

ON DECOMPOSITION OF PSEUDO BL-ALGEBRAS

ANATOLIJ DVUREČENSKIJ* — TOMASZ KOWALSKI**

*Dedicated to Prof. Charles W. Holland on the occasion of his 75th birthday**(Communicated by Sylvia Pulmannová)*

ABSTRACT. We show that under some conditions, imposed on coatoms and maximal idempotents of a pseudo BL-algebra, we can decompose a pseudo BL-algebra M as an ordinal sum and we show that then M is linearly ordered. We investigate pseudo BL-algebras with a unique coatom a and with a maximal idempotent, and analyze two main situations: either $a^n = a^{n+1}$ holds for some $n \geq 1$, or $a^n > a^{n+1}$ hold for any $n \geq 1$. We note that there exist (subdirectly irreducible) algebras with two coatoms that are not linearly ordered, so the restriction to a single coatom is natural.

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1. Introduction

In 1958, Chang [Cha] presented MV-algebras, the algebraic semantics for many-valued logic of Łukasiewicz. In 1986, strengthening a link discovered already by Chang, Mundici [Mun] showed that every MV-algebra is an interval in an Abelian unital ℓ -group (G, u) , where u is a fixed strong unit u , i.e., an element $u \geq e$ such that given $g \in G$, there is an integer $n \geq 1$ such that $u^{-n} \leq g \leq u^n$. Mundici also showed that MV-algebras are generated by the MV-algebra of the real interval $[0, 1]$. Then Hájek [Haj] introduced BL-algebras: an algebraic semantics of basic fuzzy logic, they are generated by continuous

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t-norms on the interval $[0, 1]$ and their residuals. Hoops, which can be thought of as certain subreducts of BL-algebras, were originally introduced by Bosbach, [Bos1, Bos2]. In 1990s, the commutativity assumption of the monoid operation \odot was dropped, and there appeared a non-commutative variant, often named after the commutative version, preceded by an adjective *pseudo*. Thus, pseudo MV-algebras were established independently in [GeIo] and in [Rac] (under the name of generalized MV-algebras). For them an analogue of Mundici's theorem was proved in [Dvu1] showing that a pseudo MV-algebra is always an interval in a unital ℓ -group (G, u) but not necessarily Abelian. Moreover, the categorical equivalence was established, too. This resulted in a new opportunity for studying both algebras and unital ℓ -groups as essentially the same structure (but note that the class of all unital ℓ -groups is not a variety because it is not closed under direct products) and to use an equational language for the class of unital ℓ -groups. This was used for example in [DvHo, DvHo1, DDT]. In [DGI1, DGI2], the authors presented pseudo BL-algebras, and in [GLP], pseudo hoops. It is possible to study these structures (except hoops and pseudo hoops) very elegantly within the theory of residuated lattices (see e.g. [GaTs]), in particular, as subclasses of GBL-algebras.

Recently Jipsen and Montagna contributed greatly to the theory of GBL-algebras, showing that every finite GBL-algebra has to be commutative [JiMo] (this was known only for finite pseudo MV-algebras, [Dvu1]). Then this was generalized for n -potent GBL-algebras, i.e., those satisfying the equality $x^n = x^{n+1}$, [JiMo1, JiMo2]. Moreover, in [JiMo], there was an interesting construction of a pseudo BL-algebra that is not a pseudo MV-algebra, yet comes from an ℓ -group.

In [AgMo], it was shown that every linearly ordered hoop is an ordinal sum of Wajsberg hoops. This was generalized for linearly ordered pseudo hoops, in this case we obtain an ordinal sum of negative cones or intervals in ℓ -groups, see [Dvu4].

In the present paper, we focus on the problem of an ordinal sum decomposition of subdirectly irreducible pseudo BL-algebras having one coatom, a , and having a maximal idempotent element, c . Such an idempotent element allows us to construct an ordinal sum. We show that under these assumptions pseudo BL-algebras must be linearly ordered and we discuss two situations:

- (i) $a^n = a^{n+1}$ for some $n \geq 1$,
- (ii) $a^n > a^{n+1}$ for any $n \geq 1$.

In the first case, the interval $[c, 1]$ is a commutative finite Wajsberg chain, and in the second one, $[c, 1]$ is either a commutative Wajsberg chain isomorphic to the Chang MV-algebra or the BL-algebra $\{\perp\} \cup \mathbb{Z}^-$ that is not an MV-algebra.

We recall that there are subdirectly irreducible algebras with two coatoms that are not linearly ordered.

The paper is organized as follows: Basic notions like pseudo BL-algebras, pseudo hoops, filters, etc., are presented in Section 2. Section 3 deals with basic properties of idempotent elements and their applications, including certain conditions under which a pseudo BL-algebra is commutative. The main decomposition results are in Section 4.

2. Basic notions

In contrast to multiplication and divisions used in the theory of residuated lattices (and denoted respectively by $a \cdot b$, $a \backslash b$ and b / a), we will use here multiplication denoted by \odot and so-called Heyting arrows $a \rightsquigarrow b$ and $a \rightarrow b$. In this notation, a *pseudo BL-algebra*, introduced in [DGI1, DGI2], is an algebra $M = (M; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 2, 0, 0 \rangle$ satisfying the following conditions:

- (i) $(M; \odot, 1)$ is a monoid (need not be commutative),
i.e., \odot is associative with neutral element 1.
- (ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice;
- (iii) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$, $x, y \in M$;
- (iv) $(x \rightarrow y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x)$, $x, y \in M$;
- (v) $(x \rightarrow y) \vee (y \rightarrow x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x)$, $x, y \in M$
(prelinearity condition).

To spare parentheses, we give \odot higher binding priority than \rightarrow or \rightsquigarrow , and those higher than \vee and \wedge . It can be shown that M is distributive as a lattice.

We say that a pseudo BL-algebra M is a *BL-algebra* if $x \odot y = y \odot x$ for all $x, y \in M$. This is equivalent with the statement that $\rightarrow = \rightsquigarrow$.

Let M be a pseudo BL-algebra. Let us define two unary operations (negations) $-$ and \sim on M such that $x^- := x \rightarrow 0$ and $x^\sim := x \rightsquigarrow 0$ for any $x \in M$. It is easy to show that

$$x \odot y = 0 \iff y \leq x^\sim \iff x \leq y^-.$$

Moreover, \odot distributes over \vee and \wedge from both sides.

A pseudo BL-algebra M is a *pseudo MV-algebra* iff $a^{\sim-} = a = a^{-\sim}$ for each $a \in M$. For more details on pseudo MV-algebras see [GeIo, Rac, Dvu1, Dvu2]. A pseudo BL-algebra M is *good* if the negations commute, i.e., $a^{\sim-} = a^{-\sim}$ for

each $a \in M$. For example, every pseudo MV-algebra is good and so is every linearly ordered pseudo BL-algebra, [Dvu3]. It was an open problem, [DGI2, Problem 3.21], whether every pseudo BL-algebra is good. It was answered in the negative in [DGK, Lemma 4.10].

We recall that a *pseudo hoop* is an algebra $(M; \odot, \rightarrow, \rightsquigarrow, 1)$ of type $\langle 2, 2, 2, 0 \rangle$ such that, for all $x, y, z \in M$,

- (i) $x \odot 1 = x = 1 \odot x$ and \odot is associative;
- (ii) $x \rightarrow x = 1 = x \rightsquigarrow x$;
- (iii) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$;
- (iv) $(x \odot y) \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z)$;
- (v) $(x \rightarrow y) \odot x = (y \rightarrow x) \odot y = x \odot (x \rightsquigarrow y) = y \odot (y \rightsquigarrow x)$.

