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A VORONOVSKAYA-TYPE FORMULA FOR SMK OPERATORS VIA STATISTICAL CONVERGENCE

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ABSTRACT. In this paper, we obtain a statistical Voronovskaya-type theorem for the Szász-Mirakjan-Kantorovich (SMK) operators by using the notion of A-statistical convergence, where A is a non-negative regular summability matrix.

1. Introduction

The well-known Korovkin approximation theorem ([1, 18]) for a sequence of positive linear operators, say $\{L_n(f;x)\}$, is mainly based on the existence of the limit $\lim\limits_{n\to\infty} L_n(f;x) = f(x)$. In this case, a natural question arises: what should we do when the above limit fails? This question has been investigated in some recent papers [2, 3, 4, 5, 6, 7, 8, 9, 14, 16] with the help of some convergence methods that are stronger than the usual convergence method, such as *statistical convergence*, *A-statistical convergence*, and *ideal convergence*.

However, the similar question is also valid for a Voronovskaya-type theorem which has an important role in the approximation theory (see [19]). In the present paper, we give an answer to this question for the Szász-Mirakjan-Kantorovich (or, briefly, SMK) operators.

Before proceeding further, we recall some notations and basic definitions on the A-statistical convergence.

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Let $A := (a_{jn})$, be an infinite summability matrix. For a given sequence $x := (x_n)$, the A-transform of x, denoted by $Ax := ((Ax)_j)$, is given by $(Ax)_j = \sum_{n=1}^{\infty} a_{jn}x_n$ provided the series converges for each $j \in \mathbb{N}$, the set of all natural numbers. We say that A is regular if $\lim Ax = L$ whenever $\lim x = L$ ([15]). If $A = (a_{jn})$ is a non-negative regular summability matrix, then we say that a sequence $x = (x_n)$, is A-statistically convergent to L provided that for every $\varepsilon > 0$,

$$\lim_{j\to\infty} \sum_{n\colon |x_n - L| \geq \varepsilon} a_{jn} = 0\,. \tag{1.1}$$

In this case, we write $\mathrm{st}_A\text{-}\lim x = L$ ([11]). Notice that if we replace $A = (a_{jn})$ by the Cesàro matrix C_1, then A-statistical convergence immediately reduces to the concept of statistical convergence (see [10, 12]). On the other hand, if A is the identity matrix, then A-statistical convergence coincides with the ordinary convergence. It is not hard to see that every convergent sequence is A-statistically convergent. This follows from definition (1.1) and the well-known regularity conditions of the matrix A introduced by Silverman and Toeplitz (see [15]). However, Kolk [17] proved that A-statistical convergence is stronger than convergence when $A = (a_{jn})$ is a regular summability matrix such that $\lim_{j\to\infty} \max_{n\in\mathbb{N}}\{a_{jn}\} = 0$.

Let (b_n) be a sequence of positive real numbers such that

$$\lim_{n\to\infty} \frac{b_n}{n} = 0. \tag{1.2}$$

Then, consider the following modified Szász-Mirakjan (SM) operators:

$$S_n(f;x) := \mathrm{e}^{-nx/b_n} \sum_{k=0}^{\infty} f\left(\frac{kb_n}{n}\right) \frac{(nx)^k}{k!b_n^k}, \tag{1.3}$$

where f is a real valued function defined on $\mathbb{R}_0 := [0,\infty)$; $0 \leq x < \infty$. These operators with this structure is similar to the Bernstein-Chlodowsky operators given in [13].

Assume that $A = (a_{jn})$ be a non-negative regular summability matrix. Now consider the following conditions on the sequence (b_n) instead of (1.2):

$$\mathrm{st}_A\text{-}\lim_{n\to\infty} \frac{b_n}{n} = 0. \tag{1.4}$$

In this case, notice that the conditions in (1.4) are weaker than (1.2), because every convergent sequence is A-statistically convergent to the same value, but its converse is not true. For example, if we define a sequence (b_n) such that

$$b_n := \begin{cases} n, & \text{if } n \text{ is a square,} \\ \sqrt{n}, & \text{otherwise} \end{cases}$$

then observe that (1.4) holds true with the choice of $A = C_1$, the Cesàro matrix; but $\left(\frac{b_n}{n}\right)$ is a non-convergent sequence in the usual sense.

