^\gamma w_n}{c_n} \leq \frac{N^{\gamma+1} w_N}{n c_N} - \frac{1}{n} \sum_{s=N}^{n-1} \frac{s^{\gamma+1}}{c_s} Q_s + L^{\gamma(\gamma+1)} \left(1 - \frac{N}{n} \right).$$

Since $w_{n+1} \le w_n$, we get

$$\frac{n^\gamma w_{n+1}}{c_n} \le \frac{N^{\gamma+1} w_N}{n c_N} - \frac{1}{n}\sum_{s=N}^{n-1} \frac{s^{\gamma+1}}{c_s} Q_s + L^{\gamma(\gamma+1)}\left(1 - \frac{N}{n}\right).$$

Taking the lim sup of both sides as $n \to \infty$, we obtain

$$R \le -q_* + L^{\gamma(\gamma+1)} = -q_* + L^{\gamma(\gamma+1)},$$

which implies that

$$R \le -q_* + 2^{\gamma(\gamma+1)}.$$

Using this and the inequality (3.23), we get

$$p_* \le r_* - r_*^{1+\frac{1}{\gamma}} \le r_* \le R \le -q_* + 2^{\gamma(\gamma+1)}.$$

Therefore

$$p_* + q_* \le 2^{\gamma(\gamma+1)},$$

which contradicts (3.16). The proof is complete. □

From Theorem 3.1, we have the following results immediately.

Corollary 3.1.1. *Assume that* (h$_1$)–(h$_5$) *and* (2.3) *hold. Furthermore assume that* $g(n) > n$, *and* $\Delta c_n \ge 0$. *Let* x_n *be a solution of* (1.4). *If*

$$\liminf_{n\to\infty} \frac{n^\gamma}{c_n} \sum_{s=n+1}^{\infty} Q_s > 2^{\gamma(\gamma+1)}. \tag{3.25}$$

Then either x_n *oscillates or* $\lim_{n\to\infty} z_n = 0$.

Corollary 3.1.2. *Assume that* (h$_1$)–(h$_5$) *and* (2.3) *hold. Furthermore assume that* $g(n) > n$, *and* $\Delta c_n \ge 0$. *Let* x_n *be a solution of* (1.4). *If*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{s=N}^{n} \frac{s^{\gamma+1}}{c_s} Q_s > 2^{\gamma(\gamma+1)}. \tag{3.26}$$

Then either x_n *oscillates or* $\lim_{n\to\infty} z_n = 0$.

In the following, we give some examples to illustrate the main results when $g(n) > n$.

Example 1. Consider the advanced equation

$$\Delta^3\left(x_n + \frac{1}{2}x_{n-2}\right) + \frac{\alpha}{(n-1)^2(n+1)^{(2)}} x(n^2) = 0, \qquad \text{for} \quad n > 1. \tag{3.27}$$

Here $\gamma = 1$, $K = 1$, $d_n = c_n = 1$, $p_n = 1/2$, $\tau = 2$, $g(n) = n^2$, and $q_n = \alpha/\left((n-1)^2(n+1)^{(2)}\right)$ where α is a positive constant. In this case it is clear that the conditions (h_1)–(h_5) hold. To apply Corollary 3.1 it remains to prove that (3.25) hold. In this case, we see that

$$C_n = (n-1), \quad D_n = n(n-1), \quad \text{and} \quad Q_n = \frac{\alpha}{2(n+1)^{(2)}}.$$

So that the condition (3.25) reads

$$\begin{aligned}
\liminf_{n\to\infty} \frac{n^\gamma}{c_n} \sum_{s=n+1}^{\infty} Q_s &= \alpha \liminf_{n\to\infty} n \sum_{s=n+1}^{\infty} \frac{1}{2(s+1)^{(2)}} \\
&= \frac{\alpha}{2} \liminf_{n\to\infty} n \sum_{s=n+1}^{\infty} \Delta\left(\frac{-1}{s}\right) \\
&= \frac{\alpha}{2} \liminf_{n\to\infty} n \frac{1}{n+1} = \frac{\alpha}{2}.
\end{aligned}$$

Then by Corollary 3.1, if $\alpha > 8$ the solution x_n of the equation (3.27), either oscillates or $\lim\limits_{n\to\infty} \left(x_n + \frac{1}{2}x_{n-2}\right) = 0$. Note that the results in the above mentioned papers cannot be applied on (3.27), since $g(n) = n^2 > n$.

