

REGULAR ELEMENTS IN GENERALIZED  
HERMITIAN ALGEBRAS

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ABSTRACT. A generalized Hermitian (GH) algebra is a special Jordan algebra that is at the same time a spectral order-unit space. In this paper we characterize the von Neumann regular elements in a GH-algebra, relate maximal pairwise commuting subsets of the algebra to blocks in its projection lattice, and prove a Gelfand-Naimark type representation theorem for commutative GH-algebras.

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## 1. Introduction

This paper is a continuation of [8] in which we introduced and launched a study of *generalized Hermitian* (GH) *algebras*. Motivation for the study of GH-algebras can be found in [8, Section 1].

We begin by recalling the definition and some of the basic properties<sup>1</sup> of a GH-algebra  $G$ . In Section 3 we review the spectral properties of  $G$ , and in Section 4 we characterize its (*von Neumann*) *regular elements* and show that elements with finite spectrum (*simple elements*) are regular. In Section 5, we relate the maximal commutative subalgebras of  $G$  to the so-called *blocks* in the projection lattice of  $G$ , and we obtain a Gelfand-Naimark type structure theorem for commutative GH-algebras.

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<sup>1</sup>These properties are not necessarily reviewed in the order in which they were proved in [8].

In this article,  $\mathbb{N}$  is the set of positive integers and  $\mathbb{R}$  is the ordered field of real numbers. We recall the definition of a GH-algebra [8, Definition 2.1].

**DEFINITION 1.1.** A *generalized Hermitian (GH) algebra* is a subgroup  $G$  of the additive group of a ring  $R$  with unity 1 such that:

- (i)  $G$  is a partially ordered abelian group ([11, pp. 1–4]) with positive cone  $G^+ = \{g \in G : 0 \leq g\}$  such that  $1 \in G^+$  and 1 is an order unit in  $G$  ([11, p. 4]).
- (ii) If  $a, b \in G^+$ , then  $ab = ba \implies ab \in G^+$ .
- (iii) If  $a, b \in G^+$ , then  $aba \in G^+$  and  $aba = 0 \implies ab = ba = 0$ .
- (iv) If  $g \in G$ , then  $g^2 \in G^+$ .
- (v) There exists an element  $\frac{1}{2} \in G^+$  such that  $\frac{1}{2} + \frac{1}{2} = 1$ .
- (vi) If  $g, h \in G$ , then  $gh^2g = 0 \implies gh = hg = 0$ .
- (vii) Each ascending sequence  $g_1 \leq g_2 \leq \dots$  of pairwise commuting elements of  $G$  that is bounded above in  $G$  has a supremum (least upper bound)  $g$  in  $G$ , and  $g$  commutes with every element in  $G$  that commutes with every  $g_i$ ,  $i = 1, 2, \dots$ .

The ring  $R$  is called the *enveloping ring*<sup>2</sup> of the GH-algebra  $G$ .

The prototypic example of a *noncommutative* GH-algebra is the partially ordered Jordan Banach algebra  $\mathbb{G}(\mathfrak{H})$  of bounded Hermitian operators on a Hilbert space  $\mathfrak{H}$  of dimension 2 or more with the ring  $\mathbb{B}(\mathfrak{H})$  of all bounded operators on  $\mathfrak{H}$  as its enveloping ring.

The prototypic example of a *commutative* GH-algebra is as follows: Let  $X$  be a nonempty set and let  $\mathbb{R}^{[X]}$  be the partially ordered commutative and associative linear algebra of all bounded functions  $f: X \rightarrow \mathbb{R}$  with pointwise operations and pointwise partial order. Regarded as an additive partially ordered abelian group,  $\mathbb{R}^{[X]}$  is a GH-algebra with the ring  $\mathbb{R}^{[X]}$  as its enveloping ring.

Both of the prototypic examples are actually archimedean partially ordered vector spaces over  $\mathbb{R}$ . In fact, by [8, Theorem 4.2], we have the following.

**THEOREM 1.2.** A GH-algebra  $G$  can be organized into an order-unit space, i.e., an archimedean partially ordered real vector space with order unit 1 ([1, p. 69]), and as such, it is a formally-real special Jordan algebra with respect to the Jordan product  $(g, h) \mapsto \frac{1}{2}(gh + hg)$ .

<sup>2</sup>The detailed structure of the ring  $R$  will not concern us here as  $R$  is just a convenient environment in which to study  $G$ .

**DEFINITION 1.3.** Let  $G$  be a GH-algebra. An additive subgroup  $H$  of  $G$  is a *GH-subalgebra* of  $G$  iff<sup>3</sup>

- (1)  $1 \in H$ ,
- (2)  $h \in H \implies \frac{1}{2}h \in H$ ,
- (3)  $h \in H \implies h^2 \in H$ , and
- (4) if an ascending sequence  $h_1 \leq h_2 \leq \dots$  of pairwise commuting elements of  $H$  has a supremum  $h$  in  $G$ , then  $h \in H$ .

**THEOREM 1.4.** Let  $G$  be a GH-algebra with enveloping ring  $R$ , and let  $H$  be a GH-subalgebra of  $G$ . Then, with the partial order induced from  $G$ ,  $H$  is a GH-algebra with enveloping ring  $R$ .

*Proof.* Assume that  $H$  is an additive subgroup of  $G$  that satisfies conditions (1)–(4) in Definition 1.3, and give  $H$  the partial order induced from  $G$ . We have to prove that  $H$  satisfies conditions (i)–(vii) in Definition 1.1. Clearly,  $H$  is a subgroup of the additive group of  $R$  and  $H$  is a partially ordered abelian group with positive cone  $H^+ = H \cap G^+$ . As  $1 \in H$  and  $1$  is an order unit in  $G$ , it follows that  $1$  is an order unit in  $H$  so condition (i) holds.

If  $h, k \in H$ , then  $\frac{1}{2}(hk + kh) = \frac{1}{2}((h + k)^2 - h^2 - k^2) \in H$  by (3). Let  $a, b \in H^+ = H \cap G^+$ . If  $ab = ba$ , then  $ab = \frac{1}{2}(ab + ba) \in H$  and  $ab \in G^+$  by condition (ii), so  $ab \in H \cap G^+ = H^+$ , and  $H$  also satisfies condition (ii). Put  $k = \frac{1}{2}(ab + ba)$ . Then  $k \in H$ ,  $ak + ka \in H$ , and  $aba = ak + ka - \frac{1}{2}(a^2b + ba^2) \in H$ . As  $a, b \in G^+$ , we have  $aba \in H \cap G^+ = H^+$  by condition (iii); moreover,  $aba = 0 \implies ab = ba = 0$ , whence  $H$  satisfies condition (iii) as well. That  $H$  satisfies condition (iv) follows directly from (3), and (2) implies that  $\frac{1}{2} = \frac{1}{2} \cdot 1 \in H$ , so condition (v) also holds for  $H$ . Condition (vi) is inherited from  $G$  by  $H$ , and by (4), condition (vii) is also inherited from  $G$ .  $\square$

