

# STRONGLY CONCAVE FUNCTIONS ON ROOT SYSTEMS

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**ABSTRACT.** Let  $F$  be a local complete field with discrete valuation, and let  $G$  be the group of  $F$ -points of a quasisplit group over  $F$ . Let  $\mathcal{B}$  be the Bruhat-Tits building of  $G$  and let  $\mathcal{A}$  be an apartment of  $\mathcal{B}$ ; we establish a link between bounded closed subsets of  $\mathcal{A}$  and a special kind of functions on the relative root system  $\Phi$  of  $G$ , which will be called strongly concave functions. This paper is devoted to the study of such functions.

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## 1. Introduction

Let  $F$  be a local complete field with discrete valuation; let  $\mathcal{O}$  be its ring of integers,  $\mathfrak{p}$  the maximal ideal of  $\mathcal{O}$ ,  $K = \mathcal{O}/\mathfrak{p}$  its residual field.

Let  $\underline{G}$  be a connected reductive quasisplit algebraic group defined over  $F$ , and let  $G = \underline{G}(F)$  be the group of  $F$ -points of  $\underline{G}$ . In the sequel, we will assume that  $\underline{G}$  doesn't have any normal subgroup which is isogenous to a group of the form  $R_{F'/F}(\underline{H})$ , with  $F' \supsetneq F$  and  $\underline{H}$  being a connected reductive group defined over  $F'$ ; since we have  $R_{F'/F}(\underline{H})(F) = \underline{H}(F')$  and since in this paper, only groups of rational points and relative root systems will matter, we don't lose any generality by making such an assumption.

Let  $G^1$  be the subgroup of  $G$  generated by its parahoric subgroups;  $G^1$  is an open normal subgroup of  $G$ , and the quotient  $G/G^1$  is abelian; it is finite (resp. trivial) if and only if  $G$  is semisimple (resp. semisimple and simply connected).

Let  $\mathcal{B}$  be the Bruhat-Tits building of  $G$ , and let  $\mathcal{A}$  be the apartment of  $\mathcal{B}$  associated to some given maximal split torus  $S$  of  $G$ ;  $\mathcal{A}$  is canonically isomorphic as an affine space to  $(X_*(S)/X_*(Z)) \otimes \mathbb{R}$ , where  $Z = \underline{Z}(F)$ ,  $\underline{Z}$  being the split component of the center of  $\underline{G}$ .

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Let  $T$  be the centralizer of  $S$  in  $G$  and let  $\Phi$  be the (relative) root system of  $G$  relatively to  $S$ . It is well-known (see [BT, I.6.4]) that to each open compact subgroup of  $G^1$  containing the unique parahoric subgroup  $K_T$  of  $T$  can be attached a bounded closed subset of  $\mathcal{A}$ , that is the set of elements of  $\mathcal{A}$  it fixes. However, the correspondence is generally not 1–1. To these subgroups can also be attached functions from  $\Phi$  to  $\mathbb{R}$ , called *quasi-concave* functions.

Let  $E$  be a bounded closed subset of  $\mathcal{A}$ ; the subgroup of elements of  $G^1$  fixing it pointwise is the *connected fixator*  $G_E$  of  $E$ . In Section 3, we will prove that the functions associated to connected fixators are a special kind of quasi-concave functions, called *strongly concave* functions, which will first be introduced in Section 2.

The Section 4 will be devoted to some results about root systems. In Section 5, we will investigate for a more practically usable definition of strongly concave functions, which will however depend on  $G$ .

The motivation to this work is that knowing exactly which subgroups of  $G$  are connected fixators of subsets of  $\mathcal{A}$  is of some help when we try to use the Bruhat-Tits machinery to deal with smaller open compact subgroups of  $G$  than the parahoric subgroups, for example in the context of representations or invariant distributions on  $G$  of any level.

## 2. Definitions and first properties

Let  $T$  be the unique maximal torus of  $G$  which contains  $S$ , and for every  $\alpha \in \Phi$ , let  $U_\alpha$  be the root subgroup of  $G$  associated to  $\alpha$ . Let  $\phi = (\phi_\alpha)_{\alpha \in \Phi}$  be a valuation on the root datum  $(T, (U_\alpha)_{\alpha \in \Phi})$ ; since the valuation on  $F$  is discrete, for every  $\alpha \in \Phi$ , the image of the valuation  $\phi_\alpha$  on  $U_\alpha$  is of the form  $(c(\alpha) + v(\alpha)\mathbb{Z}) \cup \{+\infty\}$ , where  $c(\alpha)$ ,  $v(\alpha)$  are elements of  $\mathbb{R}$ ,  $v(\alpha)$  being positive. For every  $\alpha$  and every element  $c$  of  $\mathbb{R}$ , let  $U_{\alpha,c}$  be the subgroup of the elements  $u$  of  $U_\alpha$  such that  $\phi_\alpha(u) \geq c$ .

Let's recall some other definitions from [BT, I.6.4]. A function  $f$  from  $\Phi$  to  $\mathbb{R}$  is *concave* if it satisfies the following condition: for every  $\alpha_1, \dots, \alpha_r \in \Phi$  whose sum is an element  $\alpha$  of  $\Phi$ , we have  $f(\alpha) \leq \sum_{i=1}^r f(\alpha_i)$ . According to [BT, I.6.4.5], this is equivalent to say that  $f$  satisfies the following two conditions, which are quite easy to check:

- for every  $\alpha \in \Phi$ ,  $f(\alpha) + f(-\alpha) \geq 0$ ;
- for every  $\alpha, \beta \in \Phi$  such that  $\alpha + \beta \in \Phi$ ,  $f(\alpha + \beta) \leq f(\alpha) + f(\beta)$ .

Let  $K_T$  be the unique parahoric subgroup of  $T$ , and let  $U_f$  (resp.  $G_f$ ) be the subgroup of  $G$  generated by the  $U_{\alpha, f(\alpha)}$ ,  $\alpha \in \Phi$  (resp. the  $U_{\alpha, f(\alpha)}$  and  $K_T$ ). The function  $f$  is said to be *optimal* if for every function  $f'$  on  $\Phi$  such that

$f' > f$ ,  $U_{f'}$  is strictly contained in  $U_f$ , or equivalently,  $G_{f'}$  is strictly contained in  $G_f$ ; since the valuation is discrete, for every open compact subgroup  $K$  of  $G$  containing  $K_T$ , there exists an optimal function  $f$  on  $\Phi$  such that  $K = G_f$ .

The function  $f$  is said to be *quasi-concave* if it satisfies the following properties for every  $\alpha \in \Phi$ :

- $f(\alpha) + f(-\alpha) \geq 0$ ;
- if  $2\alpha \notin \Phi$ ,  $U_f \cap U_\alpha = U_{\alpha, f(\alpha)}$ ;
- if  $2\alpha \in \Phi$ ,  $U_f \cap U_\alpha = U_{\alpha, f(\alpha)} U_{2\alpha, f(2\alpha)}$ .

As another well-known result ([BT, I.6.4.8]), every concave function is quasi-concave, but the converse is not always true.

We'll slightly enlarge the notion of concavity, in order to cover the cases where  $\underline{G}$  doesn't split over any unramified extension of  $F$ ; in such cases, it is easy to see that there exist optimal quasi-concave functions which are not concave. For example, assume the residual characteristic of  $K$  is not 2. Let  $F'$  be a ramified quadratic extension of  $F$ , and set  $G = SU_3(F')$ ; its relative root system  $\Phi$  is of type  $BC_1$ . Let  $\alpha$  be a short root, and let  $f$  be the function on  $\Phi$  such that  $f(\alpha) = f(-\alpha) = 0$  and  $f(2\alpha) = f(-2\alpha) = 1$ ;  $f$  is quasi-concave (for a suitable choice of the valuation,  $f$  is the quasi-concave optimal function associated to  $SU_3(\mathcal{O}_{F'})$ , where  $\mathcal{O}_{F'}$  is the ring of integers of  $F'$ ), but not concave.

We'll say  $f$  is *pseudo-concave* if it satisfies the following condition:

**CONDITION 1.**

- for every  $\alpha \in \Phi$ ,  $f(\alpha) + f(-\alpha) \geq 0$ ;
- for every  $\alpha, \beta \in \Phi$  such that  $\alpha + \beta \in \Phi$ , we have:

$$f(\alpha + \beta) < f(\alpha) + f(\beta) + v(\alpha + \beta).$$

It is immediate from the definitions that if  $\underline{G}$  splits over an unramified extension of  $F$  and  $f$  is optimal, it is pseudo-concave if and only if it is concave.

**LEMMA 2.1.** *Assume  $f$  is optimal and  $f(\alpha) + f(-\alpha) \geq 0$  for every  $\alpha \in \Phi$ . Then  $f$  is pseudo-concave if and only if we have:*

$$f(\alpha + \beta) \leq f(\alpha) + f(\beta) + v(\alpha + \beta) - \text{Inf}(v(\alpha), v(\beta))$$

for every  $\alpha, \beta \in \Phi$  such that  $\alpha + \beta \in \Phi$ .