If \odot is commutative (equivalently $\rightarrow = \rightsquigarrow$), M is said to be a *hoop*. If we set $x \leq y$ iff $x \rightarrow y = 1$ (this is equivalent to $x \rightsquigarrow y = 1$), then \leq is a partial order such that $x \wedge y = (x \rightarrow y) \odot x$ and M is a \wedge -semilattice. Every pseudo BL-algebra is a pseudo hoop.

We say that a pseudo hoop M is (i) *bounded* if there is a least element 0 , otherwise, M is *unbounded*, and (ii) *cancellative*, if $x \odot y = x \odot z$ and $s \odot x = t \odot x$ imply $y = z$ and $s = t$.

Let now G be an ℓ -group (written multiplicatively and with a neutral element e). On the negative cone $G^- = \{g \in G : g \leq e\}$ we define: $x \odot y := xy$, $x \rightarrow y := (yx^{-1}) \wedge e$, $x \rightsquigarrow y := (x^{-1}y) \wedge e$, for $x, y \in G^-$. Then $(G^-; \odot, \rightarrow, \rightsquigarrow, e)$ is an unbounded (whenever $G \neq \{e\}$) cancellative pseudo hoop.

If $u \geq e$ is a strong unit (i.e., an order unit) in G , we define on $[u^{-1}, e]$ operations $x \odot y := (xy) \vee (u^{-1})$, $x \rightarrow y := (yx^{-1}) \wedge e$, $x \rightsquigarrow y := (x^{-1}y) \wedge e$, for $x, y \in [u^{-1}, e]$. Then $([u^{-1}, e]; \odot, \rightarrow, \rightsquigarrow, u^{-1}, e)$ is a bounded pseudo hoop (i.e., a pseudo BL-algebra). If on $[e, u]$ we define $x \odot_1 y := (xu^{-1}y) \vee e$ and $x \rightarrow_1 y := (yx^{-1}u) \wedge u$, $x \rightsquigarrow_1 y := (ux^{-1}y) \wedge u$, then $([e, u]; \odot_1, \rightarrow_1, \rightsquigarrow_1, e, u)$ is a bounded pseudo hoop (i.e., a pseudo BL-algebra).

For any $x \in M$ and any integer $n \geq 0$ we define x^n inductively: $x^0 := 1$ and $x^n := x^{n-1} \odot x$ for $n \geq 1$.

A subset F of a pseudo hoop is said to be a *filter* if

- (i) $x, y \in F$ implies $x \odot y \in F$, and
- (ii) $x \leq y$ and $x \in F$ imply $y \in F$.

We denote by $\mathcal{F}(M)$ the set of all filters of M . According to [GLP, Prop 3.1], a subset F is a filter iff

- (i) $1 \in F$, and
- (ii) $x, x \rightarrow y \in F$ implies $y \in F$ ($x, x \rightsquigarrow y \in F$ implies $y \in F$),
i.e., F is a *deductive system*.

If $a \in M$, then the filter, $F(a)$, generated by a is the set

$$F(a) = \{x \in M : x \geq a^n \text{ for some } n \geq 1\}.$$

A filter F is normal if $x \rightarrow y \in F$ iff $x \rightsquigarrow y \in F$. This is equivalent to $a \odot F = F \odot a$ for any $a \in M$; here $a \odot F = \{a \odot h : h \in F\}$ and $F \odot a = \{h \odot a : h \in F\}$. If F is a normal filter, we define $x \theta_F y$ iff $x \rightarrow y \in F$ and $y \rightarrow x \in F$, then θ_F is a congruence on M , [GLP, Prop 3.13]. Moreover, there is a one-to-one correspondence between congruences and normal filters, which gives rise to an isomorphism between the lattice of congruences and the lattice of normal filters. In addition, the variety of pseudo hoops (pseudo BL-algebras) is arithmetical.

A pseudo hoop M is said to be *Wajsberg* if, for all $x, y \in M$,

- (W1) $(x \rightarrow y) \rightsquigarrow y = (y \rightarrow x) \rightsquigarrow x$;
- (W2) $(x \rightsquigarrow y) \rightarrow y = (y \rightsquigarrow x) \rightarrow x$.

Pseudo Wajsberg hoops correspond exactly to integral GMV-algebras in the terminology of [GaTs].

A pseudo hoop M is said to be *basic* if, for all $x, y, z \in M$,

- (B1) $(x \rightarrow y) \rightarrow z \leq ((y \rightarrow x) \rightarrow z) \rightarrow z$;
- (B2) $(x \rightsquigarrow y) \rightsquigarrow z \leq ((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z$.

It is straightforward to verify that any linearly ordered pseudo hoop and hence any representable pseudo hoop is basic. We recall that a pseudo hoop is *representable* if it is a subdirect product of linearly ordered pseudo hoops.

Let us recall the notion of an ordinal sum of pseudo hoops. For pseudo hoops M_0 and M_1 such that $M_0 \cap M_1 = \{1\}$, we set $M = M_0 \cup M_1$. On M we define the operations \odot , \rightarrow and \rightsquigarrow as follows

$$x \odot y = \begin{cases} x \odot_i y & \text{if } x, y \in M_i, i = 0, 1, \\ x & \text{if } x \in M_0 \setminus \{1\}, y \in M_1, \\ y & \text{if } x \in M_1, y \in M_0 \setminus \{1\}, \end{cases}$$

$$x \rightarrow y = \begin{cases} x \rightarrow_i y & \text{if } x, y \in M_i, i = 0, 1, \\ y & \text{if } x \in M_1, y \in M_0 \setminus \{1\}, \\ 1 & \text{if } x \in M_0 \setminus \{1\}, y \in M_1, \end{cases}$$

and

$$x \rightsquigarrow y = \begin{cases} x \rightsquigarrow_i y & \text{if } x, y \in M_i, i = 0, 1, \\ y & \text{if } x \in M_1, y \in M_0 \setminus \{1\}, \\ 1 & \text{if } x \in M_0 \setminus \{1\}, y \in M_1. \end{cases}$$

Then M with $1, \odot, \rightarrow$ and \rightsquigarrow is a pseudo hoop called the *ordinal sum* of M_0 and M_1 and we denote it by $M = M_0 \oplus M_1$. The construction of course can be extended to arbitrary many system of pseudo hoops.

For example, if c is an idempotent element (see the next lines) of a linearly ordered pseudo hoop M , then $M = M_0 \cup M_1$, where $M_1 = [c, 1]$ and $M_0 = (M \setminus M_1) \cup \{1\}$, see e.g. Theorem 4.2.

3. Idempotents of pseudo hoops

An element a of a pseudo hoop M is said to be an *idempotent* if $a \odot a = a$. We denote by $\text{Id}(M)$ the set of all idempotents of M .