Using Theorem 1.4, we obtain a host of examples of GH-algebras simply by forming GH-subalgebras of the prototypic GH-algebras  $\mathbb{G}(\mathfrak{H})$  and  $\mathbb{R}^{[X]}$ . For instance, if  $\mathbb{A}$  is a  $*$ -subalgebra of  $\mathbb{B}(\mathfrak{H})$ ,  $1 \in \mathbb{A}$ , and  $\mathbb{A}$  is closed in the weak operator topology (i.e.,  $\mathbb{A}$  is a von Neumann algebra acting on  $\mathfrak{H}$ ), then  $\mathbb{A} \cap \mathbb{G}(\mathfrak{H})$  is a GH-subalgebra of  $\mathbb{G}(\mathfrak{H})$ . More generally, the self-adjoint part of any AW\*-algebra ([15]) is a GH-algebra. If  $X$  is a compact, Hausdorff, and basically disconnected space (i.e., the Stone space of a  $\sigma$ -complete Boolean algebra), then the set  $C(X, \mathbb{R})$  consisting of the continuous functions in  $\mathbb{R}^{[X]}$  is a GH-subalgebra of  $\mathbb{R}^{[X]}$ . Also, if  $\mathcal{B}$  is a field of subsets of  $X$ , then the set  $RV^b(X, \mathcal{B}, \mathbb{R})$  consisting of the bounded measurable functions (bounded random variables) in  $\mathbb{R}^{[X]}$  is a GH-subalgebra of  $\mathbb{R}^{[X]}$ .

<sup>3</sup>As usual, ‘iff’ is an abbreviation for ‘if and only if’.

If  $G$  is a GH-algebra, it is obvious that the intersection of GH-subalgebras of  $G$  is again a GH-subalgebra of  $G$ ; hence any subset  $A \subseteq G$  “generates” a GH-subalgebra of  $G$ , namely the intersection of all the GH-subalgebras  $H$  of  $G$  such that  $A \subseteq H$ .

**STANDING ASSUMPTION 1.5.** In the sequel, we assume that  $G$  is a GH-algebra with enveloping ring  $R$ , and we regard  $G$  both as an order-unit space with order unit 1 and as a special Jordan algebra (Theorem 1.4). As is customary, if  $\lambda \in \mathbb{R}$ , we shall identify the real number  $\lambda$  with  $\lambda \cdot 1 \in G$ . The “unit interval” in  $G$  is denoted by  $E := \{e \in G : 0 \leq e \leq 1\}$ ,<sup>4</sup> and the set of all idempotent elements in  $G$  is denoted by  $P := \{p \in G : p = p^2\}$ . To avoid trivialities, we assume that  $G \neq \{0\}$ , i.e.,  $0 < 1$ .

Following Ludwig [16], elements  $e \in E$  are called *effects*, and elements  $p \in P$  are called *projections*. We have

$$0, 1 \in P \subseteq E \subseteq G^+ \subseteq G \subseteq R,$$

and we understand that  $G^+$ ,  $E$ , and  $P$  are partially ordered by the restrictions of the partial order  $\leq$  on  $G$ .

If  $g, h \in G$ , we define  $gCh$  to mean that  $g$  commutes with  $h$ , i.e., that  $gh = hg$ . If  $A \subseteq G$ , we also define  $C(A)$ , called the *commutant of  $A$  in  $G$* , by  $C(A) := \{g \in G : (\forall a \in A)(gCa)\}$ . The subgroup  $C(G)$  of  $G$  is called the *center* of  $G$ , and of course  $G$  is said to be *commutative* iff  $G = C(G)$ . The set  $CC(A) := C(C(A))$  is called the *bicommutant of  $A$  in  $G$* , and if  $g \in CC(h) := CC(\{h\})$ , we say that  $g$  *double commutes* with  $h$ .

**LEMMA 1.6.** *If  $A \subseteq G$ , then  $C(A)$  is a GH-subalgebra of  $G$ .*

**P r o o f.** Clearly  $C(A)$  is a subgroup of  $G$  that satisfies conditions (1)–(3) in Definition 1.3. To prove that (4) is satisfied, suppose that  $(h_n)_{n \in \mathbb{N}}$  is an ascending sequence of pairwise commuting elements of  $C(A)$  and that  $h$  is the supremum of  $(h_n)_{n \in \mathbb{N}}$  in  $G$ . By condition (vii) in Definition 1.1,  $h \in CC(\{h_n : n \in \mathbb{N}\}) \subseteq C(A)$ , and therefore (4) holds.  $\square$

Let  $g, h \in G$ . By [8, Lemma 3.2] we have  $ghg \in G$  and  $gCh \implies gh \in G$ ; moreover,  $g^n \in G$  for all  $n \in \mathbb{N}$ , so  $G$  is closed under the formation of real polynomials in its elements.

The unit interval  $E \subseteq G$  forms a so-called *interval effect algebra* ([2]) with some special multiplicative properties inherited from  $G$  (see [8, Lemma 3.3]); moreover,  $G^+ = \{ne : n \in \mathbb{N}, e \in E\}$  ([8, Lemma 4.6]). By [8, Lemma 5.4], the set  $P$  of projections with the partial order inherited from  $G$  and with  $p \mapsto 1 - p$  as the orthocomplementation, forms a  $\sigma$ -complete orthomodular lattice (a  $\sigma$ -OML)

<sup>4</sup>The notation  $:=$  means ‘equals by definition’.

([13]). The effect algebra  $E$  is a convex subset of the order-unit space  $G$ , and  $P$  is precisely the set of extreme points of  $E$  ([8, Theorem 4.3]).

If  $p, q \in P$ , we use the usual notation  $p \vee q$  and  $p \wedge q$  for the supremum and infimum, respectively, of  $p$  and  $q$  in the  $\sigma$ -OML  $P$ . More generally, if  $Q \subseteq P$ , we use the notation  $\bigvee Q$  (respectively,  $\bigwedge Q$ ) for an existing supremum (respectively, infimum) of  $Q$  in  $P$ . By definition,  $p$  is *orthogonal to*  $q$ , in symbols  $p \perp q$  iff  $p \leq 1 - q$ , i.e., iff  $p + q \in P$ . By [8, Lemma 5.5 (iii)],  $p \perp q$  iff  $pq = 0$ , in which case  $p + q = p \vee q$ . Recall that elements  $p, q \in P$  are called *Mackey compatible* iff there are pairwise orthogonal elements  $p_1, q_1, r \in P$  such that  $p = p_1 \vee r$  and  $q = q_1 \vee r$ . By [8, Lemma 5.5(v)],  $p$  and  $q$  are Mackey compatible iff  $pCq$ .

## 2. Carrier projection, square root, absolute value, positive part, and norm

By [8, Lemma 3.2 (i)], if  $g, h \in G$ , then  $gh = 0 \iff hg = 0$ , so it is not necessary to distinguish between left and right annihilation for elements of  $G$ . If  $g \in G$ , then by [8, Theorem 5.2], there is a uniquely determined projection  $g^\circ \in P$ , called the *carrier projection of (or for)  $g$* , such that for all  $h \in G$ ,  $gh = 0 \iff g^\circ h = 0$ ; moreover,  $g^\circ \in CC(g)$ . The carrier projection  $g^\circ$  of  $g$ , which enables us to deal efficiently with the question of which elements  $h \in G$  annihilate  $g$ , can be characterized as the smallest projection  $p \in P$  such that  $g = pg = gp$  ([6, Lemma 3.4 (i)]).