**Proof.** Since  $f$  is optimal,  $f(\alpha + \beta) - f(\alpha) - f(\beta)$  is a multiple of the g.c.d. of  $v(\alpha + \beta)$ ,  $v(\alpha)$  and  $v(\beta)$ , hence:

$$f(\alpha + \beta) \leq f(\alpha) + f(\beta) + v(\alpha + \beta) - \text{g.c.d.}(v(\alpha + \beta), v(\alpha), v(\beta)).$$

Moreover, we see by considering the different possible cases (see [BT, II.4.1.16]) that  $v(\alpha), v(\beta), v(\alpha + \beta)$  can take at most two different values, one of which is a multiple of the other one, hence:

$$\text{g.c.d.}(v(\alpha + \beta), v(\alpha), v(\beta)) = \text{Inf}(v(\alpha + \beta), v(\alpha), v(\beta)).$$

Finally, we remark that  $\alpha + \beta$  is at least as long as the shortest one between  $\alpha$  and  $\beta$ . Since, still by [BT, II.4.1.16], the longest a root  $\gamma$  is, the greater  $v(\gamma)$  is, we have  $v(\alpha + \beta) \geq \inf(v(\alpha), v(\beta))$ , which concludes the proof.  $\square$

We'll use this lemma to prove the following result:

**PROPOSITION 2.2.** *Assume  $f$  is optimal. Then  $f$  is pseudo-concave if and only if it satisfies the following condition: for every  $\alpha_1, \dots, \alpha_r \in \Phi$  whose sum is an element  $\alpha$  of  $\Phi$ , we have:*

$$f(\alpha) < \sum_{i=1}^r f(\alpha_i) + v(\alpha).$$

**Proof.** First we remark that if  $\underline{G}$  splits over an unramified extension of  $F$ , the proposition is simply the corresponding result about concave functions; we'll then assume it is not the case. Note first that the root system  $\Phi$  is then not simply-laced.

Assume  $f$  is pseudo-concave; we'll show the condition of the proposition is true by induction on  $r$ . The cases  $r = 1$  and  $r = 2$  being obvious, assume  $r > 2$ . Let  $i$  be such that  $(\alpha, \alpha_i) > 0$ ; assume  $i = r$ . Assume first  $\alpha_r = \alpha$ ; we will then check the following inequality:

$$\sum_{i=1}^{r-1} f(\alpha_i) \geq 0,$$

which will imply the result. If  $r = 3$ , it is obvious; assume then  $r > 3$ . First we remark that if there exists a nonempty proper subset  $I$  of  $\{1, \dots, r-1\}$  such that for every  $i \in I$  and for every  $j \notin I$ ,  $(\alpha_i, \alpha_j) = 0$ , then the sum of the  $\alpha_i$ ,  $i \in I$ , is zero and the result follows from the induction hypothesis applied to them and to the  $\alpha_i$ ,  $i \notin I$ ; assume then no such subset exists. Let  $S$  be the subset of the  $i \in \{1, \dots, r-1\}$  such that  $\alpha_i$  is short (or nondivisible if  $\Phi$  is of type  $BC_n$ ), and let  $\xi$  be the sum of all such  $\alpha_i$ ; since the sum of the long (resp. divisible) roots is  $-\xi$ , there exists  $i \in S$  and  $j \notin S$  such that  $(\alpha_i, \alpha_j) < 0$ ; but then  $\alpha_i + \alpha_j$  is a short (resp. divisible) root, and we then have  $f(\alpha_i + \alpha_j) \leq f(\alpha_i) + f(\alpha_j)$  by the previous lemma; we may then apply the induction hypothesis to obtain the desired result.

Assume now  $\alpha_r \neq \alpha$ . Then  $\alpha - \alpha_r$  is a root, and we have by induction hypothesis:

$$f(\alpha - \alpha_r) < \sum_{i=1}^{r-1} f(\alpha_i) + v(\alpha - \alpha_r).$$

We deduce from the pseudo-concavity of  $f$  and the preceding lemma the following inequality:

$$f(\alpha) < \sum_{i=1}^r f(\alpha_i) + v(\alpha - \alpha_r) + v(\alpha) - \inf(v(\alpha_r), v(\alpha - \alpha_r)).$$

If we have  $v(\alpha_r) \geq v(\alpha - \alpha_r)$ , the result is proved. Assume this is not the case;  $\alpha_r$  is then strictly shorter than  $\alpha - \alpha_r$ . Let's consider the different cases:

- Assume first  $\alpha_r$  and  $\alpha - \alpha_r$  generate a subsystem (not necessarily closed) of type  $B_2$  of  $\Phi$ : we easily see that the conditions  $\alpha_r$  short,  $\alpha - \alpha_r$  long and  $\alpha \in \Phi$  imply that  $\alpha_r$  and  $\alpha - \alpha_r$  form a basis of this subsystem; but then we must have  $(\alpha, \alpha_r) = 0$  which contradicts our previous assumption.
- Assume now  $\alpha_r$  and  $\alpha - \alpha_r$  generate a subsystem of type  $G_2$  of  $\Phi$ ; we reach a contradiction the same way as above.
- Assume finally  $\alpha_r$  and  $\alpha - \alpha_r$  generate a subsystem of type  $BC_1$  of  $\Phi$ . The assumptions imply then  $\alpha - \alpha_r = -2\alpha_r$ , hence  $\alpha = -\alpha_r$ , which contradicts the fact that  $(\alpha, \alpha_r) > 0$ .

Hence  $f$  satisfies the condition of the proposition. Since the other implication is obvious, the proposition is proved.  $\square$

Now we'll introduce a new kind of functions, which will be the main subject of the paper. Let  $f$  be a function on  $\Phi$ ;  $f$  is said to be *strongly concave* if it satisfies the following condition: for every  $\alpha_1, \dots, \alpha_k \in \Phi$  and every  $\alpha \in \Phi$  such that  $\alpha = \sum_{i=1}^k \lambda_i \alpha_i$ , with the  $\lambda_i$  being elements of  $\mathbb{Q}_+^*$ , we have:

$$f(\alpha) < \sum_{i=1}^k \lambda_i f(\alpha_i) + v(\alpha).$$

**PROPOSITION 2.3.** *Assume once again  $f$  is optimal.*

- *If  $f$  is concave, it is pseudo-concave; the converse is true when  $\underline{G}$  splits over an unramified extension of  $F$ .*
- *If  $f$  is strongly concave, it is pseudo-concave.*
- *If  $f$  is pseudo-concave, it is quasi-concave.*

**Proof.** The first two assertions are obvious; let's prove the third one. By eventually changing the valuation on the root datum of  $G$ , we may assume  $f$  is integer-valued. For every  $\alpha \in \Phi$ , let  $\mathcal{F}_\alpha$  be the set of families  $(\alpha_i)_{1 \leq i \leq k}$  of roots such that  $\sum_{i=1}^k \alpha_i = \alpha$ ; using the commutator relations ([Chev] and

[BT, Appendix A]), we obtain:

$$U_\alpha \cap U_f \subset U_{\alpha, \inf_{(\alpha_1, \dots, \alpha_k) \in \mathcal{F}_\alpha} \sum_{i=1}^k f(\alpha_i)}$$

if  $2\alpha$  is not a root, and:

$$\begin{aligned} U_\alpha \cap U_f &\subset U_{\alpha, \inf_{(\alpha_1, \dots, \alpha_k) \in \mathcal{F}_\alpha} \sum_{i=1}^k f(\alpha_i)} (U_{2\alpha} \cap U_f) \\ &= U_{\alpha, \inf_{(\alpha_1, \dots, \alpha_k) \in \mathcal{F}_\alpha} \sum_{i=1}^k f(\alpha_i)} U_{2\alpha, \inf_{(\alpha_1, \dots, \alpha_k) \in \mathcal{F}_{2\alpha}} \sum_{i=1}^k f(\alpha_i)} \end{aligned}$$

if  $2\alpha$  is a root.

Let  $c_0$  be the smallest element of  $c(\alpha) + v(\alpha)\mathbb{Z}$  which is greater than or equal to  $\inf_{(\alpha_1, \dots, \alpha_k) \in \mathcal{F}_\alpha} \sum_{i=1}^k f(\alpha_i)$ ; we have, assuming  $2\alpha$  is not a root:

$$U_\alpha \cap U_f \subset U_{\alpha, c_0}.$$

Since  $f$  is strongly concave and optimal, we have  $f(\alpha) < \sum_{i=1}^k f(\alpha_i) + v(\alpha)$  for every  $(\alpha_1, \dots, \alpha_k) \in \mathcal{F}_\alpha$ , hence  $f(\alpha) \leq c_0$ , and finally:

$$U_\alpha \cap U_f \subset U_{\alpha, f(\alpha)};$$

the other inclusion is an obvious consequence of the definitions. The case where  $2\alpha$  is a root is similar.  $\square$

Note that a pseudo-concave function is in general not strongly concave. Assume for example  $\underline{G}$  is a split group of type  $B_n$ , and  $f$  is the function defined by  $f(\alpha) = 0$  if  $\alpha$  is long, and  $f(\alpha) = 1$  if  $\alpha$  is short. The fact that the sum of two long roots is never a short root implies that  $f$  is concave, but the half-sum of two long roots  $\alpha$  and  $\beta$  may be a short root, in which case we have:

$$f\left(\frac{1}{2}\alpha + \frac{1}{2}\beta\right) = 1 = \frac{1}{2}f(\alpha) + \frac{1}{2}f(\beta) + 1,$$

hence  $f$  is not strongly concave.

### 3. Strongly concave functions and the Bruhat-Tits building

The purpose of this section is to prove that optimal strongly concave functions on  $\Phi$  are in 1–1 correspondence with nonempty bounded closed subsets of  $\mathcal{A}$ .

Recall that a subset  $E$  of  $\mathcal{A}$  is *closed* (in the sense of [BT]) if it satisfies the following conditions:

- $E$  is closed in  $\mathcal{A}$  for the usual topology of  $\mathcal{A}$  as a  $\mathbb{R}$ -affine space;

- $E$  is convex;
- $E$  is a union of facets of  $\mathcal{B}$ .