PROPOSITION 3.1. *Let a be an idempotent of a pseudo hoop M , then for any $x \in M$ we have*

- (i) $a \odot x = a \wedge x = x \odot a$;
- (ii) $a \rightarrow x = a \rightsquigarrow x$;
- (iii) *the filter $F(a)$ generated by a is normal, and $F(a) = [a, 1] := \{z \in M : a \leq z \leq 1\}$;*
- (iv) *an element $a \in M$ is idempotent if and only if $F(a) = [a, 1]$;*
- (v) *if M is finite, then M has the least element, M is a lattice, each filter F is normal, and $F = [a, 1]$ for some idempotent $a \in M$;*
- (vi) *if $a, b \in \text{Id}(M)$ and $a \vee b$ is defined in M , then $a \vee b \in \text{Id}(M)$;*
- (vii) *if M is a lattice, then $\text{Id}(M)$ is a sublattice of M ;*
- (viii) *if $a, b \in \text{Id}(M)$, then $F(a \wedge b) = F(a) \vee F(b)$, where $F(a) \vee F(b)$ denotes the filter generated by $F(a) \cup F(b)$, and if $a \vee b \in M$, then $F(a \vee b) = F(a) \cap F(b)$;*
- (ix) *if $a, b \in \text{Id}(M)$, M is subdirectly irreducible and if $a \vee b = 1$, then $a = 1$ or $b = 1$*
- (x) *if $x \leq a \leq y$, then $x \odot y = x = y \odot x$;*
- (xi) *if in M there is no incomparable element with a , then for $x < a \leq y$, we have $y \rightarrow x = x = y \rightsquigarrow x$.*

P r o o f.

(i) Calculate $a \odot x \leq a \wedge x = a \odot (a \rightsquigarrow x) = a \odot a \odot (a \rightsquigarrow x) = a \odot (a \wedge x) \leq a \odot x$.

In a similar way we prove $x \odot a = a \wedge x$.

(ii) $z \leq a \rightarrow x$ iff $z \odot a \leq x$ iff $a \odot z \leq x$ iff $z \leq a \rightsquigarrow x$, i.e. (ii) holds.

(iii) $F(a) = \{z \in M : z \geq a^n \text{ for some } n \geq 1\} = [a, 1]$. Hence, $x \rightarrow y \in F(a)$ iff $a \leq x \rightarrow y$ iff $a \odot x \leq y$ iff $x \odot a \leq y$ iff $a \leq x \rightsquigarrow y$ iff $x \rightsquigarrow y \in F(a)$ proving $F(a)$ is normal.

(iv) This follows from the fact that $F(a) = \{x \in M : x \geq a^n \text{ for some } n \geq 1\}$.

(v) M admits the least element $0 = \bigwedge \{a : a \in M\}$. Given $a, b \in M$ there is only finitely many upper bounds u_1, \dots, u_n of a, b . Then $u = u_1 \wedge \dots \wedge u_n = a \vee b$.

If F is a filter, then there is idempotent a such that $F = [a, 1]$. Indeed, let $a = \bigwedge \{x \in F\}$, then $a \leq a \odot a \leq a$. In view of (iii), F is normal.

(vi) \odot is distributive with respect to existing \vee , [GLP, Lem. 2.9], hence $a \vee b \in \text{Id}(M)$.

(vii) It follows from (i) and (vi).

(viii) It follows from (iii).

(ix) By (viii), we have $\{1\} = F(1) = F(a \vee b) = F(a) \cap F(b)$. Then $F(a) = \{1\}$ or $F(b) = \{1\}$ so that $a = 1$ or $b = 1$.

(x) We have $x = x \wedge a = x \odot a \leq x \odot y \leq x$ and similarly $x = y \odot x$.

(xi) We have $y \rightarrow x \geq x$. If $y \rightarrow x \geq a$, then $x = x \wedge y = (y \rightarrow x) \odot y \geq a \odot y = a$, a contradiction. Therefore, $y \rightarrow x < a$, and from $x \leq y \rightarrow x = (y \rightarrow x) \odot a \leq (y \rightarrow x) \odot y = x$, we have $y \rightarrow x = x$. Similarly, $y \rightsquigarrow x = x$. \square

We note that [JiMo] showed that every finite pseudo hoop is commutative. Here we present another proof for finite basic pseudo hoops.

PROPOSITION 3.2. *Every finite basic pseudo hoop is a commutative BL-algebra. In particular, every finite pseudo BL-algebra is commutative.*

P r o o f. Let M be a finite basic pseudo hoop. By Proposition 3.1(v), M has a least element, 0, M is a lattice, and every filter F of M is normal, in particular, every polar $z^\perp := \{a \in M : a \vee z = 1\}$ (it is clear that any polar is a filter) is normal, which in view of [DGK, Prop 3.5] entails that M is representable. Thus, M is a pseudo BL-algebra which is a subdirect product of finite linear pseudo hoops.

To prove the statement, it is now sufficient to assume that M is a finite linear pseudo hoop. But then M is a finite bounded pseudo BL-algebra, and according to [DvHy, Cor 4.4], M is commutative, consequently, every finite representable pseudo hoop is commutative. \square

Using the same techniques we can extend Proposition 3.2 for every *multipotent* pseudo basic hoop M . We recall that

- (i) an element $x \in M$ is *n-potent* if $x^n = x^{n+1}$,
- (ii) M is *n-potent* if every element $x \in M$ is *n-potent*,
- (iii) M is multipotent if for every element $x \in M$ there is an integer $n \geq 1$ such that x is *n-potent*.

It is known that *n-potency* implies commutativity for GBL-algebras (see [JiMo1, Theorem 24]). Here we show the same for multipotent pseudo BL-algebras. Notice that our case is a generalization (multipotency instead of *n-potency*) of a restriction (prelinearity assumed) of the result we mentioned above. Our proof however is quite different from the one in [JiMo1].

THEOREM 3.3. *Every multipotent pseudo basic hoop M is commutative.*

Proof. Let F be a filter of M . Multipotency entails that every element $x \in F$ dominates some idempotent $a \in F$. Let $y \in M$, we show that the left conjugate of x by y , i.e. $\lambda_y(x) := y \rightsquigarrow (x \odot y)$, is in F . In fact, we have $x \odot y \geq a \odot y = a \wedge y$. Then $y \rightsquigarrow (x \odot y) \geq y \rightsquigarrow (a \wedge y) = y \rightsquigarrow a \geq a$ proving $y \rightsquigarrow (x \odot y) \in F$. Similarly, for the right conjugate $\rho_y(x) := y \rightarrow (y \odot x) \in F$. It is easy to show that a filter F is normal iff $\lambda_y(F) \subseteq F$ and $\rho_y(F) \subseteq F$ for any $y \in M$. Therefore, F is normal and, consequently, every polar z^\perp is normal which means, [DGK, Prop. 3.5], that M is representable.

Then M is a subdirect product of linearly ordered pseudo basic hoops, and since all homomorphisms preserve multipotency, we can assume that M is a linearly ordered multipotent basic pseudo hoop. Due to [Dvu4], M can be decomposed as an ordinal sum of pseudo Wajsberg hoops, and each of them is ordinal sum indecomposable. Let M be such an algebra. Since M has an idempotent strictly below 1, it has to be the least element, and so M is a Wajsberg algebra, hence it is term equivalent to a pseudo MV-algebra. Multipotency now yields that M has no co-infinitesimals, i.e. no element $b \in M \setminus \{1\}$ such that $b^n \geq b^-$ for any $n \geq 1$. This means that M is an Archimedean pseudo MV-algebra, so that by [Dvu1], M is commutative. \square

We recall that there are finite hoops which are not basic, consequently are not BL-algebras. Indeed, take the pasting $M = M_0 \oplus M_1$ of two finite nonlinear MV-algebras M_0 and M_1 , this is not a BL-algebra.

We note that every pseudo BL-algebra is a basic pseudo hoop and every basic pseudo hoop preserves the prelinearity condition [GLP, Prop 4.6, Lem 4.5] $(x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1 = (x \rightarrow y) \vee (y \rightarrow x)$ for all $x, y \in M$. Expressed in

another way, this means that basic pseudo hoops are term equivalent to 0-free subreducts of pseudo BL-algebras.