By [8, Theorem 4.5], if  $0 \leq g \in G$ , there exists a unique element in  $G$ , called the *square root of  $g$*  and denoted by  $g^{1/2}$ , such that  $0 \leq g^{1/2}$  and  $(g^{1/2})^2 = g$ ; moreover,  $g^{1/2} \in CC(g)$ . If  $g, h \in G$ ,  $gCh$ , and  $0 \leq g \leq h$ , then  $g^2 \leq h^2$  and  $g^{1/2} \leq h^{1/2}$  ([8, Lemma 4.8]). If  $g \in G$ , then  $0 \leq g^2$ , and the *absolute value of  $g$*  is defined by  $|g| := (g^2)^{1/2}$ .

Let  $g \in G$ . We denote and define the *positive part of  $g$*  by  $g^+ := \frac{1}{2}(|g| + g)$ . By [8, Theorem 5.7 (xiii)], we have  $0 \leq g^+$ , and it is clear that  $g^+ \in CC(g)$ . We define  $g^- := (-g)^+$ , and note that  $0 \leq g^- \in CC(g)$ . Also, since  $|-g| = |g|$  we have  $g^- = \frac{1}{2}(|g| - g)$ , whence  $g = g^+ - g^-$ . By [8, Corollary 5.8],  $g^+$  and  $g^-$  are uniquely determined by the three properties  $g = g^+ - g^-$ ,  $g^+g^- = 0$ , and  $0 \leq g^+ + g^-$ .

We define the *signum of  $g \in G$*  by  $s := (g^+)^\circ - (g^-)^\circ$ . As  $g^+ \in CC(g)$  and  $(g^+)^\circ \in CC(g^+)$ , it follows that  $(g^+)^\circ \in CC(g)$ ; likewise,  $(g^-)^\circ \in CC(g)$ , and consequently  $s \in CC(g)$ . Moreover,  $g^\circ = s^2$ ,  $|g| = sg = gs$ , and  $g = s|g| = |g|s$ , the latter equation being called the *polar decomposition of  $g$*  ([8, Theorem 6.3]).

**THEOREM 2.1.** *Let  $H$  be a GH-subalgebra of  $G$ . Then  $H$  is a (real) vector subspace of  $G$ . Moreover, if  $h, k \in H$ , then  $0 \leq k \implies k^{1/2} \in H$  and  $|h|, h^+, h^-, h^\circ \in H$ .*

**Proof.** By Theorem 1.4,  $H$  is a GH-algebra in its own right, hence by Theorem 1.2,  $H$  is a vector space over  $\mathbb{R}$ . It is clear that  $H$  is a rational vector subspace of  $G$ , and since each real number is the supremum of an ascending sequence of rational numbers, it follows that  $H$  is a real vector subspace of  $G$ . If  $0 \leq k \in H$ , then  $k$  has a square root  $r \in H$ , and by the uniqueness of square roots,  $k^{1/2} = r \in H$ . Let  $h \in H$ . Then  $h^2 \in H$ , whence  $|h| = (h^2)^{1/2} \in H$ , and consequently  $h^+ = \frac{1}{2}(|h| + h)$ ,  $h^- = \frac{1}{2}(|h| - h) \in H$ . By duality,  $H$  is closed under the formation of existing infima in  $G$  of descending sequences in  $H$ ; hence it follows from the construction of  $h^\circ$  as in [8, Lemma 5.1 and Theorem 5.2] that  $h^\circ \in H$ .  $\square$

We omit the straightforward verification of the following theorem (see [7, Theorem 2.3]).

**THEOREM 2.2.** *If  $v \in P$ , then*

$$vGv = \{vgv : g \in G\} = \{h \in G : h = vh = hv\} = \{h \in G : h^\circ \leq v\}$$

*is again a GH-algebra<sup>5</sup> with enveloping ring  $vRv$ . Suppose that  $h \in vGv$ . Then  $0 \leq h \implies h^{1/2} \in vGv$ ; also  $h^\circ, h^+, |h| \in vGv$  and they are respectively the carrier, the positive part, and the absolute value of  $h$  as calculated in  $vGv$ .*

Since  $G$  is an order-unit space with order unit 1, it is a normed real vector space under the *order-unit norm*  $\|\cdot\|$  defined for every  $g \in G$  by

$$\|g\| = \inf\{\lambda \in \mathbb{R} : 0 < \lambda \text{ and } -\lambda \leq g \leq \lambda\}$$

([1, Proposition II.1.2]). Let  $g, h \in G$ . Then

$$-\|g\| \leq g \leq \|g\| \text{ and } -h \leq g \leq h \implies \|g\| \leq \|h\|$$

([1, Proposition II.1.2], [11, Proposition 7.12 (c)]). If  $0 \neq p \in P$ , then  $\|p\| = 1$  and  $\|pgp\| \leq 1$  ([8, Theorem 6.4 (iv) and (v)]); moreover,  $\|g^2\| = \|g\|^2$  and  $\|\frac{1}{2}(gh + hg)\| \leq \|g\|\|h\|$  ([8, Theorem 6.4 (iii) and (viii)]).

### 3. Spectral resolution and the spectrum

In [8], we showed that  $G$  is a so-called *spectral order-unit space* ([7]) and that as a consequence, the spectral theory developed in [7, Sections 3–5] is available,

<sup>5</sup>If  $v = 0$ , then  $vGv = \{0\}$  is a “degenerate” GH-algebra in which the unity is 0.

*mutatis mutandis*, for  $G$ . Here we briefly review some of the basic definitions and results.

If  $g \in G$  and  $\lambda \in \mathbb{R}$ , we define the projections

$$p_\lambda := 1 - ((g - \lambda)^+)^{\circ} \quad \text{and} \quad d_\lambda := 1 - (g - \lambda)^{\circ}.$$

The family  $(p_\lambda)_{\lambda \in \mathbb{R}}$  is called the *spectral resolution* of  $g$ , the projection  $d_\lambda$  is called the  $\lambda$ -*eigenprojection* for  $g$ , and  $\lambda$  is an *eigenvalue* of  $g$  iff  $d_\lambda \neq 0$ . We also define the *lower* and *upper spectral bounds* for  $g$  by

$$L := \sup\{\lambda \in \mathbb{R} : \lambda \leq g\} \quad \text{and} \quad U := \inf\{\lambda \in \mathbb{R} : g \leq \lambda\}, \quad \text{respectively.}$$

By [7, Theorem 3.1],  $-\infty < L \leq U < \infty$  and  $\|g\| = \max\{|L|, |U|\}$ .

**STANDING ASSUMPTION 3.1.** In what follows, we assume that  $g \in G$ ,  $(p_\lambda)_{\lambda \in \mathbb{R}}$  is the spectral resolution of  $g$ ,  $(d_\lambda)_{\lambda \in \mathbb{R}}$  is the family of eigenprojections for  $g$ , and the lower and upper spectral bounds for  $g$  are  $L$  and  $U$ , respectively.