For every nonempty bounded subset  $E$  of  $\mathcal{A}$ , let  $f_E$  be the optimal concave function such that  $G_{f_E}$  is the connected fixator of  $\mathcal{A}$ . If  $E'$  is the closure of  $E$ , that is the smallest closed subset of  $\mathcal{A}$  containing  $E$ , we have  $f_{E'} = f_E$ .

**PROPOSITION 3.1.** *Let  $f$  be an optimal concave function on  $\Phi$ ;  $f$  is strongly concave if and only if the group  $G_f$  is the full connected fixator of the set of points of  $\mathcal{A}$  it fixes.*

**Proof.** Let  $E$  be any nonempty bounded closed subset of  $\mathcal{A}$ ; we'll check that the associated optimal quasi-concave function  $f_E$  is strongly concave. Recall that for every  $\alpha \in \Phi$ , we have:

$$f_E(\alpha) = \inf\{a \in \mathbb{R} \mid (\forall x \in E)(\alpha(x) \leq a)\}.$$

Let  $\alpha_1, \dots, \alpha_r$  be elements of  $\Phi$  and  $\lambda_1, \dots, \lambda_r$  be elements of  $\mathbb{Q}^*$  such that  $\alpha = \sum_{i=1}^r \lambda_i \alpha_i$ ; for every  $x \in E$ , we must then have:

$$\alpha(x) = \sum_{i=1}^r \lambda_i \alpha_i(x) \leq \sum_{i=1}^r \lambda_i f_E(\alpha_i),$$

and since  $f_E(\alpha)$  is the smallest element of  $c(\alpha) + v(\alpha)\mathbb{Z}$  which is greater than or equal to every  $\alpha(x)$ ,  $x \in E$ , we obtain:

$$f_E(\alpha) < \sum_{i=1}^r \lambda_i f_E(\alpha_i) + v(\alpha);$$

hence  $f_E$  is strongly concave.

To prove the converse, we only have to check that if  $f$  is an optimal strongly concave function on  $\Phi$  and  $f'$  is a quasi-concave function on  $\Phi$  which is strictly smaller than  $f$ , if  $E$  (resp.  $E'$ ) is the subset of  $\mathcal{A}$  which is fixed by  $G_f$  (resp.  $G_{f'}$ ),  $E'$  is strictly contained in  $E$ . Let  $\alpha$  be an element of  $\Phi$  such that  $f'(\alpha) < f(\alpha)$ , and let  $E_m$  be the subset of elements  $x$  of  $E$  such that  $\alpha(x)$  is maximal:  $E_m$ , being the intersection of  $E$  with a closed hyperplane of  $\mathcal{A}$ , is itself closed, and thus contains at least one summit of  $\mathcal{A}$ ; let  $x_0$  be such a summit.

For every  $\beta \in \Phi$ , let  $H_{\beta, f}$  be the hyperplane of  $\mathcal{A}$  defined by the equation  $\beta(x) = f(\beta)$ , and let  $\Pi$  be the subset of the elements  $\beta$  of  $\Phi$  such that  $E \cap H_{\beta, f}$  is nonempty; since  $E$  is closed in  $\mathcal{A}$  and bounded,  $\Pi$  satisfies the following conditions:

- every element of  $\Phi$  is a linear combination with rational coefficients of elements of  $\Pi$ ;
- $E$  is precisely the set of elements  $x$  of  $\mathcal{A}$  such that  $\beta(x) \leq f(\beta)$  for every  $\beta \in \Pi$ .

Let's consider the subset  $\Pi_{x_0}$  of elements  $\beta$  of  $\Pi$  such that  $\beta(x_0) = f(\beta)$ ; since  $x_0$  belongs to the boundary of  $E$ ,  $\Pi_{x_0}$  is nonempty. Let  $C$  be the cone of elements  $x$  of  $\mathcal{A}$  such that for every  $\beta \in \Pi_{x_0}$ ,  $\beta(x) \leq f(\beta)$ ;  $C$  is the smallest cone with summit  $x_0$  containing  $E_m$ , and is thus contained in the half-apartment of  $\mathcal{A}$  defined by  $\alpha(x) \leq \alpha(x_0)$ , which implies that there exists a decomposition such as:

$$\alpha = \sum_{\beta \in \Pi_{x_0}} \lambda_\beta \beta,$$

with the  $\lambda_\beta$  being elements of  $\mathbb{Q}_+$ . Since  $f$  is strongly concave and  $f'(\alpha) < f(\alpha)$ , we obtain:

$$\begin{aligned} f'(\alpha) &\leq f(\alpha) - v(\alpha) < \sum_{\beta \in \Pi_{x_0}} \lambda_\beta f(\beta) \\ &= \sum_{\beta \in \Pi_{x_0}} \lambda_\beta \beta(x_0) = \alpha(x_0); \end{aligned}$$

hence  $x_0 \notin E'$ ; therefore  $E' \subsetneq E$ , which shows the desired assertion.  $\square$

**COROLLARY 3.2.** *Let  $f, f'$  be two strongly concave functions. Then  $\text{Sup}(f, f')$  is strongly concave.*

**Proof.** Let  $E$  (resp.  $E'$ ) be the set of points of  $\mathcal{A}$  fixed by  $G_f$  (resp.  $G_{f'}$ );  $G_{\text{Sup}(f, f')}$  is precisely the connected fixator of  $E \cup E'$ , hence of its closure, and the result follows from the proposition.  $\square$

According to the previous proposition, if  $f$  is any quasi-concave function on  $\Phi$  and if  $E$  is the set of points fixed by  $G_f$ , the function  $f_E$  is precisely the largest strongly concave function on  $\Phi$  which is smaller than  $f$ . We'll now give an expression of  $f_E$  relatively to  $f$ .

For every  $\alpha \in \Phi$  and every  $a \in \mathbb{R}$ , we'll define  $\text{ceil}_\alpha(a)$  as the smallest element of  $c(\alpha) + v(\alpha)\mathbb{Z}$  which is greater than or equal to  $a$ .

Now we'll give an expression of  $f_E$  in function of  $f$ . In fact we will prove the following slightly more general result: let  $\Psi$  be a root subsystem of  $\Phi$  (not necessarily closed) of the same rank as  $\Phi$ , and such that the restriction of the function  $\alpha \mapsto v(\alpha)$  to  $\Psi$  coincides with the corresponding function on  $\Psi$ ; we have:

**PROPOSITION 3.3.** *Let  $f$  be any quasi-concave function on  $\Psi$ . Let  $E$  be the set of points of  $\mathcal{A}$  fixed by  $G_f$ , and let  $f'$  be the function on  $\Phi$  defined by:*

$$f'(\alpha) = \inf_{(\alpha_i, \lambda_i)_{1 \leq i \leq r} \mid \sum_{i=1}^r \lambda_i \alpha_i = \alpha} \text{ceil}_\alpha \left( \sum_{i=1}^r \lambda_i f(\alpha_i) \right);$$

*then  $f' = f_E$ .*



**P r o o f.** It is obvious from the definition that  $f'(\alpha) \leq f(\alpha)$  for every  $\alpha \in \Psi$ ; the set of points of  $\mathcal{A}$  fixed by  $G_{f'}$  is then contained in  $E$ . Assume there exists  $x \in E$  which is not fixed by  $G_{f'}$ , and let  $\alpha$  be an element of  $\Psi$  such that  $\alpha(x) > f'(\alpha)$ ; we then have  $f'(\alpha) < f(\alpha)$ . Write:

$$f'(\alpha) = \text{ceil}_\alpha \left( \sum_{i=1}^r \lambda_i f(\alpha_i) \right)$$

for suitable  $(\alpha_i, \lambda_i)$ ; we obtain:

$$\alpha(x) = \sum_{i=1}^r \lambda_i \alpha_i(x) > \sum_{i=1}^r \lambda_i f(\alpha_i),$$

hence  $\alpha_i(x) > f(\alpha_i)$  for at least one  $i$ , which contradicts the fact that  $x \in E$ . Therefore, the set of points fixed by  $G_{f'}$  is precisely  $E$ .

According to the previous proposition, it only remains to check that  $f'$  is strongly concave. Let  $(\alpha_i, \lambda_i)_{1 \leq i \leq r}$  be such that  $\alpha = \sum_{i=1}^r \lambda_i \alpha_i \in \Phi$ , and for every  $i$ , write, for suitable  $(\beta_{ij}, \mu_{ij})_{1 \leq j \leq s_i}$ :

$$f'(\alpha_i) = \text{ceil}_{\alpha_i} \left( \sum_{j=1}^{s_i} \mu_{ij} f(\beta_{ij}) \right).$$

We have:

$$\alpha = \sum_{i,j} \lambda_i \mu_{ij} \beta_{ij},$$

hence:

$$\begin{aligned} f'(\alpha) &\leq \text{ceil}_\alpha \left( \sum_{i,j} \lambda_i \mu_{ij} f(\beta_{ij}) \right) \\ &\leq \text{ceil}_\alpha \left( \sum_{i=1}^r \lambda_i f'(\alpha_i) \right), \end{aligned}$$

which proves the result.  $\square$

#### 4. Some results on root systems

Let  $\Psi$  be a subsystem of  $\Phi$ ; we'll say  $\Psi$  is *parahoric* (resp. *maximal parahoric*) in  $\Phi$  if there exists a basis  $R = (\beta_1, \dots, \beta_n)$  of  $\Phi$  such that  $\Psi$  is the closed subsystem generated by a subset (resp. a subset of cardinal  $n$ ) of  $\{\beta_1, \dots, \beta_n, \beta_m\}$ , where  $\beta_m$  is the smallest root of  $\Phi$  relatively to  $R$ .