LEMMA 3.4. *Let M be a basic pseudo hoop, $a \in \text{Id}(M)$ and $x, y \in M$. Then*

- (i) *if $x \geq a$ and $x \vee (x \rightsquigarrow a) = 1$, then $x \in \text{Id}(M)$;*
- (ii) *$(x \vee y) \rightsquigarrow (x \wedge y) = (x \rightsquigarrow (x \wedge y)) \wedge (y \rightsquigarrow (x \wedge y))$ and $(x \wedge y) \rightsquigarrow (x \wedge y) = (x \rightsquigarrow (x \wedge y)) \vee (y \rightsquigarrow (x \wedge y))$;*
- (iii) *if M is bounded and $x \wedge y = 0$, then the full DeMorgan laws hold, i.e., $(x \wedge y)^\sim = x^\sim \vee y^\sim$, $(x \vee y)^\sim = x^\sim \wedge y^\sim$.*

All the statements above hold also with \rightsquigarrow replaced by \rightarrow and \sim replaced by $-$.

Proof. For (i) calculate $x = x \odot (x \vee (x \rightsquigarrow a)) = (x \odot x) \vee (x \odot (x \rightsquigarrow a)) = (x \odot x) \vee (x \wedge a) = (x \odot x) \vee a = x \odot x$, where the last equality follows by observing that since $x \geq a$ and a is idempotent, we get $x \odot x \geq a \odot x = a \wedge x = a$. The first statement of (ii) holds in any pseudo hoop, see [GLP, Prop 2.8] which also implies $x \rightsquigarrow (x \wedge y) = x \rightsquigarrow y$ and the same for \rightarrow . For the second calculate

$$\begin{aligned} (x \wedge y) \rightsquigarrow (x \wedge y) &= 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) \\ &= (x \rightsquigarrow (x \wedge y)) \vee (y \rightsquigarrow (y \wedge x)) \end{aligned}$$

where the first line follows by prelinearity. Then (iii) follows from (ii). \square

LEMMA 3.5. *Let M be a basic pseudo hoop and $a, b \in \text{Id}(M)$. Then $a \rightsquigarrow b$ and $b \rightsquigarrow a$ are idempotent as well. Similarly, $a \rightarrow b$ and $b \rightarrow a$ are also idempotent.*

Proof. Observe first that $a \wedge (a \rightsquigarrow (a \wedge b)) = a \odot (a \rightsquigarrow (a \wedge b)) = a \wedge b$. Putting $u = a \rightsquigarrow (a \wedge b)$ we have $a \wedge u = a \wedge b$ and so $a \rightsquigarrow (a \wedge u) = u$. Then, calculate

$$\begin{aligned} 1 &= (a \rightsquigarrow (a \wedge u)) \vee (u \rightsquigarrow (a \wedge u)) \\ &= (a \rightsquigarrow (a \wedge b)) \vee (u \rightsquigarrow (a \wedge u)) \\ &= u \vee (u \rightsquigarrow (a \wedge b)) \end{aligned}$$

where the second equality follows by Lemma 3.4 (ii). Moreover $u \geq a \wedge b$, and $a \wedge b$ is idempotent by Proposition 3.1 (vii). So Lemma 3.4 (i) applies to u , yielding $u \in \text{Id}(M)$ as we claimed.

By symmetry, $a \rightarrow b$ and $b \rightarrow a$ are also idempotent, if a and b are, so the following corollary is immediate. \square

COROLLARY 3.6. *Idempotent elements of a basic pseudo hoop M form a commutative subalgebra of M .*

Let $\text{Id}_n(M)$ be the set of n -potent elements of M . Then $\text{Id}(M) \subseteq \text{Id}_n(M) \subseteq \text{Id}_{n+1}(M)$. If $a, b \in \text{Id}_n(M)$, then $a \wedge b, a \odot b \in \text{Id}_n(M)$ and $(a \wedge b)^n = a^n \wedge b^n = (a \odot b)^n$. Indeed, we have $(a \odot b)^n \leq (a \wedge b)^n \leq a^n \wedge b^n = a^n \odot b^n$. On the other hand, $(a \wedge b)^n \geq (a \odot b)^n \geq (a^n \odot b^n)^n = a^n \odot b^n$. Similarly, $(a \wedge b)^{n+1} = (a \odot b)^{n+1} = a^n \wedge b^n = (a \odot b)^n = (a \wedge b)^n$.

We note, that $\text{Id}_1(M) = \text{Id}(M)$ is in view of Corollary 3.6 a subalgebra of M but this is not always the case for $\text{Id}_n(M)$ when $n > 1$. Indeed, let $M = \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (n, 0))$ be an MV-algebra where $n > 3$ and choose $a = (1, k)$. Then $a^2 = 0 = a^3$ but $b = a \rightarrow 0 = (n-1, -k) \notin \text{Id}_2(M)$ while $b^2 = (n-2, -2k) > b^3 = (n-3, -3k)$.

LEMMA 3.7. *Let M be a basic pseudo hoop. If M is subdirectly irreducible, then $\text{Id}(M)$ is a chain.*

Proof. Let $a, b \in \text{Id}(M)$. We will show that $a \leq b$ or $b \leq a$ holds. By prelinearity, we have $(a \rightsquigarrow b) \vee (b \rightsquigarrow a) = 1$. By Lemma 3.5, $a \rightsquigarrow b$ and $b \rightsquigarrow a$ are idempotent, so $F(a \rightsquigarrow b) = [a \rightsquigarrow b, 1]$ and $F(b \rightsquigarrow a) = [b \rightsquigarrow a, 1]$. It follows from Proposition 3.1(viii) that $F(a \rightsquigarrow b) \cap F(b \rightsquigarrow a) = \{1\}$, so at least one of them must be trivial. Thus, $a \rightsquigarrow b = 1$ or $b \rightsquigarrow a = 1$ and therefore $a \leq b$ or $b \leq a$ holds true. \square

We are now going to apply the observations made above to a subdirectly irreducible basic pseudo hoop. We begin with another lemma.

LEMMA 3.8. *Let M be a basic pseudo hoop such that $\text{Id}(M)$ is linearly ordered. Then for each $a, b \in \text{Id}(M)$ with $b > a$ we have $b \rightsquigarrow a = a = b \rightarrow a$. In addition, if $y \in M$ and $y \geq b$, then $y \rightarrow a = a = y \rightsquigarrow a$.*

Proof. Since $\text{Id}(M)$ is linear, $b \rightsquigarrow a$ is comparable with b . If $b \leq b \rightsquigarrow a$, then $b \odot b \leq a$, which is a contradiction, so $b > b \rightsquigarrow a \geq a$. Then, $b \wedge (b \rightsquigarrow a) = b \odot (b \rightsquigarrow a) = b \wedge a = a$, where the first equality holds because b is idempotent. The proof of $b \rightarrow a = a$ is symmetric.

Suppose now $y \geq b$. Then $a \leq y \rightarrow a \leq b \rightarrow a = a$. Similarly, the same holds for \rightsquigarrow . \square

We recall that a pseudo hoop M is *Gödel* if every element of M is idempotent.