In [8, Section 7], it is shown that  $(p_\lambda)_{\lambda \in \mathbb{R}}$  and  $(d_\lambda)_{\lambda \in \mathbb{R}}$  have the expected properties. For instance,  $p_\lambda, d_\lambda \in CC(g)$  and  $g$  can be written as a norm-convergent Riemann-Stieltjes type integral  $g = \int_{L-0}^U \lambda dp_\lambda$ .

For our purposes in this article, the following two theorems (see [8, Theorems 7.4, 7.6]) will be useful.

**THEOREM 3.2.** *There exists an ascending sequence  $g_1 \leq g_2 \leq \dots$  in  $CC(g)$  such that each  $g_n$  is a finite linear combination of projections in the family  $(p_\lambda)_{\lambda \in \mathbb{R}}$  and  $g_n \rightarrow g$  in norm. Moreover,  $g$  is the supremum of  $(g_n)_{n \in \mathbb{N}}$  in  $G$  and  $g \in CC(\{g_1, g_2, \dots\})$ .*

**THEOREM 3.3.** *Let  $g, h \in G$  and let  $A \subseteq G$ . Then:*

- (i)  $h C g \iff h C p_\lambda$  for all  $\lambda \in \mathbb{R}$ .
- (ii)  $g C h$  iff every projection in the spectral resolution of  $g$  commutes with every projection in the spectral resolution of  $h$ .
- (iii)  $C(C(A) \cap P) = CC(A)$ .

**DEFINITION 3.4.** Let  $\alpha, \rho \in \mathbb{R}$ . We say that  $\rho$  belongs to the *resolvent set* of  $a$  iff there exists  $0 < \varepsilon \in \mathbb{R}$  such that  $p_\lambda$  is constant for  $\lambda$  in the open interval  $(\rho - \varepsilon, \rho + \varepsilon)$ . The *spectrum* of  $a$ , in symbols,  $\text{spec}(a)$ , is defined to be the complement in  $\mathbb{R}$  of the resolvent set of  $a$ .

In the following theorem we collect a few useful facts about  $\text{spec}(g)$  (see [8, Theorem 7.9] and [7, Theorem 4.2]).

**THEOREM 3.5.**

- (i) If  $\gamma, \mu \in \mathbb{R}$ , then  $\text{spec}(\gamma g + \mu) = \{\gamma\alpha + \mu : \alpha \in \text{spec}(g)\}$ .
- (ii)  $\text{spec}(g)$  is a closed nonempty subset of the closed interval  $[L, U] \subseteq \mathbb{R}$ .
- (iii)  $L = \inf(\text{spec}(g)) \in \text{spec}(g)$ ,  $U = \sup(\text{spec}(g)) \in \text{spec}(g)$ .
- (iv)  $\|g\| = \sup\{|\alpha| : \alpha \in \text{spec}(g)\}$ .
- (v)  $0 \leq g \iff \text{spec}(g) \subseteq [0, \infty)$ .
- (vi) Every isolated point of  $\text{spec}(a)$  is an eigenvalue of  $a$  and every eigenvalue of  $a$  belongs to  $\text{spec}(a)$ .

#### 4. Invertible, regular, and simple elements

We maintain the notation in Standing Assumption 3.1. As usual,  $g$  is said to be *invertible* (in  $G$ ) iff there is an element  $h \in G$  such that  $gh = hg = 1$ . If such an  $h$  exists, it is unique, it is called the *inverse* of  $g$ , and it is written as  $g^{-1} := h$ .

**LEMMA 4.1.** *Let  $g \in G$  with  $0 \leq g$ . Then  $g$  is invertible iff there exists  $0 < \varepsilon \in \mathbb{R}$  such that  $\varepsilon \leq g$ . Moreover, if  $g$  is invertible, then  $0 \leq g^{-1} \in CC(g)$ .*

**Proof.** The proof of [4, Lemma 7.1] goes through *verbatim* as it requires only condition (vii) in Definition 1.1 rather than the stronger Vigier condition; hence  $g$  is invertible iff there exists  $M \in \mathbb{N}$  such that  $1 \leq Mg$ , i.e.,  $0 < 1/M \leq g$ . Conversely, if  $0 < \varepsilon \in \mathbb{R}$  with  $\varepsilon < g$ , then  $1 < Mg$  for any  $M \in \mathbb{N}$  with  $1/\varepsilon \leq M$ .  $\square$

**COROLLARY 4.2.** *If  $g \in G$ , then  $1 + g^2$  is invertible.*

**THEOREM 4.3.** *Let  $g \in G$ . Then the following conditions are mutually equivalent:*

- (i)  $g$  is invertible.
- (ii)  $|g|$  is invertible.
- (iii) There exists  $0 < \varepsilon \in \mathbb{R}$  such that  $\varepsilon \leq |g|$ . Moreover, if  $g^{-1}$  exists, then  $g^{-1} \in CC(g)$  and the signum  $s$  of  $g$  satisfies  $s^2 = g^o = 1$ .

**Proof.** Let  $s$  be the signum of  $g$ . As  $s \in CC(g)$  and  $|g| \in CC(g)$ , the desired equivalences follow from Lemma 4.1 and the obvious facts that if  $g^{-1}$  exists, then  $|g|^{-1} = sg^{-1}$ , and if  $|g|^{-1}$  exists, then  $g^{-1} = s|g|^{-1}$ . Also, if  $g^{-1}$  exists, it is clear that if  $h \in G$ , then  $gh = 0 \iff h = 0$ , so  $g^o = 1$ , and therefore  $s^2 = g^o = 1$ .  $\square$



**DEFINITION 4.4.**

- (i)  $g$  is *von Neumann regular* iff there exists  $k \in G$  such that  $gk, kg \in G$  and  $gkg = g$ .
- (ii)  $g$  is *regular* iff there exists  $0 < \varepsilon \in \mathbb{R}$  such that  $\varepsilon g^\circ \leq |g|$  (cf. [7, Definition 4.2, Lemma 4.2]).

**THEOREM 4.5.** *The following conditions are mutually equivalent:*

- (i)  $g$  is *von Neumann regular*.
- (ii) *There exists  $r \in g^\circ G g^\circ$  such that  $gr = rg = g^\circ$ .*
- (iii)  *$g$  is invertible in the GH-algebra  $g^\circ G g^\circ$ .*
- (iv)  $g$  is *regular*.

**Proof.** We note that  $g = gg^\circ = g^\circ g \in g^\circ G g^\circ$ .

(i)  $\iff$  (ii). Assume (i). Then there exists  $k \in G$  with  $kg, gk \in G$  and  $gkg = g$ , whence  $(kg)^2 = kgkg = kg$ , so  $p := kg \in P$  and  $g = gp$ . For all  $h \in G$ , we have  $gh = 0 \implies kgh = 0 \implies ph = 0 \implies gph = 0 \implies gh = 0$ , i.e.,  $gh = 0 \iff ph = 0$ , whence  $p = g^\circ$ . We also have  $gk \in G$  with  $(gk)^2 = gkgk = gk$ , so  $q := gk \in P$  and  $g = qg$ . Arguing as above, we find that  $hg = 0 \iff hq = 0$ , whereupon  $gk = q = g^\circ = p = kg$ . Consequently,  $g^\circ C k$ , and with  $r := g^\circ k = kg^\circ$ , we have  $r \in g^\circ G g^\circ$  with  $gr = g^\circ = rg$ , i.e., (ii) holds, proving that (i)  $\implies$  (ii). Conversely, if (ii) holds, then  $grg = g^\circ g = g$  with  $gr = rg \in G$ , so (i) holds.