The motivation of this terminology is the following one: when  $\underline{G}$  splits over an unramified extension of  $F$ , if  $H$  is a parahoric (resp. maximal parahoric)

subgroup of  $G$  and  $H^1$  its first congruence subgroup, the root system of the group  $H/H^1$  (which is the group of  $K$ -points of a reductive  $K$ -group) is then a parahoric (resp. maximal parahoric) subsystem of  $\Phi$ ; conversely, every parahoric subsystem of  $\Phi$  can be described in such a way. Note that  $\Phi$  is a maximal parahoric subsystem of itself, and that every Levi subsystem of  $\Phi$  is also parahoric.

By [Dyn, Theorem 5.5] (for reduced root systems, but it is quite easy to extend the proof to systems of type  $BC_n$ ), every proper maximal closed subsystem of  $\Phi$  is maximal parahoric. The converse is not true: in some cases ( $\Phi$  of type  $E_7$ ,  $E_8$  and  $F_4$ ), there exist maximal parahoric subsystems of  $\Phi$  which are neither  $\Phi$  itself nor a maximal proper closed subsystem.

Let's show the following two results on maximal parahoric subsystems:

**PROPOSITION 4.1.** *Let  $\Psi$  be a proper closed root subsystem of  $\Phi$ , let  $\alpha$  be an element of  $\Phi$  which doesn't belong to  $\Psi$ , and let  $\Phi'$  be the subsystem of  $\Phi$  generated by  $\alpha$  and  $\Psi$ . Assume  $\Psi$  and  $\Phi'$  are of the same rank; then  $\Psi$  is a maximal parahoric subsystem of  $\Phi'$ .*

**Proof.** We can obviously assume  $\Phi = \Phi'$  and  $\Phi'$  is irreducible. Suppose first  $\Phi'$  is reduced, and let  $(\beta_1, \dots, \beta_r)$  be a basis of  $\Psi$ . If  $(\alpha, \beta_i) \leq 0$  for every  $i$ , according to [Dyn, §5], there exists an  $i$  such that  $\alpha$  and the  $\beta_j$ ,  $j \neq i$ , form a basis of  $\Phi'$ , and that  $\beta_i$  is the smallest root in  $\Phi'$  relatively to this basis; hence  $\Psi$  is parahoric in  $\Phi'$ . If now  $(\alpha, \beta_i) > 0$  for some  $i$ , then  $\alpha - \beta_i$  is a root; we thus may replace  $\alpha$  by  $\alpha - \beta_i$  and iterate; after a finite number of iterations, we reach an  $\alpha$  such that  $(\alpha, \beta_i) \leq 0$  for every  $i$ , and we are then reduced to the previous case.

Suppose now  $\Phi'$  is of type  $BC_n$ . Assume first  $\alpha$  is not a short root. Let  $\Phi_1$  be the closed subsystem of type  $C_n$  contained in  $\Phi'$ , that is the subsystem of the roots  $\gamma \in \Phi'$  such that  $2\gamma \notin \Phi'$ ; consider the intersection  $\Psi_1 = \Psi \cap \Phi_1$ . Since  $\alpha$  belongs to  $\Phi_1$  but not to  $\Psi_1$ , according to the case of type  $C_n$ ,  $\Psi_1$  is a proper maximal parahoric subsystem of  $\Phi_1$ , hence a subsystem of type  $C_i \times C_{n-i}$  for some  $i$ . Since for every  $\alpha \in \Psi$  not contained in  $\Phi_1$ ,  $2\alpha$  belongs to  $\Psi$ , hence to  $\Psi_1$ ,  $\Psi$ , being closed hence contained in a parahoric subsystem of  $\Phi'$ , must be of type either  $C_i \times C_{n-i}$  or  $BC_i \times C_{n-i}$ . The first one doesn't contain any short root, hence is impossible since  $\Phi'$  wouldn't then contain any short root either, and the second one is parahoric in  $\Phi'$ , as required.

Assume now  $\alpha$  is a short root, and  $\Psi_1 \neq \Phi_1$ , hence  $\Psi_1$  is of type  $C_i \times C_{n-i}$  for some  $i$ ; according to the above discussion, we only have to show that  $\Psi \supsetneq \Psi_1$ . Assume  $\Psi = \Psi_1$ ; since  $2\alpha \in \Psi_1$ ,  $\alpha$  is a linear combination with rational coefficients of the elements of one of the two connected components of  $\Psi_1$ , say for example of the component  $C_i$ . But then the root system generated by  $\alpha$  and  $\Psi$  is of type  $BC_i \times C_{n-i}$ , which is strictly contained in  $\Phi$ , hence a contradiction.

Assume now  $\alpha$  is a short root and  $\Psi$  contains  $\Phi_1$ ; since  $\Phi_1$  itself is maximal parahoric in  $\Phi'$ , it implies  $\Psi = \Phi_1$ . Since  $\Phi_1$  and  $\Phi'$  have the same Weyl group,

$\Psi$  has the same too, and since this group acts transitively on the short roots of  $\Phi'$  (by [Bou, I, Proposition 11]), if  $\Psi$  contains one of them, it must contain all of them, and in particular  $\alpha$ , which is impossible. Hence the result.  $\square$

**PROPOSITION 4.2.** *Assume  $\Phi$  is simply-laced. Let  $\Psi, \alpha, \Phi'$  be defined as in the previous proposition. There exists then a basis  $(\beta_1, \dots, \beta_r)$  of  $\Psi$  and  $i \in \{1, \dots, r\}$  such that  $\alpha$  and the  $\beta_j$ ,  $j \neq i$ , form a basis of  $\Phi'$  and that  $\beta_i$  is the smallest root of  $\Phi'$  relatively to that basis.*

**Proof.** Let  $(\beta'_1, \dots, \beta'_r)$  be any basis of  $\Psi$ ; we know by the proof of the previous proposition that there exists  $\alpha' \in \Phi$  such that the assertion of the proposition is true for  $\alpha'$  and the  $\beta'_i$ . Moreover, let  $Q'$  (resp.  $R$ ) be the subgroup of  $X^*(T)$  generated by  $\Phi'$  (resp.  $\Psi$ ); the quotient  $Q'/R$  is a finite cyclic group generated by the image of  $\alpha'$ , and setting  $a$  to be its order, we can identify it with  $\mathbb{Z}/a\mathbb{Z}$  by setting the image of the smallest root  $\beta'_m$  of  $\Phi'$  relatively to  $(\beta_1, \dots, \beta_r)$  to be 1.

Assume first the image of  $\alpha$  in  $Q'/Q$  is also 1. Then we have:

$$\alpha = \alpha' + \sum_{i=1}^k \lambda_i \beta'_i,$$

the  $\lambda_i$  being integers. If all of them are zero there is nothing to prove; assume some of them are nonzero. Then  $\alpha \neq \alpha'$ , hence  $(\alpha, \alpha') < (\alpha, \alpha)$  since  $\Phi'$  is simply-laced; hence:

$$\left( \sum_{i=1}^k \lambda_i \beta'_i, \alpha \right) > 0.$$

Hence there exists  $i_0$  such that  $(\alpha, \lambda_{i_0} \beta'_{i_0}) > 0$ . Setting  $\varepsilon_{i_0} = \frac{\lambda_{i_0}}{|\lambda_{i_0}|}$ , we see that since  $\Phi'$  is simply-laced, we then have  $\alpha + \varepsilon_{i_0} \beta'_{i_0} = s_{\beta'_{i_0}}(\alpha)$ ; by replacing  $(\beta_1, \dots, \beta_r)$  by their image by  $s_{\beta'_{i_0}}$ , we see that the assertion is true for  $\alpha$  if and only if it is true for  $\alpha + \varepsilon_{i_0} \beta'_{i_0}$ , and we conclude by induction on the sum of the  $|\lambda_i|$ .

Assume now the image of  $\alpha$  is not 1. If it is  $-1$ , by replacing  $\alpha'$  and the  $\beta'_i$  by their opposites, we are reduced to the previous case. The only case when it is neither 1 nor  $-1$  is when  $\Phi'$  is of type  $E_8$ ,  $\Psi$  of type  $A_4 \times A_4$  (hence  $a = 5$ ) and the image of  $\alpha$  is 2 or 3; in this case, we easily see that there exists an element  $w$  of the Weyl group of  $\Phi'$  which stabilizes  $\Psi$  and switches its connected components, and that the action of  $w$  on  $Q'/R$  also switches the subsets  $\{1, -1\}$  and  $\{2, 3\}$ ; we are then reduced to the previous cases once again.  $\square$

Note that we can easily find counterexamples to the result of that proposition when  $\Phi$  is not simply-laced.

We'll also use the following lemma:

**LEMMA 4.3.** *Let  $\chi$  be an element of the subgroup  $Q$  of  $X^*(S)$  generated by  $\Phi$ , and let  $\alpha_1, \dots, \alpha_r$  be linearly independent roots generating  $\Phi$  as a root system and such that  $\chi$  is of the form  $\chi = \sum_{i=1}^r \mu_i \alpha_i$ ; then the  $\mu_i$  are integers.*

**Proof.** Let  $(\beta_1, \dots, \beta_r)$  be any basis of  $\Phi$ . For every  $i$ , write:

$$\alpha_i = \sum_{j=1}^r \nu_{ij} \beta_j.$$

We then have:

$$\chi = \sum_{j=1}^r \left( \sum_{i=1}^r \mu_i \nu_{ij} \right) \beta_j.$$

Since  $\chi$  belongs to  $Q$ , for every  $j$ ,  $\sum_{i=1}^r \mu_i \nu_{ij}$  is an integer; moreover, as an immediate consequence of [Bou, Proposition 28], the matrix  $(\nu_{ij})_{i,j}$  is invertible in  $GL_r(\mathbb{Z})$ ; hence the  $\mu_i$  are integers, as required.  $\square$

**COROLLARY 4.4.** *Let  $\chi$  be any element of  $X^*(S)$ , and let  $z$  be an integer such that  $\chi$  belongs to  $\frac{1}{z}Q$ . Let  $\alpha_1, \dots, \alpha_r$  and  $\mu_1, \dots, \mu_r$  be defined as in the above proposition; then the  $\mu_i$  are elements of  $\frac{1}{z}\mathbb{Z}$ .*

## 5. Criteria of strong concavity

The purpose of this section is to give more usable criterions than the bare definition in order to check if some given function on  $\Phi$  is strongly concave, just as [BT, I, Proposition 6.4.5] does for concave functions.

