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LATTICE OF RETRACTS OF MONOUNARY ALGEBRAS

Danica Jakubíková-Studenovská* — Jozef Pócs**

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ABSTRACT. We investigate lattices of retracts of monounary algebras. Semimodularity and concepts related to semimodularity (M-symmetry and Mac Lane's condition) are dealt with. Further, we give a description of all connected monounary algebras with modular retract lattice.

1. Introduction

The notion of retract was investigated in many areas of mathematics, first for topological spaces, later for algebraic structures as groups, lattices, posets etc. The importance of the notion of retract is well known and is commonly appreciated. Retracts of monounary algebras were first studied in [4]; this paper was inspired by the investigation of retracts of posets ([2]).

The investigation of monounary algebras has been shown to be useful tool for studying some questions concerning algebras of arbitrary type ([9]). Novotný [10] remarks that constructions of homomorphism of general algebras can be reduced to constructions of homomorphisms of monounary algebras. Similarly, it is possible to apply constructions of retracts of monounary algebras for obtaining all retracts of any algebra.

We investigate lattices of retracts of monounary algebras. A retract is a subalgebra of a given algebra A, though a lattice of retracts of A need not be a sublattice of a subalgebra lattice of A in general. A lot is known about subalgebra lattices of algebras, many authors investigated properties of subalgebra lattices of monounary and unary algebras, too (cf., e.g., [1], [11], [12]).

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The aim of the present paper is to study properties of retract lattices of monounary algebras. For a monounary algebra (A, f) we denote by $\mathbf{R}(A, f)$ the system of all retracts of (A, f) ordered by inclusion and by $\mathbf{R}^{\emptyset}(A, f)$ the system of all retracts enriched with the empty set. It is proved that $\mathbf{R}(A, f)$ is closed under arbitrary unions. Necessary and sufficient conditions under which $\mathbf{R}(A, f)$ contains a least element are described. The main goal of the paper is to investigate, for a lattice $\mathbf{R}^{\emptyset}(A, f)$, semimodularity, concepts related to semimodularity (M-symmetry and Mac Lane's condition) and modularity. We show that if (A, f) is a connected monounary algebra, then $\mathbf{R}^{\emptyset}(A, f)$ is M-symmetric, hence it is semimodular, but in general it need not fulfil Mac Lane's condition. A criterion for $\mathbf{R}^{\emptyset}(A, f)$ to be modular is proved, and the criterion yields that if $\mathbf{R}^{\emptyset}(A, f)$ is modular, then it is distributive, too. Moreover, we give a description of a connected monounary algebra with modular retract lattice via forbidden configuration of its elements.

2. Preliminaries

First we recall some basic notions. By a monounary algebra we understand a pair (A, f) where A is a nonempty set and $f\colon A \longrightarrow A$ is a mapping.

Let (A, f) be a monounary algebra. A nonempty subset M of A is said to be a *retract* of (A, f) if there is a mapping φ of A onto M such that φ is an endomorphism of (A, f) and $\varphi(x) = x$ for each $x \in M$. The mapping φ is then called a *retraction endomorphism* corresponding to the retract M.

A monounary algebra (A, f) is called *connected* if for arbitrary elements $x, y \in A$ there are non-negative integers n, m such that $f^n(x) = f^m(y)$. A maximal connected subalgebra of a monounary algebra is called a *(connected) component*.

An element $x \in A$ is referred to as cyclic, if there exists a positive integer n such that $f^n(x) = x$. In this case the set $\left\{x, f^1(x), f^2(x), \dots, f^{n-1}(x)\right\}$ is said to be a cycle.

For $X \subseteq A$ we denote by $[X]$ subalgeba generated by the set X. If $X = \{x\}$ we use the notation $[x]$ instead of $[\{x\}]$.

The notion of *degree* $s_f(x)$ of an element $x \in A$ was introduced in [7] (cf. also [8]) for describing homomorphisms of monounary algebras as follows. Let us denote by A^{∞} the set of all elements $x \in A$ such that there exists a sequence $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$ of elements belonging to A with the property $x_0 = x$ and $f(x_n) = x_{n-1}$ for each $n \in \mathbb{N}$. Further, we put $A^0 = \left\{x \in A : \ f^{-1}(x) = \emptyset\right\}$. Now we define a set $A^{\lambda} \subseteq A$ for each ordinal λ by induction. Assume that we have defined A^{α} for each ordinal $\alpha < \lambda$. Then we put

$$A^{\lambda} = \left\{ x \in A - \bigcup_{\alpha<\lambda} A^{(\alpha)} : \ f^{-1}(x) \subseteq \bigcup_{\alpha<\lambda} A^{(\alpha)} \right\}.$$

The sets A^{λ} are pairwise disjoint. For each $x \in A$, either $x \in A^{\infty}$ or there is an ordinal λ with $x \in A^{\lambda}$. In the former case we put $s_f(x) = \infty$, in the latter we set $s_f(x) = \lambda$. We put $\lambda < \infty$ for each ordinal λ.

