VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-011-0028-z
Math. Slovaca **61** (2011), No. 4, 561–570

A RELATION OF WEAK MAJORIZATION AND ITS APPLICATIONS TO CERTAIN INEQUALITIES FOR MEANS

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(*Communicated by Ján Borsík*)

ABSTRACT. A relation of weak majorization for n-dimensional real vectors is established, the result is then used to derive some inequalities involving the power mean, the arithmetic mean and the geometric mean in n variables.

1. Introduction

Over the years, the theory of majorization as a powerful tool has widely been applied to the related research areas of pure mathematics and the applied mathematics (see [1], [15]). A good survey on the theory of majorization was given by Marshall and Olkin in [4]. Recently, the authors have given considerable attention to the applications of majorization in the field of inequalities, for details, we refer the reader to our papers [2], [3], [5]–[14] and [16]–[19].

In this paper, we shall establish a weak majorization relation for positive real numbers $x_1, x_2, \dots, x_n$ with $x_1 x_2 \cdots x_n \geq 1$, and discuss the Schur-convexity of

2010 Mathematics Subject Classification: Primary 26D15; Secondary 26E60, 52A40.
Keywords: weak majorization, inequality, Schur-concavity, elementary symmetric function, power mean, arithmetic mean, geometric mean.
This work was supported, in part, by the Foundation of Scientific Research Project of Fujian Province Education Department of China under Grant JK2010058, and, in part, by the Scientific Research Common Program of Beijing Municipal Commission of Education under Grant KM201111417006. Both of the authors are grateful to the referees for their helpful and constructive comments which enhanced this paper.

the elementary symmetric function. In Section 4, the result is used to derive some inequalities involving the power mean, the arithmetic mean and the geometric mean in n variables.

Throughout the paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbf{x} = (x_1, \dots, x_n)$ denotes n-tuple (n-dimensional real vector), the set of vectors can be written as

$$\begin{aligned} \mathbb{R}^n &= \big\{\mathbf{x} = (x_1, \dots, x_n) : \ x_i \in \mathbb{R}, \ \ i = 1, \dots, n\big\}, \\ \mathbb{R}^n_+ &= \big\{\mathbf{x} = (x_1, \dots, x_n) : \ x_i \geq 0, \ \ i = 1, \dots, n\big\}, \\ \mathbb{R}^n_{++} &= \big\{\mathbf{x} = (x_1, \dots, x_n) : \ x_i > 0, \ \ i = 1, \dots, n\big\}. \end{aligned}$$

Definition 1. ([15]) Let $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. For $k = 1, \dots, n$ the kth elementary symmetric function is defined as follows:

$$E_k(\mathbf{x}) = E_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \prod_{j=1}^{k} x_{i_j}.$$

The dual form of the elementary symmetric function is defined by

$$E^*_k(\mathbf{x}) = E^*_k(x_1, \dots, x_n) = \prod_{1 \leq i_1 < \ldots < i_k \leq n} \sum_{j=1}^{k} x_{i_j}.$$

Definition 2. ([4]) Let $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ and $x_{[1]} \geq \dots \geq x_{[n]}$, $y_{[1]} \geq \dots \geq y_{[n]}$ be rearrangements of $\mathbf{x}$ and $\mathbf{y}$ in a descending order, respectively. Then

(1) $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ (briefly $\mathbf{x} \prec \mathbf{y}$) if $\sum\limits_{i=1}^{k} x_{[i]} \leq \sum\limits_{i=1}^{k} y_{[i]}$ for $k = 1, 2, \dots, n-1$ and $\sum\limits_{i=1}^{n} x_i = \sum\limits_{i=1}^{n} y_i$;

(2) $\mathbf{x}$ is said to be weakly submajorized by $\mathbf{y}$ (briefly $\mathbf{x} \prec_w \mathbf{y}$) if $\sum\limits_{i=1}^{k} x_{[i]} \leq \sum\limits_{i=1}^{k} y_{[i]}$ for $k = 1, 2, \dots, n$;

(3) a function $\varphi\colon \Omega \to \mathbb{R}$ is said to be increasing on $\Omega \subset \mathbb{R}^n$ if $\mathbf{x} \geq \mathbf{y}$ implies $\varphi(\mathbf{x}) \geq \varphi(\mathbf{y})$, where $\mathbf{x} \geq \mathbf{y}$ means $x_i \geq y_i$ for all $i = 1, 2, \dots, n$;

(4) a function $\varphi\colon \Omega \to \mathbb{R}$ is said to be Schur-convex on $\Omega \subset \mathbb{R}^n$ if $\mathbf{x} \prec \mathbf{y}$ on Ω implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$.