Let's consider some family of pairs  $(\alpha_i, \lambda_i)_{1 \leq i \leq r}$ , where for every  $i$ ,  $\alpha_i$  is a root and  $\lambda_i$  a positive rational number, such that  $\alpha = \sum_i \lambda_i \alpha_i$  belongs to  $\Phi$ ; we'll prove that checking the inequality:

$$f(\alpha) < \sum_{i=1}^r \lambda_i f(\alpha_i) + v(\alpha)$$

can always be reduced to checking a few simpler particular cases.

Unfortunately, there doesn't seem to be any simple and elegant way to treat the general case at once, and we will have to treat separately the cases of the different types of root systems. We'll start by some general results and remarks anyway.

In the rest of this section, we'll assume  $\Phi$  is irreducible. The generalization of the results to the reducible case is quite obvious.

### 5.1. Some remarks about the $\alpha_i$ and $\lambda_i$

In this subsection, we'll consider more closely the inequality  $f(\alpha) < \sum_{i=1}^r \lambda_i f(\alpha_i) + v(\alpha)$ . Let  $(\cdot, \cdot)$  be a scalar product on  $X^*(S)$  invariant by the Weyl group of  $\Phi$ ; we'll prove that in order to check the strong concavity of  $f$ , we may without loss of generality make the following assumptions on the  $\alpha_i$  and the  $\lambda_i$ :

**LEMMA 5.1.**  *$f$  is strongly concave if and only if we have  $f(\alpha) < \sum_{i=1}^r \lambda_i f(\alpha_i) + v(\alpha)$  for every family  $(\alpha_1, \dots, \alpha_r)$  such that  $\sum_{i=1}^r \lambda_i \alpha_i = \alpha$  and:*

- *the  $\alpha_i$  are linearly independent;*
- *for every  $i \neq j$ , we have  $(\alpha_i, \alpha_j) \geq 0$ ; if  $\underline{G}$  splits over some unramified extension of  $F$ , we can even assume  $\alpha_i + \alpha_j$  is not a root;*
- *for every  $i$ ,  $\lambda_i < 1$ .*

**Proof.** Let's show we can assume the  $\alpha_i$  are linearly independent. Assume there exists a linear relation between them, that is:

$$\sum_{i=1}^r \mu_i \alpha_i = 0,$$

with some  $\mu_i$  being nonzero. By eventually replacing all of the  $\mu_i$  by their opposites, we may assume:

$$\sum_{i=1}^r \mu_i f(\alpha_i) \leq 0.$$

By the pseudo-concavity of  $f$ , this implies that at least one of the  $\mu_i$  is negative. Let  $i_0$  be one among the  $i$  satisfying  $\mu_i < 0$  such that  $\frac{\lambda_i}{-\mu_i}$  is minimal; we have:

$$\alpha = \sum_{i=1}^r \lambda_i \alpha_i + \frac{\lambda_{i_0}}{-\mu_{i_0}} \left( \sum_{i=1}^r \mu_i \alpha_i \right).$$

For every  $i$ , let  $\nu_i$  be the coefficient of  $\alpha_i$  in the right-hand side of the above equality; we obviously have  $\nu_i > 0$  if  $\mu_i \geq 0$ , and if  $\mu_i < 0$ :

$$\nu_i = - \left( \frac{\lambda_i}{-\mu_i} - \frac{\lambda_{i_0}}{-\mu_{i_0}} \right) \mu_i \geq 0;$$

moreover,  $\nu_{i_0} = 0$ . On the other hand, we have:

$$\sum_{i=1}^r \nu_i f(\alpha_i) = \sum_{i=1}^r \lambda_i f(\alpha_i) + \frac{\lambda_{i_0}}{-\mu_{i_0}} \sum_{i=1}^r \mu_i f(\alpha_i) \leq \sum_{i=1}^r \lambda_i f(\alpha_i);$$

the family  $((\alpha_i, \lambda_i))_{1 \leq i \leq r}$  can thus be replaced by the family  $((\alpha_i, \nu_i))_{1 \leq i \leq r, \nu_i > 0}$ , whose cardinal is strictly smaller than  $r$ . By an obvious induction, we are reduced to the case where the  $\alpha_i$  are linearly independent. Note that it implies in particular that  $\alpha$  is not a linear combination of any proper subfamily of  $(\alpha_1, \dots, \alpha_r)$ .

Now we'll show we can assume that for every  $i, j$ ,  $(\alpha_i, \alpha_j) \geq 0$ . Assume there exist  $i, j$  such that  $(\alpha_i, \alpha_j) < 0$ ; according to [Bou, 1.3, Theorem 1],  $\alpha_i + \alpha_j$  is then a root. Assume for example  $\lambda_i \geq \lambda_j$ ; we have:

$$\lambda_i \alpha_i + \lambda_j \alpha_j = (\lambda_i - \lambda_j) \alpha_i + \lambda_j (\alpha_i + \alpha_j),$$

and since  $f$  is pseudo-concave, using the Lemma 2.1:

$$\begin{aligned} & (\lambda_i - \lambda_j) f(\alpha_i) + \lambda_j f(\alpha_i + \alpha_j) \\ & \leq \lambda_i f(\alpha_i) + \lambda_j f(\alpha_j) + \lambda_j (v(\alpha_i + \alpha_j) - \inf(v(\alpha_i), v(\alpha_j))). \end{aligned}$$

Assume first  $v(\alpha_i + \alpha_j) > \inf(v(\alpha_i), v(\alpha_j))$ . Then  $\underline{G}$  doesn't split over any unramified extension of  $F$ , and  $\alpha_i + \alpha_j$  is strictly longer than either  $\alpha_i$  or  $\alpha_j$ ; we are then in one of the following cases:

- $\alpha_i, \alpha_j$  are orthogonal short roots of some subsystem of type  $B_2$  of  $\Phi$ ;
- $\alpha_i, \alpha_j$  are short roots of some subsystem of type  $G_2$  of  $\Phi$  whose sum is a long root;
- $\alpha_i = \alpha_j$  and  $2\alpha_i$  belongs to  $\Phi$ .

All these cases contradict the assumption  $(\alpha_i, \alpha_j) < 0$ .

We then have  $v(\alpha_i + \alpha_j) \leq \inf(v(\alpha_i), v(\alpha_j))$ , hence:

$$(\lambda_i - \lambda_j) f(\alpha_i) + \lambda_j f(\alpha_i + \alpha_j) \leq \lambda_i f(\alpha_i) + \lambda_j f(\alpha_j);$$

we thus may replace  $(\alpha_i, \lambda_i)$  and  $(\alpha_j, \lambda_j)$  by respectively  $(\alpha_i, \lambda_i - \lambda_j)$  and  $(\alpha_i + \alpha_j, \lambda_j)$ ; moreover, when we iterate the process, the sum of the  $\lambda_i$  is strictly decreasing while remaining a multiple of the g.c.d. of the initial  $\lambda_i$ ; this way, after a finite number of iterations, we reach a situation where  $(\alpha_i, \alpha_j) \geq 0$  for every  $i, j$ . In particular, we have for every  $i$ :

$$(\alpha, \alpha_i) = \sum_{j=1}^r \lambda_j (\alpha_j, \alpha_i) > 0$$

since  $(\alpha_i, \alpha_i)$  is always positive.

Note that if we suppose  $\underline{G}$  to be split over some unramified extension of  $F$ , we have, for given  $i, j$ ,  $v(\alpha_i + \alpha_j) - \inf(v(\alpha_i), v(\alpha_j)) = 0$ , hence  $(\lambda_i - \lambda_j) f(\alpha_i) + \lambda_j f(\alpha_i + \alpha_j) \leq \lambda_i f(\alpha_i) + \lambda_j f(\alpha_j)$ , as soon as  $\alpha_i + \alpha_j$  is a root. By the same arguments, we thus can even assume  $\alpha_i + \alpha_j$  is not a root for any  $i, j$ .