Let (A, f) be a connected monounary algebra. We say that (A, f) is *unbounded*, if

(i) $s_f(x) \neq \infty$ for each $x \in A$,

(ii) if $x \in A$, $n \in \mathbb{N}$, then there is $m \in \mathbb{N}$ such that $f^{-(m+n)}(f^m(x)) \neq \emptyset$.

The connected monounary algebra (A, f) is said to be *bounded* if (A, f) satisfies (i) and does not fulfil (ii).

Further we will use the following notation for some algebras.

As usual, the symbols $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of all integers or of positive integers, respectively. We denote by $(\mathbb{Z}, \mathrm{suc})$ and $(\mathbb{N}, \mathrm{suc})$ the algebras such that suc is the operation of the successor.

In [4] the following theorem characterizing retracts of connected monounary algebras was proved.

Theorem 2.1. *Let (A, f) be a connected monounary algebra and let (M, f) be a subalgebra of (A, f). Then M is a retract of (A, f) if and only if the following condition is satisfied:*

If $y \in f^{-1}(M)$, then there is $z \in M$ with $f(y) = f(z)$ and $s_f(y) \leq s_f(z)$.

The next theorem characterizes retracts of monounary algebras in a general (non-connected) case.

Theorem 2.2. [4] *Let (A, f) be a monounary algebra and let (M, f) be a subalgebra of (A, f). Then M is a retract of (A, f) if and only if the following conditions are satisfied:*

(a) *If $y \in f^{-1}(M)$, then there is $z \in M$ such that $f(y) = f(z)$ and $s_f(y) \leq s_f(z)$.*

(b) *For any connected component K of (A, f) with $K \cap M = \emptyset$, the following conditions are satisfied:*

  (b1) *If K contains a cycle with d elements, then there is a connected component K' of (A, f) with $K' \cap M \neq \emptyset$ and there is $n \in \mathbb{N}$ such that n/d and K' has a cycle with n elements.*

  (b2) *If K contains no cycle and x_0 is a fixed element of K, then there is $y_0 \in M$ such that $s_f(f^k(x_0)) \leq s_f(f^k(y_0))$ for each $k \in \mathbb{N} \cup \{0\}$.*

Through this paper we will use the following notation: Let (A, f) be a connected monounary algebra. For a subset $B \subseteq A$ denote by

$$B^0 = \{x \in B : \ f^{-1}(x) = \emptyset\}.$$

If (A, f) contains no cycle, we denote by

$$B^{\mathbb{Z}} = \{X \subseteq B : \ (X, f) \cong (\mathbb{Z}, \mathrm{suc})\}.$$

Lemma 2.3. *Let (A, f) be a connected monounary algebra. If (B, f), (C, f) are subalgebras of (A, f), then*

(i) $(B \cup C)^{\mathbb{Z}} = B^{\mathbb{Z}} \cup C^{\mathbb{Z}}$, $(B \cap C)^{\mathbb{Z}} = B^{\mathbb{Z}} \cap C^{\mathbb{Z}}$,

(ii) $(B \cup C)^0 = B^0 \cup C^0$, $(B \cap C)^0 = B^0 \cap C^0$.

Proof. We prove only that $(B \cup C)^{\mathbb{Z}} = B^{\mathbb{Z}} \cup C^{\mathbb{Z}}$. Since $B \subseteq B \cup C$ and $C \subseteq B \cup C$ we obtain that $B^{\mathbb{Z}} \cup C^{\mathbb{Z}} \subseteq (B \cup C)^{\mathbb{Z}}$. To prove the opposite inclusion assume that $X \in (B \cup C)^{\mathbb{Z}}$. If $X \subseteq B \cap C$ then obviously, $X \subseteq B$ and $X \subseteq C$. Suppose that $X \nsubseteq B \cap C$. Without loss of generality we may assume that there is $b \in X - C$. Then for all $x \in X$ such that $f^n(x) = b$ we have $x \notin C$. Hence we obtain that $X \subseteq B$. □

Let (A, f) be a connected monounary algebra. From Theorem 2.1 we obtain the following facts:

(i) If (A, f) contains a cycle C, then C is a retract of (A, f).

(ii) If (A, f) contains a subalgebra (M, f), $(M, f) \cong (\mathbb{Z}, \mathrm{suc})$, then M is a retract of (A, f).

(iii) Suppose that (A, f) contains no cycle. Then every retract M of (A, f) as a subalgebra is equal to a subalgebra generated by the set $\bigcup M^{\mathbb{Z}} \cup M^0$. We note that for any system X of sets, $\bigcup X = \{x : \ (\exists X' \in X)(x \in X')\}$.