In this work, using the concept of A-statistical convergence, we introduce a Kantorovich variant of the Szász-Mirakjan operators, the so-called SMK operators, by replacing $f\left(\frac{kb_n}{n}\right)$ with an integral mean of $f(x)$ over the interval $[(k+1)b_n/n,\, kb_n/n]$ as follows:

$$K_n(f;x) := \frac{n}{b_n}\sum_{k=0}^{\infty} P_{n,k}(x) \int\limits_{kb_n/n}^{(k+1)b_n/n} f(t)\, \mathrm{d}t; \qquad n \in \mathbb{N},\ \ x \in [0,\infty), \tag{1.5}$$

where (b_n) is a sequence of positive real numbers satisfying (1.4), and

$$P_{n,k}(x) := \mathrm{e}^{-nx/b_n}\frac{(nx)^k}{k!b_n^k} \qquad (k = 0,1,2,\dots). \tag{1.6}$$

With this terminology, the main topic of this paper is to give a statistical Voronovskaya type-theorem for the SMK operators given by (1.5) with the help of A-statistical convergence. Thus, it gains us more powerful results than the classical aspects in the approximation theory.

2. Statistical Voronovskaya-type theorem

As usual, let $C(\mathbb{R}_0)$ be the set of all real-valued continuous functions on $\mathbb{R}_0$. Then, we first consider the Banach lattice

$$E := \left\{ f \in C(\mathbb{R}_0) : \lim_{x\to+\infty}\frac{f(x)}{1+x^2} \text{ exists} \right\},$$

endowed with the norm

$$\|f\|_E := \sup_{x\in\mathbb{R}_0}\frac{|f(x)|}{1+x^2}.$$

Then we can state our main theorem that is a statistical Voronovskaya-type result for the operators SMK given by (1.5).

Theorem 2.1. *Let $A = (a_{jn})$ be a non-negative regular summability matrix, and let (b_n) be a sequence of positive numbers satisfying* (1.4)*. Then, for every $f \in E$ with $f', f'' \in E$, we have*

$$\operatorname{st}_A\text{-}\lim_{n\to\infty} \frac{n}{b_n}\big(K_n(f;x) - f(x)\big) = \frac{1}{2}f'(x) + \frac{1}{2}xf''(x)$$

uniformly with respect to $x \in [0,b]$ with $b > 0$.

As a special case in Theorem 2.1, if we replace the matrix A by the identity matrix, then we immediately get the following result.

Corollary 2.2. *If (b_n) is a sequence of positive numbers satisfying* (1.2)*, then, for every $f \in E$ with $f', f'' \in E$, we have*

$$\lim_{n\to\infty} \frac{n}{b_n}\big(K_n\left(f;x\right) - f\left(x\right)\big) = \frac{1}{2}f'\left(x\right) + \frac{1}{2}xf''\left(x\right)$$

uniformly with respect to $x \in [0,b]$ with $b > 0$.

3. Auxiliary results

In this section, we need to obtain some lemmas to prove Theorem 2.1. We first give the following result.

Lemma 3.1. *Let $e_i(t) := t^i$, $(i = 0,1,2,3,4)$, and let $n \in \mathbb{N}$ and $0 \le x < \infty$. Then, the SMK operators given by* (1.5) *verify the following properties:*

(i) $K_n(e_0;x) = 1$,

(ii) $K_n(e_1;x) = x + \frac{b_n}{2n}$,

(iii) $K_n(e_2;x) = x^2 + \frac{2b_n}{n}x + \frac{b_n^2}{3n^2}$,

(iv) $K_n(e_3;x) = x^3 + \frac{9b_n}{2n}x^2 + \frac{7b_n^2}{2n^2}x + \frac{b_n^3}{4n^3}$,

(v) $K_n\left(e_4;x\right) = x^4 + \frac{8b_n}{n}x^3 + \frac{15b_n^2}{n^2}x^2 + \frac{6b_n^3}{n^3}x + \frac{b_n^4}{5n^4}$.