3.2. The case when $g(n) \leq n$ and $d_n = 1$

In this subsection, we establish some sufficient conditions which guarantee that the solution x_n of (1.4) is either oscillates or $\lim\limits_{n\to\infty} z_n = \infty$ when $g(n) \leq n$. For simplification, we introduce the following notations:

$$A_* := \liminf_{n\to\infty} \frac{n^\gamma}{c_n} \sum_{s=n+1}^{\infty} A_s, \qquad B_* := \liminf_{n\to\infty} \frac{1}{n} \sum_{s=N}^{n-1} \frac{s^{\gamma+1}}{c_s} A_s,$$

$$A_n = P_n \left(\frac{h_2(g(n), n_0)}{n+1}\right)^\gamma, \qquad P_n = K q_n (1 - p_{g(n)})^\gamma.$$

Let x_n be a nonoscillatory solution of (1.4) and z_n is defined as in (2.1) such that $z_n \in C_2$. If $d_n = 1$ and $\Delta c_n \geq 0$, then we can deduce that if $z_n > 0$, then

$$\Delta z_n > 0, \quad \Delta^2 z_n > 0, \quad \text{and} \quad \Delta^3 z_n < 0. \tag{3.28}$$

We define the new quasi-differences of z_n by

$$y_n^{[0]} = z_n > 0, \quad y_n^{[1]} = \Delta z_n, \quad y_n^{[2]} = c_n \left[\Delta^2 z_n\right]^\gamma, \quad y_n^{[3]} = \Delta(y_n^{[2]}).$$

In the following, we assume that

$$\sum_{s=n+1}^{\infty} A_s < \infty.$$

Lemma 3.2. *Assume that* (h$_1$)–(h$_4$) *and* (2.3) *hold. Furthermore assume that* $d_n = 1$, $\Delta c_n \geq 0$, *and* $g(n) \leq n$. *Let* x_n *be a solution of* (1.4) *and* $z_n \in C_2$. *Define* u_n *by the Riccati substitution*

$$u_n := \frac{y_n^{[2]}}{\left(y_n^{[1]}\right)^{\gamma}}.$$

Then $u_n > 0$, *and satisfies*

$$\Delta u_n + A_n + \gamma \frac{1}{(c_n)^{\frac{1}{\gamma}}} (u_{n+1})^{1+\frac{1}{\gamma}} \leq 0, \qquad \textit{for} \quad n \geq N. \tag{3.29}$$

Proof. Let x_n be as in the statement of this theorem and without loss of generality, we may assume that there is $n_1 > n_0$ such that $x_n > 0$, $x_{n-\tau} > 0$ and $x_{g(n)} > 0$. Now, since $z_n \in C_2$ then there exists $N > n_1$ such that $z_n > 0$, $y^{[1]} = \Delta z_n > 0$, $y_n^{[2]} = c_n \left[\Delta^2 z_n\right]^{\gamma} > 0$, $y_n^{[3]} \leq 0$ for $n \geq N$. Since $\Delta c_n \geq 0$, we see that (3.28) is satisfied. From the definition of u_n, by quotient rule and continue as in the proof of Lemma 3.1, we get

$$\Delta u_n \leq -P_n \left(\frac{z_{g(n)}}{y_{n+1}^{[1]}}\right)^{\gamma} - \gamma \frac{1}{(c_n)^{\frac{1}{\gamma}}} (u_{n+1})^{1+\frac{1}{\gamma}}. \tag{3.30}$$

Now we consider the coefficient of P_n in (3.30). This coefficient can be written in the form

$$\frac{z_{g(n)}}{z_{n+1}^{[1]}} = \frac{z_{g(n)}}{y_{g(n)}^{[1]}} \frac{y_{g(n)}^{[1]}}{y_{n+1}^{[1]}}. \tag{3.31}$$

From Lemma 2.7, since $\lim\limits_{n\to\infty} g(n) = \infty$, we can choose $N_k \geq N$ such that

$$\frac{z_{g(n)}}{y_{g(n)}^{[1]}} = \frac{g(n) z_{g(n)}}{\Delta z_{g(n)}} \geq \frac{\sqrt{k} h_2(g(n), n_0)}{g(n)}, \qquad \text{for} \quad n > N_k, \tag{3.32}$$

and

$$\frac{y_{g(n)}^{[1]}}{y_{n+1}^{[1]}} = \frac{\Delta z_{g(n)}}{\Delta z_{n+1}} \geq \frac{1}{\sqrt{k}} \frac{g(n)}{(n+1)}, \qquad \text{for} \quad 0 < k < 1. \tag{3.33}$$

Then from (3.31)–(3.33), we have

$$\frac{z_{g(n)}}{\Delta z_{n+1}} \geq \frac{h_2(g(n), n_0)}{g(n)} \frac{g(n)}{n+1} = \frac{h_2(g(n), n_0)}{(n+1)}. \tag{3.34}$$

Substituting from (3.34) into (3.30), we have the inequality (3.29) and this completes the proof. □

The following theorem gives sufficient conditions for oscillation of (1.4) in the delay case.

