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UNIQUENESS AND SET SHARING OF DERIVATIVES OF MEROMORPHIC FUNCTIONS

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ABSTRACT. We prove some uniqueness theorems concerning the derivatives of meromorphic functions when they share two or three sets which will improve some existing results.

1. Introduction, definitions and results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function $h(z)$ we denote by $S(r, h)$ any quantity satisfying

$$S(r,h) = o(T(r,h)) \qquad (r \to \infty, \quad r \notin E).$$

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f - a$ and $g - a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f - a$ and $g - a$ have the same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM. We denote by $T(r)$ the maximum of $T\left(r, f^{(k)}\right)$ and $T\left(r, g^{(k)}\right)$. The notation $S(r)$ denotes any quantity satisfying

$$S(r) = o(T(r)) \qquad (r \to \infty, \quad r \notin E).$$

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Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a \in S} \big\{z : f(z) - a = 0\big\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $E_f(S) = \bigcup_{a \in S} \big\{z : f(z) - a = 0\big\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM.

It was F. Gross who first considered the uniqueness of meromorphic functions that share sets of distinct elements instead of values. In 1976 he suggested the following open question in [10]:

Question A. *Can one find two finite sets S_j $(j = 1, 2)$ such that any two non-constant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical?*

Now it is natural to ask the following question.

Question B. *Can one find two finite sets S_j $(j = 1, 2)$ such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical?*

Also for meromorphic functions in [24] the following question was asked.

Question C. *Can one find three finite sets S_j $(j = 1, 2, 3)$ such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2, 3$ must be identical?*

Nowadays a widely studied topic of the uniqueness theory has been considering the shared value problems relative to a meromorphic function sharing two or three sets and at the same time give affirmative answers to Question B and Question C under weaker hypothesis (see [3]–[4], [7], [9], [12], [16], [19], [21], [24], [27], [30]–[31] and [1]–[2], [5]–[6], [8], [17], [20], [23]–[25], [29]). So the natural query would be whether there exists similar types of unique range sets corresponding to the derivatives of two meromorphic functions. The purpose of this paper is to deal with the problem. The following two results studied the uniqueness of the derivatives of meromorphic functions in the direction of Question B.

Theorem A. ([9]) *Let $S_1 = \big\{z : z^n + az^{n-1} + b = 0\big\}$ and $S_2 = \{\infty\}$, where a, b are nonzero constants such that $z^n + az^{n-1} + b = 0$ has no repeated root and n (≥ 7), k be two positive integers. Let f and g be two non-constant meromorphic functions such that $E_{f^{(k)}}(S_1) = E_{g^{(k)}}(S_1)$ and $E_f(S_2) = E_g(S_2)$ then $f^{(k)} \equiv g^{(k)}$.*

Theorem B. ([30]) *Let S_i, $i = 1, 2$, be given as in* Theorem A *and k be a positive integer. Let f and g be two non-constant meromorphic functions such that $E_{f^{(k)}}(S_j) = E_{g^{(k)}}(S_j)$ for $j = 1, 2$, then $f^{(k)} \equiv g^{(k)}$.*

In 2003, in the direction of Question C concerning the uniqueness of derivatives of meromorphic functions Qiu and Fang obtained the following result.

Theorem C. ([23]) *Let* $S_1 = \{z : \ z^n - z^{n-1} - 1 = 0\}$, $S_2 = \{\infty\}$ *and* $S_3 = \{0\}$ *and* n (≥ 3), k *be two positive integers. Let* f *and* g *be two non-constant meromorphic functions such that* $E_{f^{(k)}}(S_j) = E_{g^{(k)}}(S_j)$ *for* $j = 1, 3$ *and* $E_f(S_2) = E_g(S_2)$, *then* $f^{(k)} \equiv g^{(k)}$.

In 2004 Yi and Lin [29] independently proved the following theorem.

Theorem D. ([29]) *Let* $S_1 = \{z : \ z^n + az^{n-1} + b = 0\}$, $S_2 = \{\infty\}$ *and* $S_3 = \{0\}$, *where* a, b *are nonzero constants such that* $z^n + az^{n-1} + b = 0$ *has no repeated root and* n (≥ 3), k *be two positive integers. Let* f *and* g *be two non-constant meromorphic functions such that* $E_{f^{(k)}}(S_j) = E_{g^{(k)}}(S_j)$ *for* $j = 1, 2, 3$, *then* $f^{(k)} \equiv g^{(k)}$.

The following examples show that in Theorems A–D, $a \neq 0$ is necessary.

Example 1.1. Let $f(z) = \mathrm{e}^z$ and $g(z) = (-1)^k \mathrm{e}^{-z}$ and $S_1 = \{z : \ z^7 - 1 = 0\}$, $S_2 = \{\infty\}$. Since $f^{(k)} - \omega^l = g^{(k)} - \omega^{7-l}$, where $\omega = \cos\frac{2\pi}{7} + \mathrm{i}\sin\frac{2\pi}{7}$, $0 \leq l \leq 6$, clearly $E_{f^{(k)}}(S_j) = E_{g^{(k)}}(S_j)$ for $j = 1, 2$, but $f^{(k)} \not\equiv g^{(k)}$.

Example 1.2. Let $f(z) = \mathrm{e}^z$ and $g(z) = (-1)^k \mathrm{e}^{-z}$ and $S_1 = \{z : \ z^3 - 1 = 0\}$, $S_2 = \{\infty\}$, $S_3 = \{0\}$. Since $f^{(k)} - \omega^l = g^{(k)} - \omega^{3-l}$, where $\omega = \cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}$, $0 \leq l \leq 2$, clearly $E_{f^{(k)}}(S_j) = E_{g^{(k)}}(S_j)$ for $j = 1, 2, 3$, but $f^{(k)} \not\equiv g^{(k)}$.