Now we'll show we can assume all of the  $\lambda_i$  are strictly smaller than 1. Suppose there exists some  $i$  such that  $\lambda_i \geq 1$ , and we are not in the trivial case where

$r = 1$ ,  $\alpha_1 = \alpha$  and  $\lambda_1 = 1$ ; since by the preceding assumptions,  $(\alpha, \alpha_i) > 0$ , according to [Bou, 1.3, Theorem 1],  $\alpha - \alpha_i$  belongs to  $\Phi$ , and by the Lemma 2.1:

$$\begin{aligned} f(\alpha) &\leq f(\alpha - \alpha_i) + f(\alpha_i) + v(\alpha) - \inf(v(\alpha - \alpha_i), v(\alpha_i)) \\ &< \sum_{j=1}^r \lambda_j f(\alpha_j) + v(\alpha) - \inf(v(\alpha - \alpha_i), v(\alpha_i)) + v(\alpha - \alpha_i). \end{aligned}$$

Assume  $v(\alpha - \alpha_i) \leq v(\alpha_i)$ . The assertion for  $\alpha$  and the  $\alpha_i, \lambda_i$  is then a consequence of the one obtained by replacing  $(\alpha_i, \lambda_i)$  by  $(\alpha_i, \lambda_i - 1)$  and  $\alpha$  by  $\alpha - \alpha_i$ ; since we see, using the proof of the Proposition 2.2, that we always can make such an assumption, by an easy induction, we are reduced to the case where all of the  $\lambda_i$  are strictly smaller to 1. Therefore, the lemma is proved.  $\square$

## 5.2. The case-by-case result

Now we'll prove by a case-by-case analysis that we only have a few quite simple conditions (depending on  $\Phi$ ) left to check to prove the strong concavity of a given  $f$ . From now on, when dealing with sums of the form  $\alpha = \sum_{i=1}^r \lambda_i \alpha_i$ , we'll assume the  $\lambda_i$  and the  $\alpha_i$  satisfy the conditions of the Lemma 5.1.

Consider the closed subsystem  $\Phi'$  (resp.  $\Psi$ ) of  $\Phi$  generated by  $\alpha$  and the  $\alpha_i$  (resp. by the  $\alpha_i$ ); according to the Proposition 4.1,  $\Psi$  is a proper maximal parahoric subsystem of  $\Phi'$ . Assume first  $\Phi = \Phi'$ ; we'll consider in turn the various possibilities.

In the sequel, we'll adopt the following conventions and notations:

- $\beta_1, \dots, \beta_r$  is a basis of  $\Phi$ , with the indices located on the Dynkin diagram the same way as in [Bou], and  $\beta_m$  is the smallest root of  $\Phi$  relatively to that basis;
- if  $S$  is some maximal split torus of  $G$ ,  $(\varepsilon_1, \dots, \varepsilon_r)$  is a basis of  $X^*(S) \otimes \mathbb{Q}$ , the notations following [Bou] once again;
- the usual conventions for root systems:  $B_1 = C_1 = A_1$ ,  $D_2 = A_1 \times A_1$ ,  $D_3 = A_3$ , and all of  $A_0$  to  $D_0$  are the trivial root system of rank 0.

Note first that  $\Phi$  cannot be of type  $A_r$  since such a root system has no proper maximal parahoric subsystems.

- Assume first  $\Phi$  is of type  $B_r$ ,  $r \geq 2$ ;  $\Psi$  is then of type  $D_p \times B_q$ , with  $p \geq 2$ ,  $q \geq 0$  and  $p + q = r$ . Suppose first  $\alpha$  is the short root  $\pm \varepsilon_i$  for some  $i \leq p$ ; by eventually replacing  $\alpha$  and the  $\alpha_i$  by their image by some element of the Weyl group of  $\Psi$ , we can assume  $\alpha = \varepsilon_1$ . For every  $i$ ,  $\alpha_i$  must be of the form  $\varepsilon_1 \pm \varepsilon_j$  for some  $j \in \{2, \dots, p\}$ ; set for example  $\alpha_1 = \varepsilon_1 + \varepsilon_2$ . In order to cancel the term in  $\varepsilon_2$  in  $\sum_i \lambda_i \alpha_i$ , some  $\alpha_i$  must then be equal to  $\varepsilon_1 - \varepsilon_2$ , and since  $\alpha$  is a linear combination of these two roots, this implies  $r = 2$ ,  $p = 2$ ,  $q = 0$ , and we obtain the following condition:

**CONDITION 2.** For every  $\alpha_1, \alpha_2 \in \Phi$  linearly independent, orthogonal and such that  $\alpha = \frac{\alpha_1 + \alpha_2}{2}$  belongs to  $\Phi$ , we have:

$$f(\alpha) < \frac{f(\alpha_1) + f(\alpha_2)}{2} + v(\alpha).$$

Suppose now  $\alpha$  is a long root;  $\Psi$  must then contain short roots, hence  $p < r$ , and we can assume  $\alpha = \varepsilon_1 + \varepsilon_r$ . Moreover, since the  $\alpha_i$  generate  $\Psi$ , at least one of them must be short, say  $\alpha_1 = \varepsilon_1$ ; since  $\lambda_1$  is strictly smaller than 1, some  $\alpha_i$  must then be of the form  $\alpha_i = \varepsilon_1 + c_j \varepsilon_j$ , with  $1 < j \leq p$  and  $c_j = \pm 1$ . But then,  $\varepsilon_1 - c_j \varepsilon_j$  must be one of the  $\alpha_i$  too, which contradicts their linear independence; hence this case is impossible.

- Assume now  $\Phi$  is of type  $C_r$ ,  $r \geq 2$ ;  $\Psi$  is then of type  $C_p \times C_q$ , with  $p, q \geq 1$  and  $p + q = r$ , and the root  $\alpha$  is a short root of the form  $\pm \varepsilon_i \pm \varepsilon_j$ ,  $i \leq p$ ,  $j > p$ . Assume for example  $\alpha = \varepsilon_1 + \varepsilon_r$ ; for every  $i$ ,  $\alpha_i$  must be one of the following:  $2\varepsilon_1$ ,  $2\varepsilon_r$ ,  $\varepsilon_1 \pm \varepsilon_j$  for some  $j \in \{2, \dots, p\}$  or  $\varepsilon_r \pm \varepsilon_j$  for some  $j \in \{p+1, \dots, r-1\}$ . Assume first  $\underline{G}$  splits over an unramified extension of  $F$ ; if  $\alpha_i = \varepsilon_1 + \varepsilon_j$ , some  $\alpha_{i'}$  must be equal to  $\varepsilon_1 - \varepsilon_j$ , which is impossible since we then have  $\alpha_i + \alpha_{i'} = 2\varepsilon_1 \in \Phi$ ; the other cases where  $\alpha_i$  is a short root can be treated similarly, and we obtain  $r = 2$ ,  $p = q = 1$  and  $\alpha_i = 2\varepsilon_i$  for  $i = 1, 2$ ; we are then in the case of Condition 2 again.

Assume now  $\underline{G}$  doesn't split over any unramified extension of  $F$ . If all of the  $\alpha_i$  are long, we obtain the same condition as above; assume now at least one of the  $\alpha_i$  is short. According to the above discussion, we may assume for example  $\alpha_1 = \varepsilon_1 + \varepsilon_2$  and  $\alpha_2 = \varepsilon_1 - \varepsilon_2$ . If  $\alpha_i = \varepsilon_1 + \varepsilon_j$  for some  $i > 2$  and some  $j$ , then  $\varepsilon_1 - \varepsilon_j$  must be one of the  $\alpha_i$  as well, which contradicts their linear independence; hence we have either  $\alpha_3 = 2\varepsilon_r$  or  $\alpha_3 = \varepsilon_r \pm \varepsilon_j$  for some  $j > p$ . In the first case, we obtain  $r = 3$ ,  $p = 2$ ,  $q = 1$ , hence the following condition:

**CONDITION 3.** For every  $\alpha_1, \alpha_2, \alpha_3 \in \Phi$  linearly independent, orthogonal and such that  $\alpha = \frac{\alpha_1 + \alpha_2 + \alpha_3}{2}$  belongs to  $\Phi$ , we have:

$$f(\alpha) < \frac{f(\alpha_1) + f(\alpha_2) + f(\alpha_3)}{2} + v(\alpha).$$

In the second case, we must have  $r = 4$ ,  $p = q = 2$  and  $\alpha_3, \alpha_4 = \varepsilon_4 \pm \varepsilon_3$ , hence:

**CONDITION 4.** For every  $\alpha_1, \dots, \alpha_4 \in \Phi$  linearly independent, orthogonal and such that  $\alpha = \frac{\alpha_1 + \dots + \alpha_4}{2}$  belongs to  $\Phi$ , we have:

$$f(\alpha) < \frac{f(\alpha_1) + \dots + f(\alpha_4)}{2} + v(\alpha).$$

- Assume now  $\Phi$  is of type  $D_r$ ,  $r \geq 4$ ;  $\Psi$  is then of type  $D_p \times D_q$ , with  $p, q \geq 2$  and  $p + q = r$ , and the root  $\alpha$  is of the form  $\pm \varepsilon_i \pm \varepsilon_j$ ,  $i \leq p$ ,  $j > p$ . Assume for example  $\alpha = \varepsilon_1 + \varepsilon_r$ ; for every  $i$ ,  $\alpha_i$  must be either  $\varepsilon_1 \pm \varepsilon_j$  for some  $j \in \{2, \dots, p\}$



or  $\varepsilon_r \pm \varepsilon_j$  for some  $j \in \{p+1, \dots, r-1\}$ . By the same argument as above, this is possible only if  $r = 4$ ,  $p = q = 2$  and  $\alpha_1, \alpha_2$  (resp.  $\alpha_3, \alpha_4$ ) are of the form  $\varepsilon_1 \pm \varepsilon_2$  (resp.  $\pm \varepsilon_3 + \varepsilon_4$ ); we are in the case of Condition 4 again.