COROLLARY 3.9. *Let M be a basic pseudo hoop. If M is subdirectly irreducible, then $\text{Id}(M)$ is (the universe of) a Gödel subalgebra of M .*

LEMMA 3.10. *Let M be a basic pseudo hoop, $a \in \text{Id}(M)$ and $x \in M$. Then $a \vee (a \rightsquigarrow x) \in \text{Id}(M)$. Moreover, if $\text{Id}(M)$ is linearly ordered, then either $a \rightsquigarrow x = x$ or $a \vee (a \rightsquigarrow x) = 1$.*

P r o o f. First we show that $a \vee (a \rightsquigarrow x) \vee ((a \vee (a \rightsquigarrow x)) \rightsquigarrow a) = 1$. To see that, calculate $(a \vee (a \rightsquigarrow x)) \rightsquigarrow a = (a \rightsquigarrow a) \wedge ((a \rightsquigarrow x) \rightsquigarrow a) = (a \rightsquigarrow x) \rightsquigarrow a$ and $a \rightsquigarrow x = a \rightsquigarrow (a \rightsquigarrow x)$. Then, by prelinearity $1 = (a \rightsquigarrow (a \rightsquigarrow x)) \vee ((a \rightsquigarrow x) \rightsquigarrow a)$ and using the identities above we get $1 = (a \rightsquigarrow x) \vee ((a \vee (a \rightsquigarrow x)) \rightsquigarrow a)$, from which the desired identity follows trivially.

Now, a is idempotent and $a \vee (a \rightsquigarrow x) \geq a$, so Lemma 3.4(i) applies. Therefore $a \vee (a \rightsquigarrow x)$ is idempotent.

Further, let $\text{Id}(M)$ be linearly ordered. Since $a \vee (a \rightsquigarrow x) \geq a$ we have two cases to consider:

Case 1. $a \vee (a \rightsquigarrow x) = a$, and so $1 = (a \vee (a \rightsquigarrow x)) \rightsquigarrow a = (a \rightsquigarrow x) \rightsquigarrow a$. From it follows immediately that $x \leq a \rightsquigarrow x \leq a$ and thus we obtain $a \rightsquigarrow x = a \wedge (a \rightsquigarrow x) = a \odot (a \rightsquigarrow x) = a \wedge x = x$.

Case 2. $a \vee (a \rightsquigarrow x) > a$, and so $a = (a \vee (a \rightsquigarrow x)) \rightsquigarrow a = (a \rightsquigarrow x) \rightsquigarrow a$. Then by prelinearity we obtain $1 = (a \rightsquigarrow (a \rightsquigarrow x)) \vee ((a \rightsquigarrow x) \rightsquigarrow a) = (a \rightsquigarrow x) \vee a$. \square

4. Decomposition of pseudo BL-algebras

In the present section, we give the main results. We present some sufficient conditions to be a pseudo BL-algebra linearly ordered. We show that under some conditions, involving the existence of a unique coatom and a maximal idempotent element, we can decompose a pseudo BL-algebra M as an ordinal sum and we show that then M is linearly ordered. We will analyze two main cases:

- (i) M has a unique coatom a that satisfies $a^n = a^{n+1}$ for some $n \geq 1$,
- (ii) M has a unique coatom a that satisfies $a^n > a^{n+1}$ for any $n \geq 1$.

As for the uniqueness of the coatom, we will present an example of a subdirectly irreducible pseudo BL-algebras with two coatoms that is not linearly ordered.

The following important lemma was proved in [JiMo, Lem 2] for generalized BL-algebras; we present it in the form suitable for pseudo BL-algebras.

LEMMA 4.1 (Principal Lemma). *Let a be a coatom in a pseudo BL-algebra M and let x be any element of M . If there is an integer $i \geq 0$ such that $x \geq a^i$, then $x = a^j$ for some $j = 0, 1, 2, \dots$*

Therefore, if a is a coatom of a pseudo BL-algebra, then

$$F(a) = \{a^i : i = 0, 1, 2, \dots\}. \quad (4.1)$$

If $a \leq b$ are two idempotents of a pseudo hoop (pseudo BL-algebra) M , then $M(a, b) := ([a, b]; \odot_b, \rightarrow_b, \rightsquigarrow_b, a, b)$ is a pseudo hoop, where $x \odot_b y = x \odot y$, $x \rightarrow_b y := (x \rightarrow y) \wedge b$, $x \rightsquigarrow_b y := (x \rightsquigarrow y) \wedge b$ for $x, y \in [a, b]$. If $a = b$, $M(a, b)$ is a trivial pseudo hoop.

THEOREM 4.2. *Suppose that M is a subdirectly irreducible pseudo BL-algebra such that*

- (i) *if a is a coatom, there is an integer $n \geq 1$ such that $a^n = a^{n+1}$,*
- (ii) *every element $x < 1$ is under some coatom, and*
- (iii) *there is a maximal idempotent $c < 1$ in M .*

Then $M = ([0, c] \cup \{1\}) \oplus [c, 1]$, M is linearly ordered and good.

Proof.

Claim 1. M has a unique coatom.

Indeed, let a and b be two different coatoms. Then there is an integer $m \geq 1$ such that a^m and b^m are idempotents. Therefore by Proposition 3.1(iii), $\{1\} = F(1) = F(a \vee b) = F(a) \cap F(b)$ and whence, $F(a) = \{1\}$ or $F(b) = \{1\}$ giving a contradiction. Thus, $a = b$.

Suppose that c is a maximal idempotent in M . Since $c \leq a$, then $c = a^n$ for some integer $n \geq 1$.

Claim 2. M has no element incomparable to c .

Suppose that v_0 is an element in M incomparable to c . Since $v_0 \leq a$, there is an element v_1 such that $v_0 = a \odot v_1$. We cannot have $v_1 = a^k$ for any $k \geq 1$ since then v_0 is comparable to c . We have $v_0 \leq v_1$ thus either $v_0 = v_1$ or $v_0 < v_1$. In the first case, we have $v_0 = a \odot v_0$ so that $v_0 = a^n \odot v_0 \leq a^n = c$ that is a contradiction. In the second case, we have that thanks to Lemma 4.1, v_1 is also an incomparable element to c . We repeat the previous procedure to find an element v_2 such that $v_1 = a \odot v_2$. Then similarly, we have again that $v_1 < v_2$ and v_2 is also incomparable to c . In addition, we have $v_0 = a \odot v_1 = a^2 \odot v_2$. Repeating this procedure yet $n - 2$ times, we find elements $v_0 < v_1 < \dots < v_n$ such that each v_i is incomparable to c and $v_i = a \odot v_{i+1}$ for $i = 0, 1, \dots, n - 1$. Then $v_0 = a \odot v_1 = a^2 \odot v_2 = \dots = a^n \odot v_n \leq a^n = c$ which proves that our assumption was wrong, and consequently, there is no element incomparable with c .

Claim 3. M is linearly ordered.

Indeed, if x and y are incomparable, then $x \rightarrow y < 1$ and $y \rightarrow x < 1$. By Claim 1 and the assumptions, $x \rightarrow y \leq a$ and $y \rightarrow x \leq a$, so that $1 = (x \rightarrow y) \vee (y \rightarrow x) \leq a$ giving a contradiction.

Take now $M_1 = [c, 1] = \{a^n, a^{n-1}, \dots, a, 1\}$, $M_0 = M \setminus M_1 \cup \{1\}$. Then M_0 is closed under \odot and if $x, y \in M_0$, then by Claim 3, M_0 is linear so that assume $x < y$. Then $x \rightarrow y = 1 \in M_0$ and $y \rightarrow x \in M_0$ otherwise $x = x \wedge y = (y \rightarrow x) \odot y = y$ by Proposition 3.1(xi) that is absurd. Similarly, M_0 is closed under \rightsquigarrow . Then M_0 is a subalgebra of M and $(M_1; \odot, \rightarrow, \rightsquigarrow, c, 1)$ is a finite pseudo BL-algebra that is linear, so that M_1 is a commutative Wajsberg algebra. By Proposition 3.1(x)-(xi), we see that the ordinal sum $M_0 \oplus M_1$ coincides with M .