We now consider the following example which establishes the sharpness of the lower bound of n in Theorems C–D.

Example 1.3. Let $f(z) = \sqrt{\alpha+\beta}\sqrt{\alpha\beta}\mathrm{e}^z$ and $g(z) = (-1)^k\sqrt{\alpha+\beta}\sqrt{\alpha\beta}\mathrm{e}^{-z}$ and $S_1 = \{\alpha+\beta, \alpha\beta\}$, $S_2 = \{\infty\}$, $S_3 = \{0\}$, where $\alpha + \beta = -a$ and $\alpha\beta = b$; a, b are nonzero complex numbers. Clearly $E_{f^{(k)}}(S_j) = E_{g^{(k)}}(S_j)$ for $j = 1, 2, 3$, but $f^{(k)} \not\equiv g^{(k)}$.

Above example obviously motivate oneself to concentrate the attention of further relaxation of the nature of sharing of the range sets than to reduce the lower bound of n in Theorems C–D.

Regarding Theorems A–B following example establishes the fact that the set S_1 can not be replaced by any arbitrary set containing six distinct elements. However it still remains open for investigations whether the degree of the equation defining S_1 in Theorem A can be reduced to six or less.

Example 1.4. Let $f(z) = \frac{1}{(\sqrt{\alpha\beta\gamma})^{k-1}}\mathrm{e}^{\sqrt{\alpha\beta\gamma}z}$ and $g(z) = \frac{(-1)^k}{(\sqrt{\alpha\beta\gamma})^{k-1}}\mathrm{e}^{-\sqrt{\alpha\beta\gamma}z}$ and $S_1 = \{\alpha\sqrt{\beta}, \alpha\sqrt{\gamma}, \beta\sqrt{\alpha}, \beta\sqrt{\gamma}, \gamma\sqrt{\alpha}, \gamma\sqrt{\beta}, \}$, $S_2 = \{\infty\}$, where α, β and γ are three nonzero distinct complex numbers. Clearly $E_{f^{(k)}}(S_j) = E_{g^{(k)}}(S_j)$ for $j = 1, 2$, but $f^{(k)} \not\equiv g^{(k)}$.

The notion of weighted sharing of values and sets as introduced in [14, 15] renders a useful tool for the purpose of relaxation of the nature of sharing the sets. We now give the definition.

Definition 1.1. ([14, 15]) Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g with multiplicity m $(\leq k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. ([14]) Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S, k)$ the set $E_f(S, k) = \bigcup_{a \in S} E_k(a; f)$.

Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

Following four theorems are the main results of the paper. All of them improve all the theorems previously mentioned.

Theorem 1.1. *Let $S_1 = \{z : \ z^n + az^{n-1} + b = 0\}$ and $S_2 = \{\infty\}$, where a, b are nonzero constants such that $z^n + az^{n-1} + b = 0$ has no repeated root and n (≥ 7), k be two positive integers. If f and g are two non-constant meromorphic functions such that $E_{f^{(k)}}(S_1, 2) = E_{g^{(k)}}(S_1, 2)$, $E_f(S_2, 0) = E_g(S_2, 0)$ then $f^{(k)} \equiv g^{(k)}$.*

Theorem 1.2. *Let S_i, $i = 1, 2, 3$, be defined as in* Theorem D *and k be a positive integer. If f and g are two non-constant meromorphic functions such that $E_{f^{(k)}}(S_1, 5) = E_{g^{(k)}}(S_1, 5)$, $E_f(S_2, \infty) = E_g(S_2, \infty)$ and $E_{f^{(k)}}(S_3, 0) = E_{g^{(k)}}(S_3, 0)$ then $f^{(k)} \equiv g^{(k)}$.*

Theorem 1.3. *Let S_i, $i = 1, 2, 3$, be defined as in* Theorem D *and k be a positive integer. If f and g are two non-constant meromorphic functions such that $E_{f^{(k)}}(S_1, 4) = E_{g^{(k)}}(S_1, 4)$, $E_f(S_2, \infty) = E_g(S_2, \infty)$ and $E_{f^{(k)}}(S_3, 1) = E_{g^{(k)}}(S_3, 1)$ then $f^{(k)} \equiv g^{(k)}$.*

Theorem 1.4. *Let S_i, $i = 1, 2, 3$, be defined as in* Theorem D *and k be a positive integer. If f and g are two non-constant meromorphic functions such that $E_{f^{(k)}}(S_1, 5) = E_{g^{(k)}}(S_1, 5)$, $E_f(S_2, 9) = E_g(S_2, 9)$ and $E_{f^{(k)}}(S_3, \infty) = E_{g^{(k)}}(S_3, \infty)$ then $f^{(k)} \equiv g^{(k)}$.*

Though we follow the standard definitions and notations of the value distribution theory available in [11], we explain some notations which are used in the paper.

Definition 1.3. ([13]) For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid = 1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r, a; f \mid \leq m)$ ($N(r, a; f \mid \geq m)$) the counting function of those a-points of f whose multiplicities are not greater (less) than m where each a-point is counted according to its multiplicity.

$\overline{N}(r, a; f \mid \leq m)$ ($\overline{N}(r, a; f \mid \geq m)$) are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r, a; f \mid < m)$, $N(r, a; f \mid > m)$, $\overline{N}(r, a; f \mid < m)$ and $\overline{N}(r, a; f \mid > m)$ are defined analogously.