• Assume now  $\Phi$  is of type  $E_6$ ;  $\Psi$  is of type either  $A_5 \times A_1$  or  $A_2 \times A_2 \times A_2$ . Suppose we are in the first case; using Proposition 4.2, we can assume  $\Psi$  is the subsystem of  $\Phi$  generated by the  $\beta_i$ ,  $i \neq 2$ , and  $\beta_m$ , and that  $\alpha = -\beta_2$ ; we then have  $\alpha_6 = \beta_m$ ,  $\lambda_6 = \frac{1}{2}$ ; moreover,  $\alpha_1, \dots, \alpha_5$  are linearly independent roots contained in the component of type  $A_5$  of  $\Psi$ , and we have the following equality:

$$\sum_{i=1}^5 \lambda_i \alpha_i = \frac{1}{2}(\beta_1 + 2\beta_3 + 3\beta_4 + 2\beta_5 + \beta_6).$$

As a consequence of Corollary 4.4, the  $\lambda_i$  must be multiples of  $\frac{1}{2}$ ; since they belong to  $]0, 1[$ , all of them are equal to  $\frac{1}{2}$ . Since every  $(\alpha_i, \alpha_j)$  is nonnegative, we then have:

$$\left( \sum_{i=1}^5 \lambda_i \alpha_i, \sum_{i=1}^5 \lambda_i \alpha_i \right) \geq \frac{5}{2}.$$

On the other hand, we have:

$$\left( \frac{1}{2}(\beta_1 + 2\beta_3 + 3\beta_4 + 2\beta_5 + \beta_6), \frac{1}{2}(\beta_1 + 2\beta_3 + 3\beta_4 + 2\beta_5 + \beta_6) \right) = \frac{3}{2},$$

hence a contradiction.

Assume now  $\Psi$  is of type  $A_2 \times A_2 \times A_2$ . We can assume  $\Psi$  is the subsystem of  $\Phi$  generated by the  $\beta_i$ ,  $i \neq 4$ , and  $\beta_m$ , and  $\alpha = -\beta_4$ ; it can then be written in the following way as a sum of elements of the subspaces of  $V$  generated by each of the components of  $\Psi$ :

$$\alpha = \frac{1}{3}(\beta_1 + 2\beta_3) + \frac{1}{3}(2\beta_5 + \beta_6) + \frac{1}{3}(2\beta_2 + \beta_m),$$

and the only way of writing it as a linear combination with positive coefficients of some family of six elements of  $\Psi$  satisfying the required conditions is then:

$$\alpha = \frac{1}{3}((\beta_1 + \beta_3) + \beta_3 + \beta_5 + (\beta_5 + \beta_6) + \beta_2 + (\beta_2 + \beta_m)),$$

which leads to the following condition:

**CONDITION 5.** For every  $\alpha_1, \dots, \alpha_6 \in \Phi$  linearly independent and such that  $\alpha = \frac{\alpha_1 + \dots + \alpha_6}{3}$  belongs to  $\Phi$ , we have:

$$f(\alpha) < \frac{f(\alpha_1) + \dots + f(\alpha_6)}{3} + v(\alpha).$$

The types  $E_7$  and  $E_8$  are treated in a similar way, and we obtain for  $E_7$  the condition:

**CONDITION 6.** For every  $\alpha_1, \dots, \alpha_7 \in \Phi$  linearly independent and such that  $\alpha = \frac{\alpha_1 + \dots + \alpha_6 + 2\alpha_7}{4}$  belongs to  $\Phi$ , we have:

$$f(\alpha) < \frac{f(\alpha_1) + \dots + f(\alpha_6) + 2f(\alpha_7)}{4} + v(\alpha).$$

For  $E_8$  we have the following three conditions:

**CONDITION 7.** For every  $\alpha_1, \dots, \alpha_8 \in \Phi$  linearly independent and such that  $\alpha = \frac{\alpha_1 + \dots + \alpha_8}{4}$  belongs to  $\Phi$ , we have:

$$f(\alpha) < \frac{f(\alpha_1) + \dots + f(\alpha_8)}{4} + v(\alpha).$$

**CONDITION 8.** For every  $\alpha_1, \dots, \alpha_8 \in \Phi$  linearly independent and such that  $\alpha = \frac{\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \dots + \alpha_8}{5}$  belongs to  $\Phi$ , we have:

$$f(\alpha) < \frac{f(\alpha_1) + f(\alpha_2) + 2f(\alpha_3) + 2f(\alpha_4) + f(\alpha_5) + \dots + f(\alpha_8)}{5} + v(\alpha).$$

**CONDITION 9.** For every  $\alpha_1, \dots, \alpha_8 \in \Phi$  linearly independent and such that  $\alpha = \frac{\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \dots + \alpha_7 + 3\alpha_8}{6}$  belongs to  $\Phi$ , we have:

$$f(\alpha) < \frac{f(\alpha_1) + 2f(\alpha_2) + 2f(\alpha_3) + f(\alpha_4) + \dots + f(\alpha_7) + 3f(\alpha_8)}{6} + v(\alpha).$$

We leave the details to the reader.

• Assume now  $\Phi$  is of type  $F_4$ ;  $\Psi$  is then of one of the following types:  $B_4$ ,  $C_3 \times A_1$ ,  $A_2 \times A_2$  or  $A_3 \times A_1$ . Assume first  $\alpha$  is a short root, say for example  $\alpha = \varepsilon_1$ ; for every  $i$ ,  $\alpha_i$  is then of the form  $\varepsilon_1 \pm \varepsilon_j$ ,  $2 \leq j \leq 4$ , if it is long, and of the form  $\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$  if it is short. In all of the previously listed cases,  $\Psi$  contains some short roots of  $\Phi$ , hence at least one of the  $\alpha_i$  must be short; say for example  $\alpha_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$ . Assume  $\alpha_2$  is short too, and  $\alpha_i + \alpha_j$  is not a root for any  $i, j$ ; we must have, up to permutation of the indices,  $\alpha_2 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4)$ , and no other  $\alpha_i$  can then be short; this implies  $\alpha_3 = \varepsilon_1 - \varepsilon_2$  and  $\alpha_4 = \varepsilon_1 - \varepsilon_3$ , and we obtain:

$$\alpha = \frac{1}{3}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4),$$

which leads to the following condition:

**CONDITION 10.** For every  $\alpha_1, \dots, \alpha_4 \in \Phi$  linearly independent and such that  $\alpha = \frac{\alpha_1 + \dots + \alpha_4}{3}$  belongs to  $\Phi$ , we have:

$$f(\alpha) < \frac{f(\alpha_1) + \dots + f(\alpha_4)}{3} + v(\alpha).$$

Assume now  $\alpha_1 + \alpha_2$  is a root (in the case where  $\underline{G}$  doesn't split over any unramified extension of  $F$ ). Set for example  $\alpha_2 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$ . Since  $\varepsilon_1 - \varepsilon_2$  cannot belong to the  $\alpha_i$  because of their linear independence, in order to cancel

the  $\varepsilon_2$  in  $\sum_i \lambda_i \alpha_i$ , we must have (up to permutation)  $\alpha_3 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4)$  and  $\alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4)$ ; we are then in the case of Condition 4.

Assume now  $\alpha_2, \alpha_3$  and  $\alpha_4$  are long, which implies (up to permutation)  $\alpha_2 = \varepsilon_1 - \varepsilon_2$ ,  $\alpha_3 = \varepsilon_1 - \varepsilon_3$  and  $\alpha_4 = \varepsilon_1 - \varepsilon_4$ ; we obtain:

$$\alpha = \frac{1}{4}(2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4),$$

hence the following condition:

**CONDITION 11.** For every  $\alpha_1, \dots, \alpha_4 \in \Phi$  linearly independent and such that  $\alpha = \frac{2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{4}$  belongs to  $\Phi$ , we have:

$$f(\alpha) < \frac{2f(\alpha_1) + f(\alpha_2) + f(\alpha_3) + f(\alpha_4)}{4} + v(\alpha);$$

Assume now  $\alpha$  is a long root, say  $\alpha = \varepsilon_1 + \varepsilon_2$ ; for every  $i$ ,  $\alpha_i$  is then one of  $\varepsilon_1, \varepsilon_2$  and  $\frac{1}{2}(\varepsilon_1 + \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$  if it is short, and of the form  $\varepsilon_i \pm \varepsilon_j$ ,  $i = 1, 2, j = 3, 4$ , if it is long. Since  $\Psi$  contains some short roots, one of the  $\alpha_i$  must be short; up to conjugacy we may assume  $\alpha_1 = \varepsilon_1$ . On the other hand, since  $\varepsilon_2$  cannot then be one of the  $\alpha_i$  (because it would imply  $\alpha = \alpha_1 + \alpha_i$ , which is impossible) and since for the same reason at most two of the roots of the form  $\frac{1}{2}(\varepsilon_1 + \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$  belong to the  $\alpha_i$ , one of them, say  $\alpha_2$ , must be long. Assume  $\alpha_3$  and  $\alpha_4$  are both short; we have, up to conjugacy,  $\alpha_2 = \varepsilon_2 + \varepsilon_3$ ,  $\alpha_3 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4)$  and  $\alpha_4 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$ , and:

$$\alpha = \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4);$$

we are then in the case of Condition 4.