If $-\varphi$ is an increasing (resp. Schur-convex) function on Ω, then φ is said to be decreasing (resp. Schur-concave) on Ω.

2. Lemmas

To prove the main results stated in Sections 3 and 4, we need the following lemmas.

Lemma 1. ([15, p. 7]) *Let* $\mathbf{x} \in \mathbb{R}_+^n$, $\mathbf{y} \in \mathbb{R}^n$ *and* $\delta = \sum_{i=1}^{n}(y_i - x_i)$. *If* $\mathbf{x} \prec_w \mathbf{y}$, *then*

$$\Big(\mathbf{x}, \underbrace{\frac{\delta}{n}, \ldots, \frac{\delta}{n}}_{n}\Big) \prec \Big(\mathbf{y}, \underbrace{0, \ldots, 0}_{n}\Big).$$

Lemma 2. ([4, p. 122]) *Let* $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$. *If* $\mathbf{x} \prec_w \mathbf{y}$, *then*

$$(\mathbf{x},\ x_{n+1}) \prec (\mathbf{y},\ y_{n+1}),$$

where $x_{n+1} = \min\{x_1, \ldots, x_n,\ y_1, \ldots, y_n\}$, $y_{n+1} = \sum_{i=1}^{n+1} x_i - \sum_{i=1}^{n} y_i$.

Lemma 3. ([15, p. 48]) *Let* $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ *be such that* $x_i, y_i \in I \subset \mathbb{R}$, $i = 1, 2, \ldots, n$. *Then*

(1) $\mathbf{x} \prec \mathbf{y}$ *if and only if* $\sum_{i=1}^{n} g(x_i) \le \sum_{i=1}^{n} g(y_i)$ *holds for all convex functions* $g\colon I \to \mathbb{R}$;

(2) $\mathbf{x} \prec \mathbf{y}$ *if and only if* $\sum_{i=1}^{n} g(x_i) \ge \sum_{i=1}^{n} g(y_i)$ *holds for all concave functions* $g\colon I \to \mathbb{R}$.

Lemma 4. ([15, p. 65]) *Let* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ *be such that* $x_i \in I \subset \mathbb{R}$, $i = 1, 2, \ldots, n$. *If* $g\colon I \to B \subset \mathbb{R}$ *is concave,* $\varphi\colon B^n \to \mathbb{R}$ *is increasing and Schur-concave, then* $\psi\colon I^n \to \mathbb{R}$ *given by* $\psi(\mathbf{x}) = \varphi\left(g\left(x_1\right), \ldots, g\left(x_n\right)\right)$ *is Schur-concave.*

Lemma 5. ([15, p. 59]) *Let* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$, $1 \le k \le n$, *then the elementary symmetric function* $E_k(\mathbf{x})$ *and its dual version* $E_k^*(\mathbf{x})$ *are increasing and Schur-concave on* $\mathbb{R}_+^n$.

3. Main results and their proofs

Our main results are given in the Theorem 1 and Corollary 2 below.

Theorem 1. *Let* $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n_{++}$, $n \geq 2$ *and* $\prod_{i=1}^{n} x_i \geq 1$. *Then*

$$\Big(\underbrace{1, \dots, 1}_{n}\Big) \prec_w (x_1, \dots, x_n)\,. \tag{1}$$

P r o o f. We show the validity of majorization relation (1) by induction.

When $n = 2$, without loss of generality, we may assume that $x_1 \geq x_2$. From $x_1, x_2 > 0$ and $x_1 x_2 \geq 1$, it follows that $x_1 \geq 1$ and $x_1 + x_2 \geq 2\sqrt{x_1 x_2} \geq 2 = 1+1$. This means that $(1, 1) \prec_w (x_1, x_2)$.

We now assume that (1) holds true for $n = k$.