Definition 1.4. We denote by $\overline{N}(r, a; f \mid = k)$ the reduced counting function of those a-points of f whose multiplicities is exactly k, where $k \geq 2$ is an integer.

Definition 1.5. ([2]) Let f and g be two non-constant meromorphic functions such that f and g share (a, k) where $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity p, a a-point of g with multiplicity q. We denote by $\overline{N}_L(r, a; f)$ the counting function of those a-points of f and g where $p > q$; each point in this counting functions is counted only once. In the same way we can define $\overline{N}_L(r, a; g)$.

Definition 1.6. ([15]) We denote $N_2(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid \geq 2)$

Definition 1.7. ([14, 15]) Let f, g share a value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly

$$\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f) \quad \text{and} \quad \overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g).$$

Definition 1.8. ([18]) Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid g = b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g.

Definition 1.9. ([18]) Let $a, b_1, b_2, \ldots, b_q \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid g \neq b_1, b_2, \ldots, b_q)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b_i-points of g for $i = 1, 2, \ldots, q$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined as follows.

$$F = \frac{\left(f^{(k)}\right)^{n-1}\left(f^{(k)}+a\right)}{-b}, \qquad G = \frac{\left(g^{(k)}\right)^{n-1}\left(g^{(k)}+a\right)}{-b}, \tag{2.1}$$

where n (≥ 2) and k are two positive integers.

Henceforth we shall denote by H, Φ and V the following three functions

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right),$$
$$\Phi = \frac{F'}{F-1} - \frac{G'}{G-1}$$

and

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right) = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}.$$

Lemma 2.1. ([15, Lemma 1]) *Let F, G share $(1,1)$ and $H \not\equiv 0$. Then*

$$N(r,1;F \mid =1) = N(r,1;G \mid =1) \leq N(r,H) + S(r,F) + S(r,G).$$

Lemma 2.2. *Let S_1, S_2 and S_3 be defined as in* Theorem 1.1 *and F, G be given by* (2.1). *If for two non-constant meromorphic functions f and g $E_{f^{(k)}}(S_1,0) = E_{g^{(k)}}(S_1,0)$, $E_f(S_2,0) = E_g(S_2,0)$, $E_{f^{(k)}}(S_3,0) = E_{g^{(k)}}(S_3,0)$ and $H \not\equiv 0$ then*

$$\begin{aligned} N(r,H) \leq{} & \overline{N}_*\big(r,0,f^{(k)},g^{(k)}\big) + \overline{N}\big(r,0;f^{(k)}+a \mid \geq 2\big) + \overline{N}\big(r,0;g^{(k)}+a \mid \geq 2\big) \\ & + \overline{N}_*(r,1;F,G) + \overline{N}_*(r,\infty;f,g) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G'), \end{aligned}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G')$ is similarly defined.

Proof. We omit the proof since the proof of the lemma can be carried out in the line of proof of [2, Lemma 2.2]. □

Lemma 2.3. ([18, Lemma 4]) *If two non-constant meromorphic functions F and G share $(1,0)$, $(\infty,0)$ and $H \not\equiv 0$ then*

$$\begin{aligned} N(r,H) \leq{} & \overline{N}(r,0;F \mid \geq 2) + \overline{N}(r,0;G \mid \geq 2) + \overline{N}_*(r,1;F,G) + \overline{N}_*(r,\infty;F,G) \\ & + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G'), \end{aligned}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G')$ is similarly defined.

Lemma 2.4. ([22]) *Let f be a non-constant meromorphic function and let*

$$R(f)=\frac{\sum\limits_{k=0}^{n} a_k f^k}{\sum\limits_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$ where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r,R(f))=dT(r,f)+S(r,f),$$

where $d=\max\{n,m\}$.

Lemma 2.5. *Let F and G be given by (2.1). If $f^{(k)}$, $g^{(k)}$ share $(0,0)$ and 0 is not a Picard exceptional value of $f^{(k)}$ and $g^{(k)}$. Then $\Phi \equiv 0$ implies $F \equiv G$.*

Proof. We omit the proof since proceeding in the same way as done in [2, Lemma 2.4] we can prove the lemma. □

Lemma 2.6. *Let F and G be given by (2.1), $n \geq 3$ an integer and $\Phi \not\equiv 0$. If F, G share $(1,m)$; f, g share (∞,l), and $f^{(k)}$, $g^{(k)}$ share $(0,p)$, where $0 \leq p < \infty$ then*

$$[(n-1)p+n-2]\overline{N}(r,0;f^{(k)} \mid \geq p+1) \leq \overline{N}_*(r,1;F,G)+\overline{N}_*(r,\infty;F,G)+S(r).$$

Proof. The lemma can be proved in the line of the proof of [2, Lemma 2.5]. □

Lemma 2.7. *Let F and G be given by (2.1) and f, g share $(\infty,0)$ and ∞ is not a Picard exceptional value of $f^{(k)}$ and $g^{(k)}$. Then $V \equiv 0$ implies $F \equiv G$.*

Proof. We omit the proof since it can be proved in the line of the proof of [2, Lemma 2.6]. □

Lemma 2.8. *Let F, G be given by (2.1) and $H \not\equiv 0$. If $f^{(k)}$, $g^{(k)}$ share $(0,p)$; f and g share (∞,l), where $0 \leq l < \infty$ and F, G share $(1,m)$, where $1 \leq m \leq \infty$ then*

$$\begin{aligned}&\{(nl+nk+n)-1\}\overline{N}(r,\infty;f \mid \geq l+1)\\ &\leq \overline{N}_*\big(r,0;f^{(k)},g^{(k)}\big)+\overline{N}\big(r,0;f^{(k)}+a\big)+\overline{N}\big(r,0;g^{(k)}+a\big)\\ &\qquad\qquad\qquad\qquad\qquad\qquad+\overline{N}_*(r,1;F,G)+S(r).\end{aligned}$$

Similar expressions hold for g, too.