(ii)  $\iff$  (iii). Evidently,  $h$  satisfies the conditions in (ii) iff  $h$  is the inverse of  $g$  in  $g^\circ G g^\circ$ .

(iii)  $\iff$  (iv). Since  $g^\circ$  is the unit element in  $g^\circ G g^\circ$ , the equivalence of (iii) and (iv) follows from Theorem 4.3.  $\square$

**COROLLARY 4.6.** *If  $g \in G$ , then  $g$  is invertible iff  $g$  is regular and  $g^\circ = 1$ .*

Translating [7, Definition 4.2] into our present context, we find that the element  $g$  is *nonsingular* iff it is regular and  $g^\circ = 1$ ; hence by Corollary 4.6,  $g$  is nonsingular iff it is invertible. If  $g$  is regular, then the (necessarily unique) inverse of  $g$  in  $g^\circ G g^\circ$  (Theorem 4.5) is called the *pseudo-inverse* of  $g$  in  $G$ . If  $g$  is regular, it is not difficult to show that the pseudo-inverse of  $g$  belongs to  $CC(g)$ .

By [7, Theorem 4.6], we have the following.

**THEOREM 4.7.**  *$g$  is regular iff both  $g^+$  and  $g^-$  are regular.*

**COROLLARY 4.8.**  *$g$  is invertible iff  $g^\circ = 1$  and both  $g^+$  and  $g^-$  are regular.*

By [7, Theorem 4.7], we have the following.

**THEOREM 4.9.** *If  $\lambda \in \mathbb{R}$ , then:*

- (i)  $g - \lambda$  is regular iff either  $\lambda$  belongs to the resolvent set of  $a$  or else  $\lambda$  is an isolated point of  $\text{spec}(a)$ .
- (ii)  $\text{spec}(a) = \{\lambda \in \mathbb{R} : a - \lambda \text{ fails to be invertible}\}$ .

**COROLLARY 4.10.** *Let  $G$  and  $H$  be  $GH$ -algebras, let  $\phi: G \rightarrow H$  be a (real) vector-space isomorphism of  $G$  onto  $H$ , and suppose that, for all  $a, b \in G$ ,  $aCb \iff \phi(a)C\phi(b)$  and  $aCb \implies \phi(ab) = \phi(a)\phi(b)$ . Then  $\phi$  is both an order isomorphism and an isometry.*

**Proof.** Assume the hypotheses. Clearly,  $\phi(1) = 1$ . If  $g \in G$ , let  $\text{spec}_G(g)$  be the spectrum of  $g$  in  $G$ , and let  $\text{spec}_H(\phi(g))$  be the spectrum of  $\phi(g)$  in  $H$ . If  $\lambda \in \mathbb{R}$ , then  $g - \lambda$  is invertible in  $G$  iff  $\phi(g - \lambda) = \phi(g) - \lambda$  is invertible in  $H$ ; hence  $\text{spec}_G(g) = \text{spec}_H(\phi(g))$  by Theorem 4.9 (ii). By Theorem 3.5 (iv),  $\|g\| = \sup\{|\alpha| : \alpha \in \text{spec}_G(g)\} = \sup\{|\alpha| : \alpha \in \text{spec}_H(\phi(g))\} = \|\phi(g)\|$ , so  $\phi$  is an isometry. Also by Theorem 3.5 (v),  $0 \leq g$  iff  $0 \leq \phi(g)$  for all  $g \in G$ , and since  $\phi$  is an additive isomorphism, both  $\phi$  and  $\phi^{-1}$  are order preserving.  $\square$

**DEFINITION 4.11.** An element in  $G$  is *simple* iff it can be written as a finite linear combination of pairwise commuting projections.

**Remark 4.12.** By Theorem 3.2, each element in  $G$  is a norm limit of an ascending sequence of simple elements; hence, the simple elements are norm dense in  $G$ .

The following theorem is a consequence of [7, Theorem 5.3].

**THEOREM 4.13.** *The simple elements in  $G$  are precisely those with finite spectrum, and each simple element  $g \in G$  has a unique representation*

$$g = \sum_{i=1}^n \alpha_i u_i \text{ with } \alpha_1 < \alpha_2 < \dots < \alpha_n \text{ in } \mathbb{R}, \quad 0 \neq u_i \in P, \quad \text{and} \quad \sum_{i=1}^n u_i = 1.$$

Moreover:

- (i)  $\text{spec}(g) = \{\alpha_i : i = 1, 2, \dots, n\}$ .
- (ii)  $u_i = d_{\alpha_i}$  for  $i = 1, 2, \dots, n$ .
- (iii)  $|g| = \sum_{i=1}^n |\alpha_i| u_i$ .
- (iv)  $\|g\| = \max\{|\alpha_i| : i = 1, 2, \dots, n\}$ .

If  $g$  is simple, we shall refer to the representation  $g = \sum_{i=1}^n \alpha_i u_i$  in Theorem 4.13 as the *spectral representation* of  $g$ .

**COROLLARY 4.14.** *Suppose that  $g$  is simple and  $g = \sum_{i=1}^n \alpha_i u_i$  is the spectral representation of  $g$ . Then  $g$  is regular, and the pseudo-inverse of  $g$  is  $r = \sum_{i=1}^n \beta_i u_i$  where  $\beta_i = 0$  if  $\alpha_i = 0$  and  $\beta_i = \alpha_i^{-1}$  if  $\alpha_i \neq 0$ .*

**COROLLARY 4.15.**  *$g$  is a projection iff  $\text{spec}(g) \subseteq \{0, 1\}$ .*

See [7, Theorem 5.4 and Corollary 5.2] for proofs of the following theorem and corollary.

**THEOREM 4.16.** *The following conditions are mutually equivalent:*

- (i)  $g$  is simple.
- (ii)  $g - \lambda$  is regular for all  $\lambda \in \mathbb{R}$ .
- (iii)  $\text{spec}(g)$  consists entirely of isolated points.
- (iv)  $\text{spec}(g)$  is finite.
- (v)  $\{p_\lambda : \lambda \in \mathbb{R}\}$  is a finite chain in the OML  $P$ .

**COROLLARY 4.17.**

- (i) *If  $G$  is finite dimensional, then  $P$  satisfies the chain conditions (i.e., there are no infinite properly ascending or descending sequences in  $P$ ).*
- (ii) *If  $P$  satisfies the chain conditions, then every element in  $G$  is simple, hence every element in  $G$  is regular.*

**Remarks 4.18.** By Corollary 4.17, if  $G$  is finite dimensional, then  $P$  satisfies chain conditions and  $G$  is regular (i.e., every element in  $G$  is regular). However, the converse fails as there exist regular Banach GH-algebras with projection lattices that satisfy the chain conditions but that are not finite dimensional, e.g., infinite dimensional *spin factors* ([18, §19]). By contrast, it is known that a regular Banach algebra must be finite dimensional ([14]).