Let $\mathbf{x} = (x_1, \dots, x_{k+1}) \in \mathbb{R}^{k+1}_{++}$ and $\prod_{i=1}^{k+1} x_i \geq 1$. Without loss of generality, we may assume that $x_1 \geq x_2 \geq \ldots \geq x_{k+1} > 0$.

If $x_{k+1} > 1$, then $x_i > 1$ for $i = 1, \dots, k+1$. It is clear that

$$\Big(\underbrace{1, \dots, 1}_{k+1}\Big) \prec_w (x_1, \dots, x_{k+1})\,.$$

If $x_{k+1} \leq 1$, then $x_1 \geq x_2 \geq \ldots \geq x_{k-1} \geq x_k x_{k+1}$. Using the above assumption we have

$$\Big(\underbrace{1, \dots, 1}_{k}\Big) \prec_w (x_1, \dots, x_{k-1}, x_k x_{k+1})\,.$$

It follows that

$$\sum_{i=1}^{t} x_i \geq t \qquad \text{for} \quad t = 1, \dots, k-1,$$

and

$$\sum_{i=1}^{k-1} x_i + x_k x_{k+1} \geq k.$$

Thus, we have

$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k-1} x_i + x_k x_{k+1} \geq k$$

and

$$\sum_{i=1}^{k+1} x_i \geq (k+1) \sqrt[k+1]{x_1 \dots x_{k+1}} \geq k+1.$$

This proves that (1) holds true for $n = k+1$, hence the proof of Theorem 1 is completed. $\square$

Remark 1. As a direct consequence of Theorem 1, we obtain the following weak majorization relations.

Corollary 1. *Let x_1, x_2, x_3 be positive real numbers. Then*

$$(1,1,1) \prec_w \left(\frac{x_2+x_3}{x_3+x_1}, \frac{x_3+x_1}{x_1+x_2}, \frac{x_1+x_2}{x_2+x_3}\right),$$

$$(1,1,1) \prec_w \left(\frac{x_1}{\sqrt{x_2x_3}}, \frac{x_2}{\sqrt{x_3x_1}}, \frac{x_3}{\sqrt{x_1x_2}}\right),$$

$$(1,1,1) \prec_w \left(\frac{\sqrt{x_2x_3}}{x_1}, \frac{\sqrt{x_3x_1}}{x_2}, \frac{\sqrt{x_1x_2}}{x_3}\right).$$

Corollary 2. *Let $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n_{++}$, $n \geq 2$ and $\prod\limits_{i=1}^{n} x_i \geq 1$. Then*

$$\Big(\underbrace{1,\dots,1}_{n}, \underbrace{A-1,\dots,A-1}_{n}\Big) \prec \Big(x_1,\dots,x_n,\underbrace{0,\dots,0}_{n}\Big), \tag{2}$$

$$\Big(\underbrace{1,\dots,1}_{n}, a\Big) \prec (x_1,\dots,x_n,x_{n+1}), \tag{3}$$

where $A = \frac{1}{n}\sum\limits_{i=1}^{n} x_i$, $a = \min\{x_1,\dots,x_n,1\}$, $x_{n+1} = a - n(A-1)$.

Proof. Using Theorem 1, Lemma 1 and Lemma 2 the majorization relations (2) and (3) follow immediately. □

4. Some applications

In this section, we show that our results can be used to establish some new inequalities for means.

Let M_α denote the power mean which is defined for positive real numbers $x_1, x_2, \dots, x_n$ as

$$M_\alpha = \begin{cases} \left(\frac{1}{n}\sum\limits_{i=1}^{n} x_i^\alpha\right)^{\frac{1}{\alpha}}, & \alpha \neq 0, \\ \left(\prod\limits_{i=1}^{n} x_i\right)^{\frac{1}{n}}, & \alpha = 0. \end{cases}$$

Clearly, $M_1 = A$ (the arithmetic mean) and $M_0 = G$ (the geometric mean).

Theorem 2. *Let* $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n_{++}, n \geq 2$ *and* $\prod_{i=1}^{n} x_i \geq 1$.