P r o o f.

(i) It follows from (1.5) and (1.6) at once.

(ii) By the definition of (1.5), we get

$$\begin{aligned} K_n(e_1;x) &= \frac{n}{b_n}\sum_{k=0}^{\infty} P_{n,k}(x) \int\limits_{kb_n/n}^{(k+1)b_n/n} t\,\mathrm{d}t \\ &= \frac{b_n}{n}\sum_{k=0}^{\infty} P_{n,k}(x)\left(k+\frac{1}{2}\right), \\ &= x+\frac{b_n}{2n}. \end{aligned}$$

(iii) Similarly, since

$$\begin{aligned} K_n(e_2;x) &= \frac{n}{b_n}\sum_{k=0}^{\infty} P_{n,k}(x) \int\limits_{kb_n/n}^{(k+1)b_n/n} t^2\,\mathrm{d}t \\ &= \frac{b_n^2}{n^2}\sum_{k=0}^{\infty} P_{n,k}(x)\left(k^2+k+\frac{1}{3}\right), \end{aligned}$$

we have

$$K_n(e_2;x) = x^2+\frac{2b_n}{n}x+\frac{b_n^2}{3n^2}.$$

(iv) Using the following equalities

$$\begin{aligned} K_n(e_3;x) &= \frac{n}{b_n}\sum_{k=0}^{\infty} P_{n,k}(x) \int\limits_{kb_n/n}^{(k+1)b_n/n} t^3\,\mathrm{d}t \\ &= \frac{b_n^3}{n^3}\sum_{k=0}^{\infty} P_{n,k}(x)\left(k^3+\frac{3k^2}{2}+k+\frac{1}{4}\right), \end{aligned}$$

we easily get

$$\begin{aligned} K_n(e_3;x) &= x^3+\frac{b_n^2}{n^2}x+\frac{3b_n}{n}x^2 \\ &\quad+\frac{3b_n}{2n}\left(x^2+\frac{b_n}{n}x\right)+\frac{b_n^2}{n^2}x+\frac{b_n^3}{4n^3} \\ &= x^3+\frac{9b_n}{2n}x^2+\frac{7b_n^2}{2n^2}x+\frac{b_n^3}{4n^3}. \end{aligned}$$

(v) Finally, we may write that

$$K_n(e_4;x) = \frac{n}{b_n}\sum_{k=0}^{\infty} P_{n,k}(x) \int\limits_{kb_n/n}^{(k+1)b_n/n} t^4 \,\mathrm{d}t$$

$$= \frac{b_n^4}{n^4}\sum_{k=0}^{\infty} P_{n,k}(x)\left(k^4+2k^3+2k^2+k+\frac{1}{5}\right).$$

Hence, after some simple calculations, we conclude that

$$K_n(e_4;x) = x^4 + \frac{8b_n}{n}x^3 + \frac{15b_n^2}{n^2}x^2 + \frac{6b_n^3}{n^3}x + \frac{b_n^4}{5n^4},$$

which completes the proof. □

Now, fix $b > 0$ and consider the lattice homomorphism $T_b\colon C(\mathbb{R}_0) \to C[0,b]$ defined by $T_b(f) := f|_{[0,b]}$ for every $f \in C(\mathbb{R}_0)$. In this case, we see from Lemma 3.1 that, for each $i = 0,1,2$,

$$\mathrm{st}_A\text{-}\lim_{n\to\infty} T_b\left(K_n(e_i;x)\right) = T_b(e_i;x)$$

uniformly with respect to $x \in [0,b]$ provided that $A = (a_{jn})$ is a non-negative regular summability matrix.