Proof. Suppose ∞ is not an e.v.P. of $f^{(k)}$ and $g^{(k)}$. Since $H \not\equiv 0$, it follows that $F \not\equiv G$. So from Lemma 2.7 we know that $V \not\equiv 0$. Since f, g share $(\infty;l)$, it follows that F, G share $\big(\infty;n(k+l)\big)$. Clearly a pole of F with multiplicity s $(\geq n(k+l)+1)$ is a pole of G with multiplicity r $(\geq n(k+l)+1)$

and vice versa. We note that F and G have no pole of multiplicity q where $n(k+l) < q < n(k+l+1)$. Also since any common pole of F and G of multiplicity $s \leq n(k+l)$ is a zero of V of multiplicity $s-1$, using Lemma 2.4 we get from the definition of V

$$\begin{aligned}
&\{n(l+k+1)-1\}\overline{N}(r,\infty;f\mid\geq l+1)\\
&\leq N(r,0;V)\\
&\leq N(r,\infty;V)+S(r,f^{(k)})+S(r,g^{(k)})\\
&\leq \overline{N}_*(r,0;f^{(k)},g^{(k)})+\overline{N}(r,0;f^{(k)}+a)+\overline{N}(r,0;g^{(k)}+a)\\
&\quad+\overline{N}_*(r,1;F,G)+S(r).
\end{aligned}$$

If ∞ is an e.v.P. of $f^{(k)}$ and $g^{(k)}$ then the lemma follows immediately. □

Lemma 2.9. *Let F, G be given by (2.1) and $V \not\equiv 0$. If f, g share (∞, l), where $0 \leq l < \infty$ and F, G share $(1, m)$ then the poles of F and G are the zeros of V and*

$$\begin{aligned}
&\{n(k+l+1)-1\}\overline{N}(r,\infty;f\mid\geq l+1)\\
&\leq \overline{N}(r,0;f^{(k)})+\overline{N}(r,0;g^{(k)})+\overline{N}(r,0;f^{(k)}+a)+\overline{N}(r,0;g^{(k)}+a)\\
&\quad+\overline{N}_*(r,1;F,G)+S(r).
\end{aligned}$$

Similar expressions hold for g also.

P r o o f. Suppose ∞ is an e.v.P. of $f^{(k)}$ and $g^{(k)}$ then the lemma follows immediately.

Next suppose ∞ is not an e.v.P. of $f^{(k)}$ and $g^{(k)}$. Now using the same argument as in Lemma 2.8 we can deduce from the definition of V that

$$\begin{aligned}
&\{n(k+1)-1\}N(r,\infty;f\mid=1)+\{n(k+2)-1\}\overline{N}(r,\infty;f\mid=2)+\dots\\
&\quad+\{n(k+l)-1\}\overline{N}(r,\infty;f\mid=l)+\{n(k+l+1)-1\}\overline{N}(r,\infty;f\mid\geq l+1)\\
&\leq N(r,0;V)\\
&\leq T(r,V)\\
&\leq N(r,\infty;V)+S(r,f^{(k)})+S(r,g^{(k)})\\
&\leq \overline{N}(r,0;f^{(k)})+\overline{N}(r,0;g^{(k)})+\overline{N}(r,0;f^{(k)}+a)+\overline{N}(r,0;g^{(k)}+a)\\
&\quad+\overline{N}_*(r,1;F,G)+S(r).
\end{aligned}$$

□

Lemma 2.10. *Let f and g be two meromorphic functions sharing $(1,m)$, where $1 \le m < \infty$. Then*

$$\begin{aligned}&\overline{N}(r,1;f)+\overline{N}(r,1;g)-N(r,1;f\mid=1)+\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;f,g)\\&\le\frac{1}{2}\left[N(r,1;f)+N(r,1;g)\right].\end{aligned}$$

Proof. Let z_0 be a 1-point of f of multiplicity p and a 1-point of g of multiplicity q.

Since f, g share $(1,m)$, we note that the 1-points of f and g up to multiplicity m are same and as a result when $p=q\le m$, z_0 is counted 2 times in the left hand side of the above inequality whereas it is counted m times in the right hand side of the same. If $p=m+1$ then the possible values of q are as follows.

(i) $q=m+1$,
(ii) $q\ge m+2$.

When $p=m+2$ then q can take the following possible values

(i) $q=m+1$,
(ii) $q=m+2$,
(iii) $q\ge m+3$.

Similar explanations hold if we interchange p and q. Clearly when $p=q\ge m+1$, z_0 is counted 2 times in the left hand side and $p\ge m+1$ times in the right hand side of the above inequality. When $p>q\ge m+1$, in view of Definition 1.7 we know z_0 is counted $m+\frac{3}{2}$ times in the left hand side and $\frac{p+q}{2}\ge m+\frac{3}{2}$ times in the right hand side of the above inequality. When $q>p$ we can explain similarly. Hence the lemma follows. □

Lemma 2.11. *Let F, G be given by (2.1) and $H\not\equiv 0$. If F, G share $(1,m)$ and f, g share (∞,k), $f^{(k)}$, $g^{(k)}$ share $(0,p)$ where $1\le m<\infty$. Then*

$$\begin{aligned}&\left(\frac{n}{2}-1\right)\left\{T\left(r,f^{(k)}\right)+T\left(r,g^{(k)}\right)\right\}\\&\le\overline{N}\left(r,0;f^{(k)}\right)+\overline{N}(r,\infty;f)+\overline{N}\left(r,0;g^{(k)}\right)+\overline{N}(r,\infty;g)\\&\quad+\overline{N}_*\left(r,0;f^{(k)},g^{(k)}\right)+\overline{N}_*(r,\infty;f,g)-\left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)\\&\quad+S\left(r,f^{(k)}\right)+S\left(r,g^{(k)}\right).\end{aligned}$$