The fact that M is good follows from [Dvu4] because every linearly ordered pseudo BL-algebra is good. \square

THEOREM 4.3. *Suppose that a is a unique coatom of a pseudo BL-algebra M . If*

- (i) *there is an integer $n \geq 1$ such that $a^n = a^{n+1}$,*
- (ii) *every element $x < 1$ is under a ,*
- (iii) *there is a maximal idempotent $c < 1$ in M ,*

then $M = ([0, c] \cup \{1\}) \oplus [c, 1]$, M is linearly ordered, subdirectly irreducible and good.

Proof. The proof follows the same steps as that of Theorem 4.2. The MV-algebra $[c, 1]$ is subdirectly irreducible so is M . \square

In what follows, we generalize Theorems 4.2–4.3. First we introduce the following BL-algebra that is not an MV-algebra.

Let $\mathbb{Z}^- = \{\dots, -2, -1, 0\}$ and let $\mathbb{Z}_\perp := \mathbb{Z}^- \cup \{\perp\}$. We endow \mathbb{Z}_\perp with \odot and \rightarrow via $x \odot y = x + y$, $x \rightarrow y = (y - x) \wedge 0$, $x \odot \perp = \perp = \perp \odot x$ and $x \rightarrow \perp = \perp$, $x \rightarrow \perp = 0$. Then $(\mathbb{Z}_\perp; \odot, \rightarrow, \perp, 0)$ is a BL-algebra that is not an MV-algebra: $x < x^{--} = \perp^- = 0$ for any $\perp \neq x < 0$.

THEOREM 4.4. *Assume M is a pseudo BL-algebra such that*

- (i) *M has a unique coatom, a ,*
- (ii) *for every $x < 1$, $x \leq a$,*
- (iii) *$a^n > a^{n+1}$ for any $n \geq 0$, and*
- (iv) *there is a maximal idempotent $c < 1$ in M .*

Then M is linearly ordered, subdirectly irreducible, and $M = M_0 \oplus M_1$ where $M_0 = [0, c] \cup \{1\}$ and $M_1 = [c, 1]$. In addition, $M_1 \cong \Gamma(\mathbb{Z} \xrightarrow{\rightarrow} \mathbb{Z}, (1, 0))$ whenever $a \rightarrow c > c$ and $M_1 \cong \mathbb{Z}_\perp$ whenever $a \rightarrow c = c$.

Proof. We recall that $F(a) = \{a^i : i \geq 0\}$. The proof of Theorem will follow the following claims.

Claim 1. M is linearly ordered.

In fact, if x and y are incomparable, then $x \rightarrow y \leq a$ and $y \rightarrow x \leq a$ so that $(x \rightarrow y) \vee (y \rightarrow x) \leq a < 1$, a contradiction. In particular, M is good.

Claim 2. Let $i < j$, then $a^i \rightarrow a^j = a^{j-i} = a^i \rightsquigarrow a^j$ and if $i \geq j$ then $a^i \rightarrow a^j = 1 = a^i \rightsquigarrow a^j$.

Indeed, we use [GLP, Lem 2.5(18)–(19)]: $a \rightarrow b \leq (a \odot c) \rightarrow (b \odot c)$ and $a \rightsquigarrow b \leq (c \odot a) \rightsquigarrow (c \odot b)$.

Then $a^i \rightarrow a^j = (1 \odot a^i) \rightarrow (a^{j-i} \odot a^i) \geq 1 \rightarrow a^{j-i} = a^{j-i}$. By Principal Lemma, $a^i \rightarrow a^j = a^k$ for some $k \leq j-i$. If $k < j-i$ then $a^j = (a^i \rightarrow a^j) \odot a^i = a^k \odot a^i = a^{k+i}$ which implies that a^j is an idempotent, a contradiction.

By a similar way, we prove $a^i \rightsquigarrow a^j = a^{j-i}$.

Claim 3. $a^i \rightarrow c, a^i \rightsquigarrow c \notin F(a)$.

If not, by Principal Lemma, $a^i \rightarrow c = a^k$ for some integer $k \geq 0$. Hence $c = c \wedge a^i = (a^i \rightarrow c) \odot a^i = a^{k+i}$ saying that a^{k+i} is idempotent, an absurd. Similarly for the second case.

Claim 4. If $a^i \rightarrow c = a^{i+1} \rightarrow c$, then $a^k \rightarrow c = a^i \rightarrow c$ for any $k \geq i$. Similarly, if $a^i \rightsquigarrow c = a^{i+1} \rightsquigarrow c$, then $a^k \rightsquigarrow c = a^i \rightsquigarrow c$ for any $k \geq i$.

At any rate, $a^{i+2} \rightarrow c = a \rightarrow (a^{i+1} \rightarrow c) = a \rightarrow (a^i \rightarrow c) = a^{i+1} \rightarrow c$, and by induction, we have the claim.

Claim 5. If $a \rightarrow c > c$, then

$$(a \rightarrow c) \rightsquigarrow c = a. \quad (4.2)$$

We have $a \leq (a \rightarrow c) \rightsquigarrow c$ and by Principal Lemma, $(a \rightarrow c) \rightsquigarrow c \in \{a, 1\}$. If $(a \rightarrow c) \rightsquigarrow c = 1$ then $c \leq a \rightarrow c \leq c$ that is a contradiction. Whence (4.2) holds.

Claim 6. If $a \rightarrow c > c$, then $a \rightarrow c$ is a cover of c , and $a \rightarrow c > 0$ if and only if $a \rightsquigarrow c > 0$ and then $a \rightarrow c = a \rightsquigarrow a$.

Indeed, assume $c \leq x \leq a \rightarrow c$ for some $x \in M$. Since $a \rightarrow c \geq x$, then $a = (a \rightarrow c) \rightsquigarrow c \leq x \rightsquigarrow c$. By Principal Lemma, $x \rightsquigarrow c \in \{a, 1\}$. If $x \rightsquigarrow c = 1$ then $c \leq x \leq c$. Assume thus $x \rightsquigarrow c = a$. Check $x = x \wedge (a \rightarrow c) = (a \rightarrow c) \odot ((a \rightarrow c) \rightsquigarrow x)$. If $(a \rightarrow c) \rightsquigarrow x = 1$ then $x \leq a \rightarrow c \leq x$. Otherwise, $(a \rightarrow c) \rightsquigarrow x \leq a$ and $c \leq x = (a \rightarrow c) \odot ((a \rightarrow c) \rightsquigarrow x) \leq (a \rightarrow c) \odot a = c$.

In a similar way, we have that

$$\text{if } a \rightsquigarrow c > c, \quad \text{then } (a \rightsquigarrow c) \rightarrow c = a$$

and also $a \rightsquigarrow c$ is a cover of c . Since M is linearly ordered, $a \rightsquigarrow c = a \rightarrow c$.

On the other hand, since M is linearly ordered, it is good, and therefore, $a = (a \rightarrow c) \rightsquigarrow c = (a \rightsquigarrow c) \rightarrow c$ which means that $a \rightarrow c > c$ iff $a \rightsquigarrow c$.