Assume now  $\alpha_3$  is short and  $\alpha_4$  is long; say for example  $\alpha_3 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$ . We have, up to conjugacy,  $\alpha_2 = \varepsilon_2 - \varepsilon_3$  and  $\alpha_4 = \varepsilon_j - \varepsilon_4$ ,  $j \in \{1, 2\}$ , from which we deduce that  $\lambda_3 = 2\lambda_2 = 2\lambda_4$ . If  $j = 1$ , considering the coefficients of  $\varepsilon_1$  and  $\varepsilon_2$  in  $\sum_i \lambda_i \alpha_i$ , we obtain:

$$\lambda_1 + \frac{1}{2}\lambda_3 + \lambda_4 = \lambda_2 + \frac{1}{2}\lambda_3,$$

hence  $\lambda_1 = 0$ , which is impossible. We must then have  $j = 2$ , hence:

$$\alpha = \frac{1}{3}(2\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4),$$

and we obtain the following condition:

**CONDITION 12.** For every  $\alpha_1, \dots, \alpha_4 \in \Phi$  linearly independent and such that  $\alpha = \frac{2\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4}{3}$  belongs to  $\Phi$ , we have:

$$f(\alpha) < \frac{2f(\alpha_1) + f(\alpha_2) + 2f(\alpha_3) + f(\alpha_4)}{3} + v(\alpha).$$

Assume finally  $\alpha_3$  and  $\alpha_4$  are both long. We may assume  $\alpha_2 = \varepsilon_2 + \varepsilon_3$ ; but then, neither  $\varepsilon_1 - \varepsilon_3$  nor  $\varepsilon_2 - \varepsilon_3$  can be equal to any  $\alpha_i$ , hence the term in  $\varepsilon_3$  of  $\sum_i \lambda_i \alpha_i$  cannot be zero; we thus obtain a contradiction.

• Assume now  $\Phi$  is of type  $G_2$ ;  $\Psi$  is then of type  $A_2$  or  $A_1 \times A_1$ . In the first (resp. the second) case, if  $\alpha_1$  and  $\alpha_2$  are linearly independent elements of  $\Psi$  such that  $(\alpha_1, \alpha_2) \geq 0$ , we have  $(\alpha_1, \alpha_2) = 1$  (resp  $(\alpha_1, \alpha_2) = 0$ ) and the only element of  $\Phi$  which is a linear combination with coefficients belonging to  $]0, 1[$  of these two roots is  $\frac{1}{3}(\alpha_1 + \alpha_2)$  (resp.  $\frac{1}{2}(\alpha_1 + \alpha_2)$ ); we thus obtain the following condition:

**CONDITION 13.** For every  $\alpha_1, \alpha_2 \in \Phi$  linearly independent and such that  $\alpha = \frac{\alpha_1 + \alpha_2}{3}$  belongs to  $\Phi$ , we have:

$$f(\alpha) < \frac{f(\alpha_1) + f(\alpha_2)}{3} + v(\alpha).$$

(resp. the Condition 2).

• Assume finally  $\Phi$  is of type  $BC_r$ ,  $r \geq 1$ ;  $\Psi$  is then of type  $C_p \times BC_q$ ,  $p \geq 1$ ,  $q \geq 0$  and  $p + q = r$ . Since all divisible roots always belong to  $\Psi$ ,  $\alpha$  is either a short root or a long nondivisible root. Assume first  $\alpha$  is short, say  $\alpha = \varepsilon_1$ ; every  $\alpha_i$  is then either  $2\varepsilon_1$  or  $\varepsilon_1 \pm \varepsilon_j$ ,  $j \leq p$ ; moreover, if  $\varepsilon_1 + \varepsilon_j$  is one of the  $\alpha_i$ , then so is  $\varepsilon_1 - \varepsilon_j$ . In the first case,  $r = 1$  and we obtain the following condition:

**CONDITION 14.** For every  $\alpha_1 \in \Phi$  such that  $\alpha = \frac{\alpha_1}{2}$  belongs to  $\Phi$ , we have:

$$f(\alpha) < \frac{f(\alpha_1)}{2} + v(\alpha).$$

In the second case, we have  $r = 2$ ,  $\underline{G}$  doesn't split over any unramified extension of  $F$  (since  $\alpha_1 + \alpha_2$  is then a root) and we obtain Condition 2.

Assume now  $\alpha$  is a long nondivisible root, say  $\alpha = \varepsilon_1 + \varepsilon_r$ ; we then have  $p < r$ . In this case,  $\Psi$  then contains some short roots, which implies that one of the  $\alpha_i$ , say  $\alpha_1$ , must be short, say for example  $\alpha_1 = \varepsilon_1$ . Since  $\lambda_1 < 1$ , there must be another  $i$  such that  $\alpha_i = \varepsilon_1 \pm \varepsilon_j$  for some  $j$ , and we reach a contradiction the same way as in the case where  $\Phi$  is of type  $B_r$  and  $\alpha$  is long.

Let's return now to the general case; we just have to check which ones of the  $\Phi'$  which actually occur in the above discussion (that is:  $B_2, C_3, C_4, D_4, E_6, E_7, E_8, F_4, G_2, BC_1$  and  $BC_2$ ) are closed subsystems of  $\Phi$ . With the help of [Dyn, Tables 9, 11], we obtain that:

- if  $\Phi$  is of type  $A_r$ ,  $r \geq 1$ , it contains none of them as a closed subsystem;
- if  $\Phi$  is of type  $B_2$  or  $B_3$ , it contains only  $B_2$ ;
- if  $\Phi$  is of type  $B_r$ ,  $r \geq 4$ , it contains only  $B_2$  and  $D_4$ ;
- if  $\Phi$  is of type  $C_3$ , it contains only  $B_2$  and itself;
- if  $\Phi$  is of type  $C_r$ ,  $r \geq 4$ , it contains only  $B_2, C_3$  and  $C_4$ ;

- if  $\Phi$  is of type  $D_r$ ,  $r \geq 4$ , it contains only  $D_4$ ;
- if  $\Phi$  is of type  $E_6$ , it contains only  $D_4$  and itself;
- if  $\Phi$  is of type  $E_7$ , it contains only  $D_4$ ,  $E_6$  and itself;
- if  $\Phi$  is of type  $E_8$ , it contains only  $D_4$ ,  $E_6$ ,  $E_7$  and itself;
- if  $\Phi$  is of type  $F_4$ , it contains only  $B_2$ ,  $C_3$ ,  $D_4$  and itself;
- if  $\Phi$  is of type  $G_2$ , it contains only itself;
- if  $\Phi$  is of type  $BC_1$ , it contains only itself;
- if  $\Phi$  is of type  $BC_2$ , it contains only  $B_2$ ,  $BC_1$  and itself;
- if  $\Phi$  is of type  $BC_3$ , it contains only  $B_2$ ,  $C_3$ ,  $BC_1$  and  $BC_2$ ;
- if  $\Phi$  is of type  $BC_r$ ,  $r \geq 4$ , it contains only  $B_2$ ,  $C_3$ ,  $C_4$ ,  $BC_1$  and  $BC_2$ ;
- if  $\Phi$  is reducible, we just have to check componentwise.

Moreover, the subsystems of type  $C_3$  and  $C_4$  only matter if  $\underline{G}$  doesn't split over any unramified extension of  $F$ ; we finally obtain the following result:

**PROPOSITION 5.2.**  *$f$  is strongly concave if and only if it satisfies the following conditions:*

- if  $\Phi$  is of type  $A_n$ , the Condition 1 (or in other words, every pseudo-concave function is strongly concave);
- if  $\Phi$  is of type  $B_2$  or  $B_3$ , the Conditions 1 and 2;
- if  $\Phi$  is of type  $B_n$ ,  $n \geq 4$ , the Conditions 1, 2 and 4;
- if  $\Phi$  is of type  $C_n$ ,  $n \geq 3$ , and  $\underline{G}$  splits over some unramified extension of  $F$ , the Conditions 1 and 2;
- if  $\Phi$  is of type  $C_3$  and  $\underline{G}$  doesn't split over any unramified extension of  $F$ , the Conditions 1, 2 and 3;
- if  $\Phi$  is of type  $C_n$ ,  $n \geq 4$ , and  $\underline{G}$  doesn't split over any unramified extension of  $F$ , the Conditions 1, 2, 3 and 4;
- if  $\Phi$  is of type  $D_n$ ,  $n \geq 4$ , the Conditions 1 and 4;
- if  $\Phi$  is of type  $E_6$ , the Conditions 1, 4 and 5;
- if  $\Phi$  is of type  $E_7$ , the Conditions 1, 4, 5 and 6;
- if  $\Phi$  is of type  $E_8$ , the Conditions 1, 4, 5, 6, 7, 8 and 9;
- if  $\Phi$  is of type  $F_4$  and  $\underline{G}$  splits over some unramified extension of  $F$ , the Conditions 1, 2, 4, 10, 11 and 12;
- if  $\Phi$  is of type  $F_4$  and  $\underline{G}$  doesn't split over any unramified extension of  $F$ , the Conditions 1, 2, 3, 4, 10, 11 and 12;
- if  $\Phi$  is of type  $G_2$ , the Conditions 1, 2 and 13;
- if  $\Phi$  is of type  $BC_1$ , the Conditions 1 and 14;

- if  $\Phi$  is of type  $BC_2$ , the Conditions 1, 2 and 14;
- if  $\Phi$  is of type  $BC_n$ ,  $n \geq 3$ , and  $\underline{G}$  splits over some unramified extension of  $F$ , the Conditions 1, 2 and 14;
- if  $\Phi$  is of type  $BC_3$  and  $\underline{G}$  doesn't split over any unramified extension of  $F$ , the Conditions 1, 2, 3 and 14;
- if  $\Phi$  is of type  $BC_n$ ,  $n \geq 4$ , and  $\underline{G}$  doesn't split over any unramified extension of  $F$ , the Conditions 1, 2, 3, 4 and 14;
- if  $\Phi$  is reducible, the conditions which are necessary for at least one of its irreducible components.

Note that in order to treat the general case, all of the fourteen conditions seem to be necessary (only thirteen if we assume  $\underline{G}$  to be split over some unramified extension of  $F$ , since the Condition 3 then doesn't show up). If we restrain ourselves to classical groups, however, only Conditions 1, 2, 3, 4 and 14 are necessary, and to treat the case of split classical groups, we only need to check Conditions 1, 2 and 4.

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