P r o o f. By the second fundamental theorem we get

$$\begin{aligned}
&T(r,F)+T(r,G)\\
&\le \overline{N}(r,1;F)+\overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;G)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)\\
&\quad -N_0(r,0;F')-N_0(r,0;G')+S(r,F)+S(r,G).
\end{aligned}\tag{2.2}$$

Using Lemmas 2.1, 2.2, 2.4 and 2.10 we see that

$$\begin{aligned}
&\overline{N}(r,1;F)+\overline{N}(r,1;G)\\
&\le \frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]+N(r,1;F\mid=1)-\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;F,G)\\
&\le \frac{n}{2}\left\{T\left(r,f^{(k)}\right)+T\left(r,g^{(k)}\right)\right\}+\overline{N}_*\left(r,0;f^{(k)},g^{(k)}\right)+\overline{N}_*(r,\infty;f,g)\\
&\quad+\overline{N}\left(r,0;f^{(k)}+a\mid\ge 2\right)+\overline{N}\left(r,0;g^{(k)}+a\mid\ge 2\right)+\overline{N}_L(r,1;F)\\
&\quad+\overline{N}_L(r,1;G)-\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;F,G)\\
&\quad+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S\left(r,f^{(k)}\right)+S\left(r,g^{(k)}\right)\\
&\le \frac{n}{2}\left\{T\left(r,f^{(k)}\right)+T\left(r,g^{(k)}\right)\right\}+\overline{N}_*\left(r,0;f^{(k)},g^{(k)}\right)\\
&\quad+\overline{N}_*(r,\infty;f,g)+\overline{N}\left(r,0;f^{(k)}+a\mid\ge 2\right)\\
&\quad+\overline{N}\left(r,0;g^{(k)}+a\mid\ge 2\right)-\left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)\\
&\quad+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S\left(r,f^{(k)}\right)+S\left(r,g^{(k)}\right).
\end{aligned}\tag{2.3}$$

Using (2.3) in (2.2) the lemma follows in view of Definition 1.5. □

Lemma 2.12. *Let F, G be given by* (2.1) *and they share* $(1,m)$. *If f, g share* (∞,l) *where $2\le m<\infty$ and $H\not\equiv 0$. Then*

$$\begin{aligned}
&\left(\frac{n}{2}-1\right)\left\{T\left(r,f^{(k)}\right)+T\left(r,g^{(k)}\right)\right\}\\
&\le 2\overline{N}\left(r,0;f^{(k)}\right)+\overline{N}(r,\infty;f)+2\overline{N}\left(r,0;g^{(k)}\right)+\overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;f,g)\\
&\quad-\left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+S\left(r,f^{(k)}\right)+S\left(r,g^{(k)}\right).
\end{aligned}$$

P r o o f. We omit the proof since using Lemmas 2.1, 2.3 and 2.10 the proof of the lemma can be carried out in the line of the proof of Lemma 2.11. □

Lemma 2.13. *Let $f^{(k)}$, $g^{(k)}$ be two non-constant meromorphic functions sharing $(0,0)$, (∞,∞). Then $\left(f^{(k)}\right)^{n-1}\left(f^{(k)}+a\right) \equiv \left(g^{(k)}\right)^{n-1}\left(g^{(k)}+a\right)$ implies $f^{(k)} \equiv g^{(k)}$, where n (≥ 2) is an integer, k is a positive integer and a is a nonzero finite constant.*

Proof. We first note that $\Theta\left(\infty; f^{(k)}\right) + \Theta\left(\infty; g^{(k)}\right) > 2 - \frac{2}{k+1} = \frac{2k}{k+1} > 0$. Now since the given condition implies $f^{(k)}$, $g^{(k)}$ share $(0;\infty)$, the lemma can be proved in the line of the proof of [17, Lemma 3]. □

Lemma 2.14. *If two meromorphic functions f, g share $(\infty,0)$ then for $n \geq 2$*

$$\left(f^{(k)}\right)^{n-1}\left(f^{(k)}+a\right)\left(g^{(k)}\right)^{n-1}\left(g^{(k)}+a\right) \not\equiv b^2,$$

where a, b are finite nonzero constants and k is a positive integer.

Proof. Noting that according to the lemma $f^{(k)}$, $g^{(k)}$ share (∞,k), we omit the proof since the proof of the lemma can be carried out in the line of the proof of [16, Lemma 5]. □

Lemma 2.15. ([28, Lemma 6]) *If $H \equiv 0$, then F, G share $(1,\infty)$. If further F, G share $(\infty,0)$ then F, G share (∞,∞).*

Lemma 2.16. *Let F, G be given by (2.1) and they share $(1,m)$. Also let $\omega_1,\omega_2,\ldots,\omega_n$ are the members of the set $S_1 = \left\{z : \ z^n + az^{n-1} + b = 0\right\}$, where a, b are nonzero constants such that $z^n + az^{n-1} + b = 0$ has no repeated root and n (≥ 3) is an integer. Then*

$$\overline{N}_*(r,1;F,G) \leq \frac{1}{m}\left[\overline{N}\left(r,0;f^{(k)}\right) + \overline{N}(r,\infty;f) - N_{\otimes}\left(r,0;f^{(k+1)}\right)\right] + S(r),$$

where $N_{\otimes}\left(r,0;f^{(k+1)}\right) = N\left(r,0;f^{(k+1)} \mid f^{(k)} \neq 0,\omega_1,\omega_2,\ldots,\omega_n\right)$.