Theorem 3.2. *Assume that* (h_1)–(h_5) *and* (2.3) *hold. Furthermore assume that* $d_n = 1$, $\Delta c_n \geq 0$, $f(u)/u^\gamma \geqslant K > 0$ *and* $g(n) \leq n$. *Let* x_n *be a solution of* (1.4). *If*

$$A_* > \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}, \tag{3.35}$$

or

$$A_* + B_* > 2^{\gamma(\gamma+1)}. \tag{3.36}$$

Then x_n *is oscillatory or* $\lim\limits_{n\to\infty} z_n = 0$.

P r o o f. The proof is similar to the proof of Theorem 3.1, by replacing w_n by u_n, and Q_n by A_n and hence is omitted. □

Corollary 3.2.1. *Assume that* (h_1)–(h_5) *and* (2.3) *hold. Furthermore assume that* $d_n = 1$, $\Delta c_n \geq 0$, *and* $g(n) \leq n$. *Let* x_n *be a solution of* (1.4). *If*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{s=N}^{n-1} \frac{s^{\gamma+1}}{c_s} A_s > 2^{\gamma(\gamma+1)}. \tag{3.37}$$

Then x_n *is oscillatory or* $\lim\limits_{n\to\infty} z_n = 0$.

Corollary 3.2.2. *Assume that* (h_1)–(h_5) *and* (2.3) *hold. Furthermore assume that* $d_n = 1$, $\Delta c_n \geq 0$, *and* $g(n) \leq n$. *Let* x_n *be a solution of* (1.4). *If*

$$\liminf_{n\to\infty} \frac{n^\gamma}{c_n} \sum_{s=n+1}^{\infty} A_s > 2^{\gamma(\gamma+1)}. \tag{3.38}$$

Then x_n *is oscillatory or* $\lim\limits_{n\to\infty} z_n = 0$.

In the following, we give some examples to illustrate the main results when $g(n) \leq n$.

Example 2. Consider the third order delay difference equation

$$\Delta^3\Big(x_n + \frac{1}{2}x_{n-3}\Big) + \frac{\alpha(n+1)}{g^2(n)h_2(g(n),1)} x_{g(n)} = 0, \quad g(n) \leq n \qquad \text{for} \quad n \geq 1. \tag{3.39}$$

Here $c_n = d_n = 1$, $\gamma = 1$, $\beta = 1$, $K = 1$ and $q_n = \alpha(n+1)/(g^2(n)h_2(g(n),1))$ where α is a positive constant. It is clear that (h_1)–(h_4) hold. To apply Corollary 3.3 it remains to prove that (h_3) and (3.37) hold. For equation (3.39), we have

$$\begin{aligned}\sum_{n=n_0}^{\infty} \frac{1}{d_n} \sum_{t=n_0}^{n-1} \left(\frac{1}{c_t} \sum_{s=n_0}^{t-1} q_s\right)^{\frac{1}{\gamma}} &\geq \sum_{n=n_0}^{\infty} \sum_{t=n_0}^{n-1} \left(\sum_{s=n_0}^{t-1} \frac{\alpha(s)}{s^2 h_2(g(s),1)}\right) \\ &= \sum_{n=1}^{\infty} \sum_{t=1}^{n-1} \left(\sum_{s=1}^{t-1} \frac{\alpha}{s h_2(g(s),1)}\right) = \infty.\end{aligned}$$

Also the condition (3.37) reads.

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n} \sum_{n=N}^{n-1} \frac{s^{\gamma+1}}{c_s} A(s) \\
&= \liminf_{n\to\infty} \frac{\alpha}{2n} \sum_{n=N}^{n-1} \frac{s^2(s+1)}{g^2(s)h_2(g(s),1)} \left(\frac{h_2(g(s),1)}{(s+1)}\right) \\
&= \liminf_{n\to\infty} \frac{\alpha}{2n} \sum_{n=1}^{n-1} \frac{s^2}{g^2(s)} \geq \liminf_{n\to\infty} \frac{\alpha}{2n} \sum_{n=1}^{n-1} 1 = \frac{\alpha}{2}.
\end{aligned}$$

Then by Corollary 3.3, the solution x_n of (3.39) is oscillatory or $\lim\limits_{n\to\infty}\left(x_n+\frac{1}{2}x_{n-3}\right) = 0$ if $\alpha > 8$.

Remark 1. It would be great of interest to consider the case when

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{c_n}\right)^{\gamma} < \infty, \qquad \sum_{n=n_0}^{\infty} \left(\frac{1}{d_n}\right) < \infty, \tag{3.40}$$

and establish some sufficient conditions for oscillation of (1.4). In this case if (2.4) holds and x_n is a nonoscillatory solution of (1.4), then $z_n \in C_0 \cup C_1 \cup C_2$. If $z_n \in C_0$, then we can prove that $\lim\limits_{n\to\infty} z_n = 0$ and if $z_n \in C_2$, we follow the proofs of Theorems 3.1 and 3.2 to establish conditions for oscillation. It remains to consider the case when $z_n \in C_1$. This will left as an open problem to the interested reader.

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