If $\alpha \geq 1$, *then*

$$M_\alpha \geq \big(1 + (A-1)^\alpha\big)^{\frac{1}{\alpha}}. \tag{4}$$

If $\alpha \geq 1$ *and* $n(A-1) \leq a$, *where* $a = \min\{x_1, \dots, x_n, 1\}$, *then*

$$M_\alpha \geq \left(1 + \frac{a^\alpha - (a - n(A-1))^\alpha}{n}\right)^{\frac{1}{\alpha}}. \tag{5}$$

Furthermore, the inequalities (4) *and* (5) *are reversed for* $0 < \alpha < 1$.

P r o o f. When $\alpha \geq 1$, the function $f(x) = x^\alpha$ is convex on $(0, +\infty)$.

By using Lemma 3, we deduce from (2) and (3) that

$$\sum_{i=1}^{n} f(x_i) + nf(0) \geq nf(1) + nf(A-1) \tag{6}$$

and

$$\sum_{i=1}^{n} f(x_i) + f\,(a - n(A-1)) \geq nf(1) + f(a). \tag{7}$$

After a simple computation the inequalities (6) and (7) may be written in the form (4) and (5), respectively.

For $0 < \alpha < 1$, the function $f(x) = x^\alpha$ is concave on $(0, +\infty)$. Using Lemma 3 and the majorization relations (2) and (3) we obtain the reverse inequalities of (4) and (5). Theorem 2 is proved. □

Corollary 3. *Let* $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n_{++}$, $n \geq 2$.

If $\alpha \geq 1$, *then*

$$M_\alpha \geq (G^\alpha + (A-G)^\alpha)^{\frac{1}{\alpha}} \geq G. \tag{8}$$

If $\alpha \geq 1$ *and* $b \geq n(A-G)$, *where* $b = \min\{x_1, \dots, x_n, G\}$, *then*

$$M_\alpha \geq \left(G^\alpha + \frac{b^\alpha - (b - n(A-G))^\alpha}{n}\right)^{\frac{1}{\alpha}} \geq G. \tag{9}$$

P r o o f. For positive numbers x_1/G, x_2/G, $\dots$, x_n/G, we have

$$\prod_{i=1}^{n} \frac{x_i}{G} = 1, \quad \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{G} = \frac{A}{G}, \quad \left(\frac{1}{n}\sum_{i=1}^{n} \left(\frac{x_i}{G}\right)^\alpha\right)^{\frac{1}{\alpha}} = \frac{M_\alpha}{G},$$

$$\min\left\{\frac{x_1}{G}, \dots, \frac{x_n}{G}, 1\right\} = \frac{b}{G}.$$

Replacing $x_1, x_2 \ldots, x_n$ by $x_1/G,\ x_2/G,\ \ldots,\ x_n/G$, respectively, in (4) and (5) we obtain

$$\frac{M_\alpha}{G} \geq \left(1+\left(\frac{A}{G}-1\right)^\alpha\right)^{\frac{1}{\alpha}} \tag{10}$$

and

$$\frac{M_\alpha}{G} \geq \left(1+\frac{\left(\frac{b}{G}\right)^\alpha-\left(\frac{b}{G}-n\left(\frac{A}{G}-1\right)\right)^\alpha}{n}\right)^{\frac{1}{\alpha}}, \tag{11}$$

respectively, which corresponds to (8) and (9). □

Theorem 3. *Let* $\mathbf{x}=(x_1,\ldots,x_n)\in\mathbb{R}^n_{++}$, $n\geq 2$, $0<\alpha\leq 1$ *and* $\prod_{i=1}^{n} x_i \geq 1$.

If $1\leq k\leq n$, *then*

$$E_k(\mathbf{x}^\alpha) \leq \sum_{i=0}^{k} C_n^i C_n^{k-i}(A-1)^{(k-i)\alpha}. \tag{12}$$

If $n+1\leq k\leq 2n$, *then*

$$\prod_{l=k-n}^{n} \left(E_l^*(\mathbf{x}^\alpha)\right)^{C_n^{k-l}} \leq \prod_{l=k-n}^{n} \left(l+(k-l)(A-1)^\alpha\right)^{C_n^l C_n^{k-l}}. \tag{13}$$

P r o o f. By Lemma 4 and Lemma 5, we conclude that $E_k(\mathbf{x}^\alpha)$ and $E_k^*(\mathbf{x}^\alpha)$ are Schur-concave on $\mathbb{R}^n_{++}$. Using the majorization relation (2) with the definition of Schur-concavity leads us to the desired inequalities (12) and (13). □

Replacing $x_1, x_2, \ldots, x_n$ by $x_1/G, x_2/G, \ldots, x_n/G$, respectively, in (12) and (13) we get the following corollary.