Proof. The proof can be carried out along the lines of the proof of [2, Lemma 2.15]. □

Lemma 2.17. ([26]) *Let F, G be two meromorphic functions sharing $(1,\infty)$ and (∞,∞). If*

$$N_2(r,0;F) + N_2(r,0;G) + 2\overline{N}(r,\infty;F) < \lambda T_1(r) + S_1(r),$$

where $\lambda < 1$ and $T_1(r) = \max\{T(r,F),T(r,G)\}$ and $S_1(r) = o(T_1(r))$, $r \to \infty$, outside a possible exceptional set of finite linear measure, then $F \equiv G$ or $FG \equiv 1$.

Lemma 2.18. *Let F, G be given by* (2.1), $n \geq 3$ *and they share* $(1, m)$. *If* $f^{(k)}$, $g^{(k)}$ *share* $(0,0)$, *and* f, g *share* (∞, l) *and* $H \equiv 0$. *Then* $f^{(k)} \equiv g^{(k)}$.

Proof. Since $H \equiv 0$, we get from Lemma 2.15, F and G share $(1, \infty)$ and (∞, ∞). If possible let us suppose $F \not\equiv G$. Then from Lemma 2.5 and Lemma 2.6 we have

$$\overline{N}(r, 0; f^{(k)}) = \overline{N}(r, 0; g^{(k)}) = S(r).$$

Again from Lemma 2.7 we get $V \not\equiv 0$ and so in view of Lemma 2.8 we have

$$\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) \leq \frac{4}{n(k+1)-1} T(r) + S(r).$$

Therefore we see that

$$\begin{aligned}
&N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, \infty; F) \\
\leq\ & 2\overline{N}(r, 0; f^{(k)}) + 2\overline{N}(r, 0; g^{(k)}) \\
&+ N_2(r, 0; f^{(k)} + a) + N_2(r, 0; g^{(k)} + a) + 2\overline{N}(r, \infty; f) \\
\leq\ & N_2\left(r, 0; f^{(k)} + a\right) + N_2\left(r, 0; g^{(k)} + a\right) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + S(r).
\end{aligned} \tag{2.4}$$

Using Lemma 2.4 we obtain

$$T_1(r) = n \max\left\{T\left(r, f^{(k)}\right), T\left(r, g^{(k)}\right)\right\} + O(1) = nT(r) + O(1). \tag{2.5}$$

So again using Lemma 2.4 we get from (2.4) and (2.5)

$$\begin{aligned}
&N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, \infty; F) \\
\leq\ & \frac{\left[2 + \frac{4}{n(k+1)-1}\right]}{n} T_1(r) + S(r).
\end{aligned}$$

Since $k \geq 1$ and $n \geq 3$, we have by Lemma 2.17, $FG \equiv 1$, which is impossible by Lemma 2.14. Hence $F \equiv G$, i.e. $\left(f^{(k)}\right)^{n-1}(f^{(k)} + a) \equiv \left(g^{(k)}\right)^{n-1}(g^{(k)} + a)$. Now the lemma follows from Lemma 2.13. □

Lemma 2.19. *Suppose F and G be defined as in* (2.1) *and $n \geq 7$ be an integer. Then $F \equiv G$ implies $f^{(k)} \equiv g^{(k)}$.*

Proof. We note that $\Theta\left(\infty; f^{(k)}\right) > 1 - \frac{1}{k+1} = \frac{k}{k+1} \geq \frac{1}{2} > \frac{2}{n-1}$, for $n \geq 7$. So the proof of the lemma can be carried out along the lines of the proof of [30, Lemma 2]. □

Lemma 2.20. *Suppose F and G be defined as in* (2.1) *and $n \geq 7$ be an integer. If f, g share (∞, k) and $H \equiv 0$. Then $f^{(k)} \equiv g^{(k)}$.*

Proof. Since $H \equiv 0$, we get from Lemma 2.15, F and G share $(1, \infty)$ and (∞, ∞). If possible, let us suppose $F \not\equiv G$. Using Lemma 2.7 and Lemma 2.9 with $l = 0$ we have

$$\begin{aligned}
& N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, \infty; F) \\
\leq\ & 2\overline{N}(r, 0; f^{(k)}) + 2\overline{N}(r, 0; g^{(k)}) \\
& + N_2(r, 0; f^{(k)} + a) + N_2(r, 0; g^{(k)} + a) + 2\overline{N}(r, \infty; f) \\
\leq\ & \frac{\left[6 + \frac{8}{nk+n-1}\right]}{n} T_1(r) + S(r).
\end{aligned}$$

So respectively using Lemmas 2.17, 2.14 we can deduce a contradiction. Hence $F \equiv G$. Now the lemma follows from Lemma 2.19. □

3. Proofs of the theorems

Proof of Theorem 1.1. Let F, G be given by (2.1). Then F and G share $(1, 2)$, $(\infty, nk + n - 1)$. We consider the following cases.

Case 1. Let $H \not\equiv 0$.