Claim 7. Let for some integer $i > 0$ $a^i \rightarrow c > a^{i-1} \rightarrow c$. Then

$$(a^i \rightarrow c) \rightsquigarrow c = a^i. \quad (4.3)$$

Indeed, we have $a^i \leq (a^i \rightarrow c) \rightsquigarrow c$. By Principal Lemma, $(a^i \rightarrow c) \rightsquigarrow c = a^j$ for some $0 \leq j \leq i$. If $j < i$, then

$$\begin{aligned} (a^i \rightarrow c) \rightsquigarrow c &= a^j \\ ((a^i \rightarrow c) \rightsquigarrow c) \rightarrow c &= a^j \rightarrow c \\ a^i \rightarrow c &= a^j \rightarrow c \end{aligned}$$

which proves that $a^i \rightarrow c = a^{i-1} \rightarrow c$, a contradiction. Similarly, if $a^i \rightsquigarrow c > a^{i-1} \rightsquigarrow c$, then

$$(a^i \rightsquigarrow c) \rightarrow c = a^i. \quad (4.4)$$

Claim 8. Given $i > 0$, $a^i \rightarrow c > a^{i-1} \rightarrow c$ if and only if $a^i \rightsquigarrow c > a^{i-1} \rightsquigarrow c$.

Indeed, let $a^i \rightarrow c > a^{i-1} \rightarrow c$ and let $a^i \rightsquigarrow c = a^{i_0-1} \rightarrow c$. There has to be the greatest integer $0 < i_0 < i$ such that $a^{i_0} \rightsquigarrow c > a^{i_0-1} \rightsquigarrow c$ otherwise, $a \rightsquigarrow c = c$ and whence $a \rightarrow c = c$ that is impossible. Because M is good, by (4.3) and (4.4), we have $a^i = (a^i \rightarrow c) \rightsquigarrow c = (a^i \rightsquigarrow c) \rightarrow c = (a^{i_0} \rightsquigarrow c) \rightarrow c = a^{i_0}$ that is impossible.

Claim 9. If $c < a \rightarrow c < a^2 \rightarrow c < \dots < a^n \rightarrow c$, then $a^i \rightarrow c$ is a cover of $a^{i-1} \rightarrow c$ for any $i = 1, \dots, n$ and

$$(a^{i+1} \rightarrow c) \odot a = a^i \rightarrow c = a \odot (a^{i+1} \rightarrow c). \quad (4.5)$$

First we have $(a^{i+1} \rightarrow c) \odot a \rightarrow (a^i \rightarrow c) = (a^{i+1} \rightarrow c) \rightarrow (a \rightarrow (a^i \rightarrow c)) = (a^{i+1} \rightarrow c) \rightarrow (a^{i+1} \rightarrow c) = 1$, i.e.

$$(a^{i+1} \rightarrow c) \odot a \leq a^i \rightarrow c. \quad (4.6)$$

In the same way we have $a \odot (a^{i+1} \rightarrow c) \rightsquigarrow (a^i \rightarrow c) = a \odot (a^{i+1} \rightsquigarrow c) \rightsquigarrow (a^i \rightsquigarrow c) = (a^{i+1} \rightsquigarrow c) \rightsquigarrow (a \rightsquigarrow (a^i \rightsquigarrow c)) = (a^{i+1} \rightsquigarrow c) \rightsquigarrow (a^{i+1} \rightsquigarrow c) = 1$ and whence

$$a \odot (a^{i+1} \rightarrow c) \leq a^i \rightarrow c. \quad (4.7)$$

We now follow mathematical induction. Suppose that for every $0 < j \leq i$ (4.5) holds and $a^j \rightarrow c$ is a cover of $a^{j-1} \rightarrow c$. Then $(a^{i+1} \rightarrow c) \odot a \geq (a^i \rightarrow c) \odot a = a^{i-1} \rightarrow c$. The linearity of M and the induction assumption yield

either $(a^{i+1} \rightarrow c) \odot a = a^{i-1} \rightarrow c$ or $(a^{i+1} \rightarrow c) \odot a = a^i \rightarrow c$. In the first case we have

$$\begin{aligned}
 a^i \rightarrow c &= (a^i \rightarrow c) \wedge (a^{i+1} \rightarrow c) \\
 &= ((a^{i+1} \rightarrow c) \rightarrow (a^i \rightarrow c)) \odot (a^{i+1} \rightarrow c) \\
 &= ((a^{i+1} \rightarrow c) \odot a \rightarrow (a^{i-1} \rightarrow c)) \odot (a^{i+1} \rightarrow c) \\
 &= ((a^{i-1} \rightarrow c) \rightarrow (a^{i-1} \rightarrow c)) \odot (a^{i+1} \rightarrow c) \\
 &= a^{i+1} \rightarrow c
 \end{aligned}$$

which is a contradiction.

Assume now $a^i \rightarrow c \leq x \leq a^{i+1} \rightarrow c$. Then $x = x \wedge (a^{i+1} \rightarrow c) = ((a^{i+1} \rightarrow c) \rightarrow x) \odot (a^{i+1} \rightarrow c)$. If $(a^{i+1} \rightarrow c) \rightarrow x = 1$, then $x \leq a^{i+1} \rightarrow c \leq x$. If $a^{i+1} \rightarrow c \rightarrow x \leq a$ then $x \leq a \odot (a^{i+1} \rightarrow c) = a^i \rightarrow c \leq x$, thus $a^{i+1} \rightarrow c$ is a cover of $a^i \rightarrow c$.

Consequently, using Claim 9 and this claim, we have also a byproduct

$$a^i \rightarrow c = a^i \rightsquigarrow c \quad (4.8)$$

for any $i \geq 0$.

Claim 10. If $a \rightarrow c > c$, then $a^i \rightarrow c > a^{i-1} \rightarrow c$ for every $i \geq 1$.

Suppose the converse, that is, let $i > 1$ be the smallest integer such that $a^{i-1} \rightarrow c < a^i \rightarrow c = a^{i+1} \rightarrow c$. Then $a^k \rightarrow c = a^i \rightarrow c$ for any $k \geq i$, Claim 4, and we have

$$\begin{aligned}
 a^{i-1} \rightarrow c &= (a^{i-1} \rightarrow c) \wedge a^i \\
 &= a^i \odot (a^i \rightarrow (a^{i-1} \rightarrow c)) \\
 &= a^i \odot (a^{2i-1} \rightarrow c) \\
 &= a^i \odot (a^i \rightarrow c) = c
 \end{aligned}$$

and this gives a contradiction.

Claim 11. If $a \rightarrow c > c$, then $[c, 1] \cong \Gamma(\mathbb{Z} \xrightarrow{\gamma} \mathbb{Z}, (1, 0))$.

By the previous Claims, $((a^i \rightarrow c) \rightarrow c) \rightsquigarrow c = ((a^i \rightsquigarrow c) \rightarrow c) \rightsquigarrow c = a^i \rightsquigarrow c = a^i \rightarrow c$ and it equals to $((a^i \rightarrow c) \rightsquigarrow c) \rightarrow c$. Moreover, if $i \geq j$, then $(a^i \rightarrow c) \rightarrow (a^j \rightarrow c) = (a^i \rightarrow c) \odot a^j \rightarrow c = (a^{i-j} \rightarrow c) \rightarrow c = (a^{i-j} \rightsquigarrow c) \rightarrow c = a^{i-j} = (a^i \rightarrow c) \rightsquigarrow (a^j \rightarrow c)$. Therefore, for all $x, y \in [c, 1]$ we have $x \rightarrow y = x \rightsquigarrow y$ so that $[c, 1]$ is commutative and from $x^{--} = x$ we conclude that $[c, 1]$ is an MV-algebra isomorphic to the Chang MV-algebra $\Gamma(\mathbb{Z} \xrightarrow{\gamma} \mathbb{Z}, (1, 0))$.