Corollary 4. *Let* $\mathbf{x}=(x_1,\ldots,x_n)\in\mathbb{R}^n_{++}$, $n\geq 2$ *and* $0<\alpha\leq 1$.

If $1\leq k\leq n$, *then*

$$E_k(\mathbf{x}^\alpha) \leq \sum_{i=0}^{k} C_n^i C_n^{k-i} G^{(i-k+C_n^k)\alpha}\,(A-G)^{(k-i)\alpha}. \tag{14}$$

If $n+1\leq k\leq 2n$, *then*

$$\prod_{l=k-n}^{n} \left(E_l^*(\mathbf{x}^\alpha)\right)^{C_n^{k-l}} \leq \prod_{l=k-n}^{n} \left(lG^\alpha+(k-l)(A-G)^\alpha\right)^{C_n^l C_n^{k-l}}. \tag{15}$$

Theorem 4. *Let* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++}$*,* $n \geq 2$*,* $\prod_{i=1}^{n} x_i \geq 1$ *and* $A \leq 1 + \frac{a}{n}$*, where* a *as in* Theorem 2*. If* $1 \leq k \leq n$ *and* $0 < \alpha \leq 1$*, then*

$$E_k(\mathbf{x}^\alpha) + (a - n(A-1))^\alpha E_{k-1}(\mathbf{x}^\alpha) \leq C_n^k + C_n^{k-1} a^\alpha \tag{16}$$

and

$$E_k^*(\mathbf{x}^\alpha) \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left((a - (A-1))^\alpha + \sum_{j=1}^{k-1} x_{i_j}^\alpha \right) \leq k^{C_n^k} \left(a^\alpha + k - 1 \right)^{C_n^{k-1}}. \tag{17}$$

P r o o f. Since $E_k(\mathbf{x}^\alpha)$ and $E_k^*(\mathbf{x}^\alpha)$ are Schur-concave on $\mathbb{R}^n_{++}$, then the majorization relation (3) with the definition of Schur-concavity yield inequalities (16) and (17). □

Again, replacing $x_1, x_2, \ldots, x_n$ by $x_1/G, x_2/G, \ldots, x_n/G$, respectively, in (16) and (17) we get the following corollary.

Corollary 5. *Let* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++}$*,* $n \geq 2$ *and* $b \geq n(A - G)$*, where* b *as in* Corollary 3*. If* $1 \leq k \leq n$ *and* $0 < \alpha \leq 1$*, then*

$$E_k(\mathbf{x}^\alpha) + (b - n(A-G))^\alpha E_{k-1}(\mathbf{x}^\alpha) \leq C_n^k G^{k\alpha} + C_n^{k-1} b^\alpha G^{(k-1)\alpha},$$

and

$$\begin{aligned} E_k^*(\mathbf{x}^\alpha) \prod_{1 \leq i_1 < \ldots < i_k \leq n} & \left((b - n(A-G))^\alpha + \sum_{j=1}^{k-1} x_{i_j}^\alpha \right) \\ & \leq k^{C_n^k} G^{\alpha C_n^k} \left(b^\alpha + (k-1) G^\alpha \right)^{C_n^{k-1}}. \end{aligned}$$

Remark 2. Theorems 2, 3, 4 and their corollaries enable us to obtain a large number of inequalities for particular choice of parameters α, n and k. For example, if we take $n = 3$, $k = 2$ in (14) and $n = 3$, $k = 5$ in (15), we get for $x_i > 0$, $i = 1, 2, 3$ and $0 < \alpha < 1$ the following interesting inequalities:

$$\left(x_1^\alpha x_2^\alpha + x_2^\alpha x_3^\alpha + x_3^\alpha x_1^\alpha \right)/3 \leq G^\alpha (A-G)^{2\alpha} + 3G^{2\alpha}(A-G)^\alpha + G^{3\alpha}, \tag{18}$$

$$\begin{aligned} & \left(x_1^\alpha + x_2^\alpha + x_3^\alpha \right) \sqrt[3]{\left(x_1^\alpha + x_2^\alpha \right)\left(x_2^\alpha + x_3^\alpha \right)\left(x_3^\alpha + x_1^\alpha \right)} \\ & \qquad \leq \left(2G^\alpha + 3(A-G)^\alpha \right)\left(3G^\alpha + 2(A-G)^\alpha \right). \end{aligned} \tag{19}$$