## 5. C-Blocks and the commutative case

A subset  $B$  of  $P$  is called a *block* of  $P$  if  $B$  is a maximal set of pairwise Mackey compatible elements ([13, Ch. 1, §4]). As two projections are Mackey compatible iff they commute, it is clear that  $B \subseteq P$  is a block of  $P$  iff  $B = C(B) \cap P$ . As  $P$  is a  $\sigma$ -OML, it is well-known that every block in  $P$  is a maximal Boolean  $\sigma$ -subalgebra of  $P$ . Following [5, Def. 5.1], a subgroup of  $G$  having the form  $C(B)$ , where  $B$  is a block in  $P$ , will be called a *C-block* in  $G$ .

**THEOREM 5.1.** *A subset  $H$  of  $G$  is a  $C$ -block of  $G$  iff  $H$  is a maximal set of pairwise commuting elements of  $G$ .*

**Proof.** If  $H \subseteq G$ , it is clear that  $H$  is a maximal set of pairwise commuting elements of  $G$  iff  $H = C(H)$ . Suppose  $H = C(B)$  for some block  $B = C(B) \cap P$  of  $P$ . Then by Theorem 3.3 (iii),  $H = C(B) = C(C(B) \cap P) = CC(B) = C(H)$ . Conversely, suppose  $H = C(H)$  and put  $B := H \cap P = C(H) \cap P$ . Then  $CC(H) = C(H) = H$  and, again by Theorem 3.3 (iii),  $B = H \cap P = CC(H) \cap P = C(C(H) \cap P) \cap P = C(B) \cap P$ .  $\square$

**COROLLARY 5.2.**  *$G$  is covered by its own  $C$ -blocks and the center  $C(G)$  is the intersection of all the  $C$ -blocks in  $G$ .*

Let  $\mathcal{A}$  be a linear algebra over  $\mathbb{R}$ . We say that  $\mathcal{A}$  is a *partially ordered linear algebra* iff the additive group of  $\mathcal{A}$  is a partially ordered abelian group, and whenever  $0 \leq a, b \in \mathcal{A}$  and  $0 \leq \lambda \in \mathbb{R}$ , we have  $0 \leq ab$  and  $0 \leq \lambda a$ . If a partially ordered linear algebra  $\mathcal{A}$  is a lattice, it is called an  $\ell$ -algebra ([9]).

**THEOREM 5.3.** *Let  $H$  be a  $C$ -block in  $G$ . Then  $H$  is a  $GH$ -subalgebra of  $G$ , and as a  $GH$ -algebra in its own right,  $H$  has the following properties:*

- (i)  $H$  is monotone  $\sigma$ -complete.
- (ii)  $H$  is a commutative and associative real Banach algebra with unity element 1.
- (iii) If  $h, k \in H$ , then  $|h|, h^+, h^-, h^\circ \in H$  and  $0 \leq k \implies k^{1/2} \in H$ .
- (iv)  $H$  is an archimedean Dedekind  $\sigma$ -complete  $\ell$ -algebra with order unit 1.
- (v) If  $g, h \in H$ , then the infimum and supremum of  $g$  and  $h$  in  $H$  are given by  $g \wedge_H h = g - (g - h)^+$  and  $g \vee_H h = g + (h - g)^+$ .
- (vi) If  $h \in H$ , then the spectral resolution and the family of eigenprojections of  $h$  are the same whether calculated in  $G$  or in  $H$ .

**Proof.** We have  $H = C(H)$ , so  $H$  is a  $GH$ -subalgebra of  $G$  by Lemma 1.6. As  $H$  satisfies condition (vii) in Definition 1.1 and the elements of  $H$  commute with each other, it is clear that (i) holds. If  $g, h \in H$ , then  $gh = hg \in G$ , and as  $g, h \in C(H)$ , it follows that  $gh \in C(H) = H$ . Therefore,  $H$  is a commutative and associative linear algebra with unity 1 over  $\mathbb{R}$ , and by (i) and [8, Theorem 6.5], it is a Banach algebra, proving (ii). Part (iii) follows from Theorem 2.1.

Obviously,  $H$  is an archimedean partially ordered algebra over  $\mathbb{R}$  and 1 is an order unit in  $H$ . To prove that  $H$  is a lattice, let  $g, h \in H$  and put  $p := ((g - h)^+)^{\circ}$ . Then by (iii),  $p \in H \cap P$ , and by [8, Theorem 5.7],  $(1 - p)(g - h) \leq 0 \leq p(g - h)$  with  $(1 - p)(g - h), p(g - h) \in H$ . Put  $a := ph + (1 - p)g$ . Then  $a \in H$  and  $a \leq g, h$ . Suppose  $b \in H$  and  $b \leq g, h$ . Then  $pb \leq ph$  and  $(1 - p)b \leq (1 - p)g$ , so  $b \leq pb + (1 - p)b \leq a$ . Thus  $a$  is the infimum of  $g$  and  $h$  in  $H$ . The existence

of the supremum of  $g$  and  $h$  in  $H$  is shown dually, hence  $H$  is a lattice. As  $H$  is monotone  $\sigma$ -complete, it is Dedekind  $\sigma$ -complete by [11, Lemma 16.7], and (iv) is proved.

Let  $g, h \in H$ . As per [3, Definition 5.2], the pseudo-meet  $g \sqcap h$  and pseudo-join  $g \sqcup h$  are defined by  $g \sqcap h := g - (g - h)^+$ ,  $g \sqcup h := g + (h - g)^+$ . By (i),  $g \sqcap h, g \sqcup h \in H$ , and by [3, Theorem 5.4 (iv)],  $g \wedge_H h = g \sqcap h$  and  $g \vee_H h = g \sqcup h$ , proving (v).

Part (vi) follows from (iii), the definition of the spectral resolution and the definition of an eigenprojection.  $\square$

**COROLLARY 5.4.** *Suppose that  $H_1$  and  $H_2$  are C-blocks in  $G$  and that  $g, h \in H_1 \cap H_2$ . Then the infimum and supremum of  $g$  and  $h$  as calculated in  $H_1$  are the same as the infimum and supremum of  $g$  and  $h$  as calculated in  $H_2$ .*

We now turn our attention to the case in which  $G$  itself is commutative, i.e.,  $G = C(G)$ .

**THEOREM 5.5.** *If  $G$  is commutative, then:*

- (i)  $P$  is a Boolean  $\sigma$ -algebra.
- (ii)  $G$  is monotone  $\sigma$ -complete.
- (iii) Under the norm  $\|\cdot\|$ ,  $G$  is a commutative and associative real Banach algebra with unity element 1.
- (iv)  $G$  is an archimedean Dedekind  $\sigma$ -complete  $\ell$ -algebra with order unit 1.
- (v)  $g \in G \implies \|g^2\| = \|g\|^2$ .
- (vi)  $g \in G \implies 1 + g^2$  is invertible in  $G$ .