Claim 12. If $a \rightarrow c = c$, then $[c, 1] \cong \mathbb{Z}_\perp$.

Indeed, by Claim 9, Claim 6 and Claim 10, we have $a^i \rightarrow c = c = a^i \rightsquigarrow c$ for any integer $i \geq 1$. Therefore, $[c, 1] \cong \mathbb{Z}_\perp$.

Claim 13. If we set $M_1 = [c, 1]$ and $M_0 = M \setminus M_1 \cup \{1\}$, then $M = M_0 \oplus M_1$ and it is subdirectly irreducible.

Since M_1 is subdirectly irreducible so is M . □

For the following remark, we present the following example of a pseudo BL-algebra which was introduced in [JiMo].

Let G be an ℓ -group written multiplicatively and with neutral element e . We denote by $(G^-)^\partial$ the dual poset of the lattice reduct of G^- (the order in $(G^-)^\partial$ is reverse to the order of G^-). Let $G^\dagger = (G^-)^\partial \cup G^- \times G^-$. Then G^\dagger is a pseudo BL-algebra, [JiMo, Lem 8], under the following operations \odot^\dagger , \rightarrow^\dagger and \rightsquigarrow^\dagger : The elements of $G^- \times G^-$ are denoted by $\langle a, b \rangle$, and we define

$$\begin{aligned}\langle a, b \rangle \odot^\dagger \langle c, d \rangle &= \langle ac, bd \rangle, \\ \langle a, b \rangle \odot^\dagger u &= a \rightarrow u, \\ u \odot^\dagger \langle a, b \rangle &= b \rightsquigarrow u, \\ u \odot^\dagger v &= e\end{aligned}$$

and

$$\begin{aligned}\langle a, b \rangle \rightsquigarrow^\dagger \langle c, d \rangle &= \langle a \rightsquigarrow c, b \rightsquigarrow d \rangle, & \langle c, d \rangle \rightarrow^\dagger \langle a, b \rangle &= \langle c \rightarrow^\dagger a, d \rightarrow^\dagger b \rangle, \\ \langle a, b \rangle \rightsquigarrow^\dagger u &= ua, & u \rightarrow^\dagger \langle a, b \rangle &= \langle e, e \rangle, \\ u \rightsquigarrow^\dagger \langle a, b \rangle &= \langle e, e \rangle, & \langle a, b \rangle \rightarrow^\dagger u &= bu, \\ u \rightsquigarrow^\dagger v &= \langle e, v \rightarrow u \rangle, & v \rightarrow^\dagger u &= \langle u \rightsquigarrow v, e \rangle.\end{aligned}$$

We recall that every G^\dagger gives a pseudo BL-algebra $(G^\dagger; \odot, \rightarrow, \rightsquigarrow, e, \langle e, e \rangle)$ that is not commutative (even if $G = \mathbb{Z}$) and is ordinal sum indecomposable. If G is subdirectly irreducible so is G^\dagger . Since the picture of G^\dagger bears some resemblance to a kite (closeness of resemblance depending on drawing skills of the reader), we will call such algebras *kites*. Kites are a source of many examples. Here is one of them.

Remark 4.5. We note that Theorem 4.4 cannot be weakened for the subdirectly irreducible case. The kite \mathbb{Z}^\dagger has two coatoms, $\langle 0, -1 \rangle$ and $\langle -1, 0 \rangle$, 0 is a unique idempotent under $\langle 0, 0 \rangle$ and \mathbb{Z}^\dagger is subdirectly irreducible. Moreover, $\langle -1, 0 \rangle^{n+1} = \langle -n-1, 0 \rangle < \langle -1, 0 \rangle^n = \langle -n, 0 \rangle$, and $\langle 0, -1 \rangle^{n+1} = \langle 0, -n-1 \rangle < \langle 0, -1 \rangle^n = \langle 0, -n \rangle$ but \mathbb{Z}^\dagger cannot be decomposed as in Theorem 4.4.

Finally, we apply the results of Theorems 4.3–4.4 in order to obtain the following generalization.

THEOREM 4.6. *Let M be a pseudo BL-algebra. Let $c_0 := 1 > c_1 > c_2 > \dots$ be the set of all nonzero idempotents of M , $i \in I$, where I is at most countable. Let a_0 be a unique coatom of M , let a_i be an element of M such that c_i is its cover, $i \in I \setminus \{0\}$, and let every element $x < 1$ is under a_0 . Let $M_i := [c_i, c_{i-1}] \cup \{1\}$ for any $i \in I$, $i > 0$. Then each M_i is either*

(i) *a finite Wajsberg algebra,*

or isomorphic with

(ii) *the Chang MV-algebra or with* (iii) *the BL-algebra \mathbb{Z}_\perp ,*

and M is a linearly ordered pseudo BL-algebra. Moreover, if $M'' := \dots \oplus M_i \oplus M_{i-1} \oplus \dots \oplus M_1$ and $M' = (M \setminus M'') \cup \{1\}$, then $M = M' \oplus M''$.

Proof. The element $a := a_0$ is a coatom of M . Then M is linearly ordered because if there are two elements x, y such that $x \rightarrow y < 1$ and $y \rightarrow x < 1$, then $1 = (x \rightarrow y) \vee (y \rightarrow x) \leq a < 1$. Then M with $a = a_0$ and $c = c_1$ satisfies the conditions of Theorem 4.3 or Theorem 4.4. Hence M_1 is either (i) a finite Wajsberg algebra, or isomorphic with (ii) the Chang MV-algebra or with (iii) the BL-algebra \mathbb{Z}_\perp .

We define the interval pseudo BL-algebra $(M[c_i, c_{i-1}]; \odot_i, \rightarrow_i, \rightsquigarrow_i, c_i, c_{i-1})$, where $M[c_i, c_{i-1}] := [c_i, c_{i-1}]$ with $\odot_i = \odot$, $x \rightarrow_i y = (x \rightarrow y) \wedge c_{i-1}$ and $x \rightsquigarrow_i y = (x \rightsquigarrow y) \wedge c_{i-1}$ for $x, y \in M[c_i, c_{i-1}]$. Since M is linear, we have that each $M[c_i, c_{i-1}]$ is linearly ordered and it together with $a = a_{i-1}$ and $c = c_i$ satisfies the conditions of Theorem 4.3 or Theorem 4.4. If $M[c_i, c_{i-1}]$ satisfies the conditions of Theorem 4.3, then $c_i = a^k$ for some integer $k \geq 1$ and $M[c_i, c_{i-1}] = \{a^k, \dots, a, c_{i-1}\}$ is a finite Wajsberg chain.

Let $M[c_i, c_{i-1}]$ satisfy the conditions of Theorem 4.4. Since $c < a$, we have $a \rightarrow_i c = a \rightarrow c$, so that $a \rightarrow_i c = c$ iff $a \rightarrow c = c$ and $a \rightarrow_i c > c$ iff $a \rightarrow c > c$. Therefore, $M[c_i, c_{i-1}]$ is either an MV-algebra isomorphic with the Chang MV-algebra or a BL-algebra isomorphic with the BL-algebra \mathbb{Z}_\perp .

In each of these case, $M_i = [c_i, c_{i-1}] \cup \{1\}$ is isomorphic with $M[c_i, c_{i-1}]$.

Combining the previous lines, M'' is a hoop. $M' = (M \setminus M'') \cup \{0\}$ is closed under \odot , \rightarrow and \rightsquigarrow (see the proof of Theorem 4.2). Hence, M' is a pseudo BL-algebra, and $M = M' \oplus M''$. \square

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