Hence, by using this and [14, Theorem 1] and by considering the statistical version of the universal Korovkin-type property (see [1, Theorem 4.1.4 (vi)]), we arrive the following statistical Korovkin-type approximation theorem for the SMK operators.

Corollary 3.2. *Let $A = (a_{jn})$ be a non-negative regular summability matrix, and let (b_n) be a sequence of positive real numbers satisfying* (1.4). *Then, for every $f \in C(\mathbb{R}_0)$, we have*

$$\mathrm{st}_A\text{-}\lim_{n\to\infty} K_n(f;x) = f(x)$$

uniformly with respect to $x \in [0,b]$ with $b > 0$.

Remark. In Corollary 3.2, if we replace the matrix A with the identity matrix and if (b_n) is a sequence of positive real numbers satisfying (1.2), then we obtain that, for any $f \in C(\mathbb{R}_0)$,

$$\lim_{n\to\infty} K_n(f;x) = f(x)$$

uniformly with respect to $x \in [0,b]$ with $b > 0$.

The next result can immediately obtained from Lemma 3.1.

Lemma 3.3. *For $0 \leq x < \infty$, let $\varphi_x(t) = t - x$. Then, for the SMK operators, we have*

(i) $K_n(\varphi_x; x) = \frac{b_n}{2n}$,

(ii) $K_n(\varphi_x^2; x) = \frac{b_n}{n} x + \frac{b_n^2}{3n^2}$,

(iii) $K_n(\varphi_x^3; x) = \frac{5b_n^2}{2n^2} x + \frac{b_n^3}{4n^3}$,

(iv) $K_n(\varphi_x^4; x) = \frac{3b_n^2}{n^2} x^2 + \frac{5b_n^3}{n^3} x + \frac{b_n^4}{5n^4}$.

Lemma 3.4. *Let $A = (a_{jn})$ be a non-negative regular summability matrix, and let (b_n) be a sequence of positive real numbers satisfying* (1.4)*. Then, we have*

$$\operatorname{st}_A\text{-}\lim_{n\to\infty} \frac{n^2}{b_n^2} K_n(\varphi_x^4; x) = 3x^2$$

uniformly with respect to $x \in [0, b]$ with $b > 0$; where the function φ_x is given as in Lemma 3.3.

P r o o f. Let $x \in [0, b]$. By Lemma 3.3 (iv), we may write that

$$\frac{n^2}{b_n^2} K_n(\varphi_x^4; x) = 3x^2 + \frac{5b_n}{n} x + \frac{1}{5}\left(\frac{b_n}{n}\right)^2,$$

which implies

$$\left|\frac{n^2}{b_n^2} K_n(\varphi_x^4; x) - 3x^2\right| \leq B\left\{\frac{b_n}{n} + \left(\frac{b_n}{n}\right)^2\right\}, \tag{3.1}$$

where $B := \max\{5b, 1/5\}$. Now, for a given $\varepsilon > 0$, define the following sets:

$$D := \left\{n : \left|\frac{n^2}{b_n^2} K_n(\varphi_x^4; x) - 3x^2\right| \geq \varepsilon\right\},$$

$$D_1 := \left\{n : \frac{b_n}{n} \geq \frac{\varepsilon}{2B}\right\},$$

$$D_2 := \left\{n : \frac{b_n}{n} \geq \sqrt{\frac{\varepsilon}{2B}}\right\}.$$

Hence, by (3.1), we easily see that $D \subseteq D_1 \cup D_2$. Then, for any $j \in \mathbb{N}$, we have

$$\sum_{n\in D} a_{jn} \leq \sum_{n\in D_1} a_{jn} + \sum_{n\in D_2} a_{jn}. \tag{3.2}$$

Taking limit as $j \to \infty$ on the both sides of (3.2) and using the fact that $\mathrm{st}_A\text{-}\lim(b_n/n) = 0$, we conclude that

$$\lim_{j\to\infty} \sum_{n\in D} a_{jn} = 0,$$

whence the result. □

4. The proof of Theorem 2.1

Now we are ready to prove Theorem 2.1.