**Proof.** As  $G$  is commutative, it is a C-block in itself; hence parts (i)–(iv) follow from Theorems 5.1 and 5.3. Parts (v) and (vi) follow from [8, Theorem 6.4 (iii)] and Corollary 4.2.  $\square$

**THEOREM 5.6.** *The following conditions are mutually equivalent:*

- (i)  $G$  is commutative.
- (ii)  $G$  is lattice ordered.
- (iii)  $G$  is an interpolation group.<sup>6</sup>
- (iv)  $P$  is a Boolean  $\sigma$ -algebra.
- (v) If  $g, h \in G$ , then  $-h \leq g \leq h \iff |g| \leq h$ .
- (vi) If  $g, h \in G$ , then  $|g + h| \leq |g| + |h|$ .

<sup>6</sup>Recall that  $G$  is an interpolation group iff, whenever  $a, b, c, d \in G$  with  $a, b \leq c, d$ , there exists  $t \in G$  such that  $a, b \leq t \leq c, d$  ([11, page 23]).

**P r o o f.** That (i)  $\implies$  (ii) follows from Theorem 5.5, and (ii)  $\implies$  (iii) is obvious. That (iii)  $\implies$  (iv) follows from [11, Theorem 8.7]. If (iv) holds, then the projections in  $P$  are pairwise Mackey compatible, hence they commute pairwise, and (i) follows from Theorem 3.3 (i). Thus we have the mutual equivalence of (i)–(iv). The equivalence of (v) and (vi) with each other and with the remaining conditions follows from [3, Theorem 5.5].  $\square$

Let  $X$  be a compact Hausdorff space. Clearly, the partially ordered commutative Banach algebra  $C(X, \mathbb{R})$  of all continuous functions  $f: X \rightarrow \mathbb{R}$  satisfies axioms (i)–(vi). The set  $P(X, \mathbb{R})$  of idempotents in  $C(X, \mathbb{R})$  consists of all characteristic set functions (indicator functions)  $\chi_K$  of compact open subsets  $K$  of  $X$ . Thus, with the partial order induced from  $C(X, \mathbb{R})$ , and with  $p \mapsto 1 - p$  as the complementation mapping,  $P(X, \mathbb{R})$  is a Boolean algebra, and the field  $\mathcal{S}$  of compact open subsets of  $X$  is isomorphic as a Boolean algebra to  $P(X, \mathbb{R})$  under the mapping  $K \mapsto \chi_K$ . Consequently,  $X$  may be identified with the Stone space of  $P(X, \mathbb{R})$  iff  $X$  is totally disconnected. As is well-known ([17, §22.4]), the basically disconnected compact Hausdorff spaces are precisely the Stone spaces of Boolean  $\sigma$ -algebras.

**THEOREM 5.7.** *If  $X$  is a compact Hausdorff space, then the following conditions are mutually equivalent:*

- (i)  $C(X, \mathbb{R})$  is a commutative GH-algebra.
- (ii)  $C(X, \mathbb{R})$  is monotone  $\sigma$ -complete.<sup>7</sup>
- (iii)  $C(X, \mathbb{R})$  is Dedekind  $\sigma$ -complete.
- (iv)  $C(X, \mathbb{R})$  has both the comparability and the Rickart projection properties.<sup>8</sup>
- (v)  $X$  is basically disconnected. Moreover, if any, hence all of these conditions hold, then  $P(X, \mathbb{R})$  is a Boolean  $\sigma$ -algebra and  $X$  is the Stone space of  $P(X, \mathbb{R})$ .

**P r o o f.** As  $C(X, \mathbb{R})$  is commutative, condition (vii) in Definition 1.1 is satisfied iff  $G(X, \mathbb{R})$  is monotone  $\sigma$ -complete; hence (i)  $\iff$  (ii). That (ii)  $\implies$  (iii) follows from [11, Lemma 16.7], and (iii)  $\implies$  (ii) is obvious, whence (ii)  $\iff$  (iii). Therefore, by [5, Theorem 4.8], conditions (i)–(v) are mutually equivalent. If  $X$  is basically disconnected, then the field  $\mathcal{S}$  of compact open subsets of  $X$  is a Boolean  $\sigma$ -algebra, and  $K \mapsto \chi_K$  is a Boolean-algebra isomorphism of  $\mathcal{S}$  onto  $P(X, \mathbb{R})$ ; hence  $X$  is the Stone space of  $P(X, \mathbb{R})$ .  $\square$

<sup>7</sup>N.B. If  $C(X, \mathbb{R})$  is monotone  $\sigma$ -complete, then the supremum  $f$  in  $C(X, \mathbb{R})$  of an ascending sequence  $f_1 \leq f_2 \leq \dots$  of functions in  $C(X, \mathbb{R})$  is not necessarily the pointwise supremum (see the proof of [11, Theorem 9.2]).

<sup>8</sup>See [8, Remark 6.1].

A real  $C^*$ -algebra ([10, page 63]) is a real Banach  $*$ -algebra  $A$  such that  $\|x^*x\| = \|x\|^2$  and  $1+x^*x$  is invertible in  $A$  for all  $x \in A$ .<sup>9</sup> If  $G$  is a commutative GH-algebra, then by Theorem 5.5,  $G$  is a commutative real  $C^*$ -algebra with the identity mapping  $g \mapsto g^* := g$  as the involution. Consequently, by [10, Theorem 11.5], there is a compact Hausdorff space  $X$  and a  $C^*$ -isomorphism of  $G$  onto  $C(X, \mathbb{R})$ . A simple and illuminating alternative proof of this result (Theorem 5.9 below) follows easily from the previous developments in this article.

**LEMMA 5.8.** *Let  $G$  be a commutative GH-algebra and let  $X$  be the Stone space of the Boolean  $\sigma$ -algebra  $P$  of projections in  $G$ . Then  $X$  is basically disconnected,  $C(X, \mathbb{R})$  is a commutative GH-algebra, and there is a Boolean isomorphism  $p \mapsto K_p$  of  $P$  onto the field of compact open subsets of  $X$ . Let  $G_0$  be the set of simple elements in  $G$ , and let  $C(X, \mathbb{R})_0$  be the set of simple elements in  $C(X, \mathbb{R})$ . Then  $G_0$  is a subalgebra of  $G$ ,  $C(X, \mathbb{R})_0$  is a subalgebra of  $C(X, \mathbb{R})$ , and there is a uniquely determined linear algebra isomorphism  $\nu: G_0 \rightarrow C(X, \mathbb{R})_0$  of  $G_0$  onto  $C(X, \mathbb{R})_0$  such that  $\nu(p)$  is the characteristic set function of  $K_p$  for every  $p \in P$ . Moreover,  $\nu$  is an isometry.*

**Proof.** The existence of a Boolean isomorphism  $p \mapsto K_p$  follows from the assumption that  $X$  is the Stone space of  $P$ . If  $p \in P$ , define  $\nu(p) \in C(X, \mathbb{R})$  to be the characteristic set function of  $K_p$ . Then  $p \mapsto \nu(p)$  is a Boolean isomorphism of  $P$  onto  $P(X, \mathbb{R})$ . If  $g \in G_0$  and  $g = \sum_{i=1}^n \alpha_i u_i$  is the spectral representation of  $g$ , define  $\nu(g) := \sum_{i=1}^n \alpha_i \nu(u_i) \in C(X, \mathbb{R})_0$ . Obviously,  $\nu: G_0 \rightarrow C(X, \mathbb{R})_0$  is a bijection.