In particular, for $\alpha = 1$ the inequalities (18) and (19) reduce to

$$\left(x_1x_2 + x_2x_3 + x_3x_1\right)/3 \leq G(A^2 + AG - G^2)$$

and

$$(x_1 + x_2 + x_3)\sqrt[3]{(x_1 + x_2)(x_2 + x_3)(x_3 + x_1)} \leq (3A - G)(G + 2A).$$

REFERENCES

[1] BULLEN, P. S.: *A Dictionary of Inequalities*. Pitman Monographs and Surveys in Pure and Applied Mathematics 97, Longman, Harlow, 1998.

[2] LI, D. M.—SHI, H. N.: *Schur-convexity and Schur-geometric convexity for a class of means*, RGMIA Res. Rep. Coll. **9** (2006), 1–9.

[3] LI, D. M.—SHI, H. N.—ZHANG, J.: *Schur-convexity and Schur-geometrically concavity of Seiffert's mean*, RGMIA Res. Rep. Coll. **11** (2008), 1–6.

[4] MARSHALL, A. W.—OLKIN, I.: *Inequalities: The Theory of Majorization and Its Applicaitons*, Academic Press, New York, 1979.

[5] SHI, H. N.: *A proposition on the theory of majorization and its applications*. Research in Inequality, Tibet People's Publishing House, Lasha, 2000 (Chinese).

[6] SHI, H. N.: *Schur-convex functions relate to Hadamard-type inequalities*, J. Math. Inequal. **1** (2007), 127–136.

[7] SHI, H. N.: *Sharpening of Zhong Kai-lai's inequality*, RGMIA Res. Rep. Coll. **10** (2007), 1–5.

[8] SHI, H. N.: *Generalizations of Bernoulli's inequality with applications*, J. Math. Inequal. **2** (2007), 101–107.

[9] SHI, H. N.—BENCZE, M.—WU, S. H.—LI, D. M.: *Schur convexity of generalized Heronian means involving two parameters*, J. Inequal. Appl. **2008** (2008), 1–9.

[10] SHI, H. N.—JIANG, Y. M.—JIANG, W. D.: *Schur-convexity and Schur-geometrically concavity of Gini mean*, Comput. Math. Appl. **57** (2009), 266–274.

[11] SHI, H. N.—WU, S. H.—QI, F.: *An alternative note on the Schur-convexity of the extended mean values*, Math. Inequal. Appl. **9** (2006), 219–224.

[12] SHI, H. N.—XU, T. Q.—QI, F.: *Monotonicity results for arithmetic means of concave and convex functions*, RGMIA Res. Rep. Coll. **9** (2006), 1–18.

[13] SHI, H. N.—WU, S. H.: *Majorized proof and refinement of the discrete Steffensen's inequality*, Taiwanese J. Math. **11** (2007), 1203–1208.

[14] SHI, H. N.—WU, S. H.: *Refinement of an inequality for generalized logarithmic mean*, Chinese Quart. J. Math. **23** (2008), 594–599.

[15] WANG, B. Y.: *Foundations of Majorization Inequalities*, Beijing Normal University Press, Beijing, 1990 (Chinese).

[16] WU, S. H.: *Generalization and sharpness of power means inequality and their applications*, J. Math. Anal. Appl. **312** (2005), 637–652.

[17] WU, S. H.: *Schur-convexity for a class of symmetric functions and its applications*, Math. Practice Theory **34** (2004), 162–172.

[18] WU, S. H.—DEBNATH, L.: *Inequalities for convex sequences and their applications*, Comput. Math. Appl. **54** (2007), 525–534.

[19] WU, S. H.—SHI, H. N.: *Majorization proofs of inequalities for convex sequences*, Math. Practice Theory **33** (2003), 132–137.

Received 2. 3. 2009
Accepted 13. 5. 2009

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