Let $f, f', f'' \in E$ and $x \in [0, b]$. Define the function Φ_x by

$$\Phi_x(t) = \begin{cases} \frac{f(t)-f(x)-(t-x)f'(x)-\frac{1}{2}(t-x)^2 f''(x)}{(t-x)^2}, & \text{if } t \neq x \\ 0, & \text{if } t = x. \end{cases}$$

Then, it is clear that $\Phi_x(x) = 0$. Also observe that the function $\Phi_x(\cdot)$ belongs to E. Hence, by Taylor's theorem we get

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2} f''(x) + (t-x)^2 \Phi_x(t).$$

Now the definition of the SMK operators implies that

$$\begin{aligned} K_n(f;x) - f(x) = f'(x) K_n(\varphi_x;x) + \frac{1}{2} f''(x) K_n\left(\varphi_x^2;x\right) \\ + K_n\left(\varphi_x \Phi_x;x\right), \end{aligned}$$

where $\varphi_x(t) = t - x$. By Lemma 3.3 (i)–(ii) we have

$$\begin{aligned} K_n(f;x) - f(x) = \frac{b_n}{2n} f'(x) + \left(\frac{b_n}{2n}x + \frac{b_n^2}{6n^2}\right) f''(x) \\ + K_n\left(\varphi_x^2 \Phi_x;x\right), \end{aligned}$$

and hence

$$\begin{aligned} \frac{n}{b_n}\{K_n(f;x) - f(x)\} = \left(\frac{1}{2}f'(x) + \frac{1}{2}x f''(x)\right) + \frac{b_n}{6n} f''(x) \\ + \frac{n}{b_n} K_n\left(\varphi_x^2 \Phi_x;x\right). \end{aligned} \tag{4.1}$$

If we apply the Cauchy-Schwarz inequality for the second term on the right-hand side of (4.1), then we see that

$$\left|K_n\left(\varphi_x^2 \Phi_x;x\right)\right| \le \sqrt{K_n(\varphi_x^4;x)}\sqrt{K_n(\Phi_x^2;x)},$$

which yields

$$\frac{n}{b_n}\left|K_n\left(\varphi_x^2\Phi_x;x\right)\right| \leq \sqrt{\frac{n^2}{b_n^2}K_n(\varphi_x^4;x)}\sqrt{K_n(\Phi_x^2;x)}. \tag{4.2}$$

Let $\eta_x(t) := \Phi_x^2(t)$. In this case, observe that $\eta_x(x) = 0$ and $\eta_x(\cdot) \in E$. Then it follows from Corollary 3.2 that

$$\mathrm{st}_A\text{-}\lim_{n\to\infty} K_n\left(\Phi_x^2;x\right) = \mathrm{st}_A\text{-}\lim_{n\to\infty} K_n\left(\eta_x;x\right) = \eta_x(x) = 0 \tag{4.3}$$

uniformly with respect to $x \in [0, b]$. Now considering (4.2) and (4.3), and also using Lemma 3.4, we immediately see that

$$\mathrm{st}_A\text{-}\lim_{n\to\infty} \frac{n}{b_n}K_n\left(\varphi_x^2\Phi_x;x\right) = 0 \tag{4.4}$$

uniformly with respect to $x \in [0, b]$. Using (4.4) in (4.1) and also considering (1.4), we have

$$\mathrm{st}_A\text{-}\lim_{n\to\infty} \frac{n}{b_n}\left\{K_n(f;x) - f(x)\right\} = \frac{1}{2}f'(x) + \frac{1}{2}xf''(x)$$

uniformly with respect to $x \in [0, b]$. So the proof is completed.

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