Suppose that  $g, h \in G_0$  with spectral representations  $g = \sum_{i=1}^n \alpha_i u_i$  and  $h = \sum_{j=1}^m \beta_j v_j$ , respectively. With the understanding that  $i = 1, 2, \dots, n$  and  $j = 1, 2, \dots, m$ , we have  $u_i = \sum_j u_i v_j$ ,  $v_j = \sum_i u_i v_j$ , and as  $G$  is commutative,  $u_i v_j = u_i \wedge v_j \in P$ . Therefore,  $\sum'_{i,j} u_i v_j = 1$ , where  $\sum'_{i,j}$  denotes the sum over all  $i$  and  $j$  for which  $u_i v_j \neq 0$ . With the same notation, it is clear that  $g + h = \sum'_{i,j} (\alpha_i + \beta_j) u_i v_j$  and  $gh = \sum'_{i,j} (\alpha_i \beta_j) u_i v_j$ . By collecting terms with the same coefficients, reordering and relabeling as necessary, we obtain the spectral representations of  $g + h$  and of  $gh$ . Repeating the same procedures with  $\nu(g)$  and  $\nu(h)$ , we find that  $\nu(g + h) = \nu(g) + \nu(h)$  and  $\nu(gh) = \nu(g)\nu(h)$ , i.e.,  $\nu$  is a linear-algebra isomorphism of  $G_0$  onto  $C(X, \mathbb{R})_0$ . The uniqueness of  $\nu$  is obvious.

<sup>9</sup>Unlike the complex case, the invertibility of  $1 + x^*x$  does not follow from the other axioms.

If  $g = \sum_{i=1}^n \alpha_i u_i$  is the spectral representation of  $g \in G_0$ , then by Theorem 4.13 the spectrum of  $g$  in  $G$  is  $\{\alpha_i : i = 1, 2, \dots, n\}$  and  $\|g\| = \max\{|\alpha_i| : i = 1, 2, \dots, n\}$ . Likewise, the spectrum of  $\nu(g)$  in  $C(X, \mathbb{R})$  is  $\{\alpha_i : i = 1, 2, \dots, n\}$ , and it follows the  $\|g\| = \|\nu(g)\|$ .  $\square$

**THEOREM 5.9.** *Let  $G$  be a commutative GH-algebra and let  $X$  be the Stone space of the Boolean  $\sigma$ -algebra  $P$  of projections in  $G$ . Then  $X$  is a compact Hausdorff basically disconnected space,  $C(X, \mathbb{R})$  is a commutative GH-algebra, and there is a linear-algebra isomorphism from  $G$  onto  $C(X, \mathbb{R})$ . Moreover, both  $G$  and  $C(X, \mathbb{R})$  are real  $C^*$ -algebras, and any linear algebra isomorphism from  $G$  onto  $C(X, \mathbb{R})$  is both an isomorphism of  $C^*$ -algebras and an order isomorphism.*

**Proof.** With the notation of Lemma 5.8, there is an isometric linear isomorphism  $\nu$  of the subalgebra  $G_0$  of  $G$  onto the subalgebra  $C(X, \mathbb{R})_0$  of  $C(X, \mathbb{R})$ . By Remark 4.12,  $G_0$  is norm dense in the commutative Banach algebra  $G$  and  $C(X, \mathbb{R})_0$  is norm dense in the commutative Banach algebra  $C(X, \mathbb{R})$ ; hence there is a unique extension of  $\nu$  to a linear-algebra isomorphism  $\phi$  of  $G$  onto  $C(X, \mathbb{R})$ . By Corollary 4.10,  $\phi: G \rightarrow C(X, \mathbb{R})$  is an isometry, hence an isomorphism of real  $C^*$ -algebras, and it is also an order isomorphism.  $\square$

**COROLLARY 5.10.** *Let  $G$  be a commutative GH-algebra. Then the following conditions are mutually equivalent:*

- (i) *Every element  $g \in G$  is regular.*
- (ii)  *$G$  is finite dimensional.*
- (iii)  *$P$  satisfies the chain conditions.*
- (iv)  *$P$  is a finite Boolean algebra.*
- (v)  *$G$  can be represented as the partially ordered Banach algebra  $\mathbb{R}^X$ , with pointwise operations and order, of all  $\mathbb{R}$ -valued functions on a finite non-empty set  $X$ .*
- (vi) *Every element in  $G$  is simple.*

**Proof.** By [14], every regular Banach algebra is finite dimensional, hence (i)  $\implies$  (ii). Assume (ii) and suppose that  $p_1 < p_2 < \dots$  is a properly ascending infinite sequence in  $P$ . Then  $\{p_{n+1} - p_n : n \in \mathbb{N}\}$  is an infinite set of nonzero pairwise orthogonal projections, hence it is linearly independent, contradicting (ii) and proving that (ii)  $\implies$  (iii). A Boolean algebra satisfies the chain conditions iff it is finite, hence (iii)  $\iff$  (iv). If  $P$  is a finite Boolean algebra, then the Stone space  $X$  of  $P$  is finite, hence (iv)  $\implies$  (v) by Theorem 5.9. That (v)  $\implies$  (vi) is obvious, and (vi)  $\implies$  (i) by Corollary 4.14.  $\square$



**THEOREM 5.11.** *The following conditions are mutually equivalent:*

- (i) *Every element in  $G$  is regular.*
- (ii) *Every C-block in  $G$  is finite dimensional.*
- (iii) *The OML  $P$  satisfies the chain conditions.*
- (iv) *Every element  $g \in G$  is simple.*

**Proof.** Suppose (i) holds, let  $H$  be a C-block in  $G$ , let  $g \in H$ , and let  $h$  be the pseudo-inverse of  $g$  in  $G$ . Then  $h \in CC(g) \subseteq H$ , and therefore  $g$  is regular in the commutative GH-algebra  $H$ , and we have (i)  $\implies$  (ii) by Corollary 5.10. The projections in an ascending or descending sequence in  $P$  commute with each other, hence every such sequence in  $P$  is contained in at least one C-block in  $G$ . Thus, if (ii) holds, then the projections in every C-block in  $G$  satisfy the chain conditions by Corollary 5.10, hence (ii)  $\implies$  (iii). Suppose that (iii) holds and let  $g \in G$ . Then there is a C-block  $H$  in  $G$  with  $g \in H$ , and as the projections in  $H$  inherit the chain conditions from  $P$ , it follows from Corollary 5.10 that  $g$  is simple in the GH-algebra  $H$ , hence  $g$  is simple in  $G$ , and we have (iii)  $\implies$  (iv). Finally, (iv)  $\implies$  (i) by Corollary 4.14.  $\square$

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