Then $F \not\equiv G$. Suppose ∞ is not an e.v.P. of $f^{(k)}$ and $g^{(k)}$. Then by Lemma 2.7 we get $V \not\equiv 0$. Hence from Lemmas 2.9 and 2.12 with $l = 0$ we obtain

$$\begin{aligned}
& \left(\frac{n}{2} - 1\right) \left\{T\big(r, f^{(k)}\big) + T\big(r, g^{(k)}\big)\right\} \\
\leq\ & 2\overline{N}\big(r, 0; f^{(k)}\big) + 2\overline{N}\big(r, 0; g^{(k)}\big) + 3\overline{N}(r, \infty; f) - \frac{1}{2}\overline{N}_*(r, 1; F, G) \\
& + S\big(r, f^{(k)}\big) + S\big(r, g^{(k)}\big) \\
\leq\ & 2T\big(r, f^{(k)}\big) + 2T\big(r, g^{(k)}\big) + \frac{3}{nk + n - 1}\left\{2T\big(r, f^{(k)}\big) + 2T\big(r, g^{(k)}\big)\right. \\
& \left. + \overline{N}_*(r, 1; F, G)\right\} - \frac{1}{2}\overline{N}_*(r, 1; F, G) + S\big(r, f^{(k)}\big) + S\big(r, g^{(k)}\big) \\
\leq\ & \left[2 + \frac{6}{nk + n - 1}\right] \left\{T\big(r, f^{(k)}\big) + T\big(r, g^{(k)}\big)\right\} + S\big(r, f^{(k)}\big) + S\big(r, g^{(k)}\big). \qquad (3.1)
\end{aligned}$$

If ∞ is an e.v.P. of $f^{(k)}$ and $g^{(k)}$ then $\overline{N}(r, \infty; f) = \overline{N}(r, \infty; f^{(k)}) \leq S(r, f^{(k)})$ and hence (3.1) automatically holds.

From (3.1) we have

$$\left[\frac{n}{2}-3-\frac{6}{nk+n-1}\right]\left\{T\left(r,f^{(k)}\right)+T\left(r,g^{(k)}\right)\right\}\le S\left(r,f^{(k)}\right)+S\left(r,g^{(k)}\right),$$

which leads to a contradiction for $n\ge 7$.
Case 2. Let $H\equiv 0$.
Now the theorem follows from Lemma 2.20. □

P r o o f o f T h e o r e m 1.2. Let F, G be given by (2.1). Then F and G share $(1,5)$, (∞,∞). We consider the following cases.
Case 1. Let $H\not\equiv 0$.
Then $F\not\equiv G$. Suppose 0, ∞ are not exceptional values Picard of $f^{(k)}$ and $g^{(k)}$. Then by Lemma 2.5 and Lemma 2.7 we get $\Phi\not\equiv 0$ and $V\not\equiv 0$. Hence from Lemmas 2.4, 2.6, 2.8, 2.11 and 2.16 we obtain

$$\begin{aligned}
&\left(\frac{n}{2}-1\right)\left\{T\left(r,f^{(k)}\right)+T\left(r,g^{(k)}\right)\right\}\\
&\le 3\overline{N}\left(r,0;f^{(k)}\right)+2\overline{N}(r,\infty;f)-\frac{7}{2}\overline{N}_*(r,1;F,G)\\
&\quad+S\left(r,f^{(k)}\right)+S\left(r,g^{(k)}\right)\\
&\le \frac{3}{n-2}\{\overline{N}_*(r,1;F,G)\}+\frac{2}{nk+n-1}\left\{T\left(r,f^{(k)}\right)+T\left(r,g^{(k)}\right)\right.\\
&\quad\left.+\overline{N}\left(r,0;f^{(k)}\right)+\overline{N}_*(r,1;F,G)\right\}-\frac{7}{2}\overline{N}_*(r,1;F,G)\\
&\quad+S\left(r,f^{(k)}\right)+S\left(r,g^{(k)}\right)\\
&\le \left[\frac{2}{nk+n-1}\right]\overline{N}\left(r,0;f^{(k)}\right)+\frac{2}{nk+n-1}\left\{T\left(r,f^{(k)}\right)+T\left(r,g^{(k)}\right)\right\}\\
&\quad+S\left(r,f^{(k)}\right)+S\left(r,g^{(k)}\right)\\
&\le \left[\frac{2}{nk+n-1}+\frac{2}{5(n-2)(nk+n-1)}\right]\left\{T\left(r,f^{(k)}\right)+T\left(r,g^{(k)}\right)\right\}\\
&\quad+S\left(r,f^{(k)}\right)+S\left(r,g^{(k)}\right).
\end{aligned}\tag{3.2}$$

If 0 or ∞ is an e.v.P. of $f^{(k)}$ and $g^{(k)}$ then (3.2) automatically holds.

From (3.2) we have

$$\begin{aligned}
&\left[\frac{n}{2}-1-\frac{2}{nk+n-1}-\frac{2}{5(n-2)(nk+n-1)}\right]\left\{T\left(r,f^{(k)}\right)+T\left(r,g^{(k)}\right)\right\}\\
&\le S\left(r,f^{(k)}\right)+S\left(r,g^{(k)}\right),
\end{aligned}$$

which leads to a contradiction for $n\ge 3$.

Case 2. Let $H \equiv 0$.
Now the theorem follows from Lemma 2.18. $\square$

Proof of Theorem 1.3. Let F, G be given by (2.1). Then F and G share $(1,4)$, (∞,∞). We consider the following cases.
Case 1. Let $H \not\equiv 0$.
With the same argument as mentioned in Theorem 1.2 we get $\Phi \not\equiv 0$ and $V \not\equiv 0$. Hence from Lemmas 2.4, 2.6 with $p=0$ and $p=1$, and from Lemmas 2.8, 2.11 and 2.16 we obtain

$$\begin{aligned}
&\left(\frac{n}{2}-1\right)\left\{T\left(r,f^{(k)}\right)+T\left(r,g^{(k)}\right)\right\}\\
&\le 2\overline{N}\left(r,0;f^{(k)}\right)+\overline{N}\left(r,0;f^{(k)}\mid\ge 2\right)+2\overline{N}(r,\infty;f)-\frac{5}{2}\overline{N}_*(r,1;F,G)\\
&\quad+S\left(r,f^{(k)}\right)+S\left(r,g^{(k)}\right)\\
&\le\left[1+\frac{2}{nk+n-1}\right]\overline{N}\left(r,0;f^{(k)}\mid\ge 2\right)\\
&\quad+\frac{2}{nk+n-1}\left\{T\left(r,f^{(k)}\right)+T\left(r,g^{(k)}\right)+\overline{N}_*(r,1;F,G)\right\}\\
&\quad-\frac{1}{2}\overline{N}_*(r,1;F,G)+S\left(r,f^{(k)}\right)+S\left(r,g^{(k)}\right)\\
&\le\left[\frac{1}{2n-3}+\frac{4(n-1)}{(2n-3)(nk+n-1)}-\frac{1}{2}\right]\overline{N}_*(r,1;F,G)\\
&\quad+\frac{2}{nk+n-1}\left\{T\left(r,f^{(k)}\right)+T\left(r,g^{(k)}\right)\right\}+S\left(r,f^{(k)}\right)+S\left(r,g^{(k)}\right)\\
&\le\left[\frac{2}{nk+n-1}+\frac{1}{4(2n-3)}+\frac{n-1}{(2n-3)(nk+n-1)}-\frac{1}{8}\right]\cdot\\
&\quad\cdot\left\{T\left(r,f^{(k)}\right)+T\left(r,g^{(k)}\right)\right\}+S\left(r,f^{(k)}\right)+S\left(r,g^{(k)}\right).
\end{aligned}\tag{3.3}$$

From (3.3) we have

$$\begin{aligned}
&\left[\frac{n}{2}-\frac{7}{8}-\frac{2}{nk+n-1}-\frac{1}{4(2n-3)}-\frac{n-1}{(2n-3)(nk+n-1)}\right]\cdot\\
&\qquad\qquad\cdot\left\{T\left(r,f^{(k)}\right)+T\left(r,g^{(k)}\right)\right\}\\
&\le S\left(r,f^{(k)}\right)+S\left(r,g^{(k)}\right),
\end{aligned}$$

which leads to a contradiction for $n\ge 3$.

Case 2. Let $H \equiv 0$.
Now the theorem follows from Lemma 2.18. □

P r o o f o f T h e o r e m 1.4. Let F, G be given by (2.1). Then F and G share $(1,5)$. Here since $f^{(k)}$ and $g^{(k)}$ share $(0,\infty)$ it follows that $\overline{N}_*\left(r,0;f^{(k)},g^{(k)}\right)=0$ We consider the following cases.
Case 1. Let $H \not\equiv 0$.
With the same argument as mentioned in Theorem 1.2 we get $\Phi \not\equiv 0$ and $V \not\equiv 0$. Hence from Lemmas 2.4, 2.6, 2.8 with $l=0$ and $l=9$, and from Lemmas 2.11 and 2.16 we obtain

$$\begin{aligned}
&\left(\frac{n}{2}-1\right)\left\{T\left(r,f^{(k)}\right)+T\left(r,g^{(k)}\right)\right\}\\
&\le 2\overline{N}\left(r,0;f^{(k)}\right)+2\overline{N}(r,\infty;f)+\overline{N}\left(r,\infty;f\mid\ge 10\right)-\frac{7}{2}\overline{N}_*(r,1;F,G)\\
&\quad+S\left(r,f^{(k)}\right)+S\left(r,g^{(k)}\right)\\
&\le \frac{2}{n-2}\overline{N}_*(r,1;F,G)+\frac{n}{n-2}\overline{N}(r,\infty;f\mid\ge 10)\\
&\quad+\frac{2}{nk+n-1}\left\{T\left(r,f^{(k)}\right)+T\left(r,g^{(k)}\right)+\overline{N}_*(r,1;F,G)\right\}\\
&\quad-\frac{7}{2}\overline{N}_*(r,1;F,G)+S\left(r,f^{(k)}\right)+S\left(r,g^{(k)}\right)\\
&\le \frac{2}{nk+n-1}\left\{T\left(r,f^{(k)}\right)+T\left(r,g^{(k)}\right)\right\}\\
&\quad+\frac{n}{(n-2)(10n+nk-1)}\left\{T\left(r,f^{(k)}\right)+T\left(r,g^{(k)}\right)+\overline{N}_*(r,1;F,G)\right\}\\
&\quad-\overline{N}_*(r,1;F,G)+S\left(r,f^{(k)}\right)+S\left(r,g^{(k)}\right)\\
&\le \left[\frac{2}{nk+n-1}+\frac{n}{(n-2)(10n+nk-1)}\right]\left\{T\left(r,f^{(k)}\right)+T\left(r,g^{(k)}\right)\right\}\\
&\quad+S\left(r,f^{(k)}\right)+S\left(r,g^{(k)}\right).
\end{aligned}\tag{3.4}$$

From (3.4) we have

$$\begin{aligned}
&\left[\frac{n}{2}-1-\frac{2}{nk+n-1}-\frac{n}{(n-2)(10n+nk-1)}\right]\left\{T\left(r,f^{(k)}\right)+T\left(r,g^{(k)}\right)\right\}\\
&\le S\left(r,f^{(k)}\right)+S\left(r,g^{(k)}\right),
\end{aligned}$$

which leads to a contradiction for $n \ge 3$.
Case 2. Let $H \equiv 0$.
Now the theorem follows from Lemma 2.18. □

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