(iv) If M is a retract of (A, f) and $A^{\mathbb{Z}} \neq \emptyset$ then $M^{\mathbb{Z}} \neq \emptyset$.

(v) Let M be a retract, and $x \in M$ be an element with $s_f(x) = \alpha$, $\alpha \in \mathrm{Ord}$, α limit. Then the set $\{s_f(y) : \ y \in f^{-1}(x) \cap M\}$ is cofinal in α.

We note that a subset B of a partially ordered set A is *cofinal* if for every a in A there is b in B such that $a \leq b$. (Note, that if α is a limit ordinal then a set $S \subseteq \alpha$ of ordinals is cofinal in α, if and only if $\sup S = \alpha$.)

3. Lattice of retracts

In what follows we will study a system of all retracts of a given monounary algebra ordered by set inclusion. First we show that this system is closed under arbitrary unions. Next we will deal with a direct decomposition of certain sublatices of a retract lattice.

Lemma 3.1. *Let (A, f) be a monounary algebra and R_s, $s \in S$ be a nonempty system of retracts of (A, f). Then $\bigcup_{s\in S} R_s$ is a retract of (A, f).*

Proof. Suppose that (A, f) is connected. Denote $R = \bigcup_{s\in S} R_s$. Let $x \in R$ be an arbitrary element and $y \in f^{-1}(x)$. There exists $t \in S$ such that $x \in R_t$, thus we have $y \in f^{-1}(R_t)$. Since R_t is a retract, there exists $z \in R_t$ with $f(y) = f(z) = x$ and $s_f(y) \leq s_f(z)$. Obviously, $z \in R$, hence R is a retract of (A, f) according to 2.1.

Further assume that (A, f) contains more than one connected component. In view of the first part of the proof, it suffices to show that each component K of (A, f) having an empty intersection with R can be homomorphically mapped into R. Let $t \in S$. Since $R_t \cap K = \emptyset$, there exists a homomorphism $\varphi\colon K \to R_t$. Evidently, φ is the desired homomorphism of K into R. □

Denote the system of all retracts of a given monounary algebra (A, f) by $R(A, f)$. Consider $\mathbf{R}(A, f) = (R(A, f), \subseteq)$ — the system of all retracts ordered by set inclusion. Since A is the retract of (A, f), $\mathbf{R}(A, f)$ contains the greatest element and forms a complete upper semilattice, i.e., every nonempty subset of retracts R_s, $s \in S$ has the least upper bound in $\mathbf{R}(A, f)$ equal to $\bigcup_{s\in S} R_s$. If $\mathbf{R}(A, f)$ contains the least element, then $\mathbf{R}(A, f)$ forms a complete lattice.

Further denote $R^{\emptyset}(A, f) = R(A, f) \cup \{\emptyset\}$ and $\mathbf{R}^{\emptyset}(A, f) = (R^{\emptyset}(A, f), \subseteq)$. According to the previous facts, $\mathbf{R}^{\emptyset}(A, f)$ always forms a complete lattice.

Let (A, f) be a monounary algebra and $A = \bigcup_{i\in I} A_i$ where (A_i, f), $i \in I$, are connected components of (A, f). For $B \in R^{\emptyset}(A, f)$ denote

$$\langle B) = \bigl\{R \in R^{\emptyset}(A, f) : \ B \subseteq R\bigr\}$$

and $\langle \mathbf{B}) = \bigl(\langle B), \subseteq\bigr)$. Clearly, $\langle \mathbf{B})$ is a sublattice of $\mathbf{R}^{\emptyset}(A, f)$. Let $B \in R(A, f)$. If for $i \in I$ we denote $B_i = B \cap A_i$, then $B_i \in R^{\emptyset}(A_i, f)$. Put

$$I_0 = \bigl\{i \in I : \ B \cap A_i \neq \emptyset\bigr\}.$$

A mapping $\varphi\colon \langle \mathbf{B}) \to \prod_{i\in I_0} \langle \mathbf{B_i}) \times \prod_{i\in I-I_0} \mathbf{R}^{\emptyset}(A_i, f)$ defined by $\varphi(R)(i) = R \cap A_i$, $i \in I$, $R \in \langle B)$ is an isotone bijection with an isotone inverse. Thus φ is a lattice isomorphism and we have the following useful lemma:

Lemma 3.2. *Let (A, f) be a monounary algebra with the system (A_i, f), $i \in I$, of connected components and let $B \in R(A, f)$ be a proper retract. Then*

$$\langle \mathbf{B}) \cong \prod_{i\in I_0} \langle \mathbf{B_i}) \times \prod_{i\in I-I_0} \mathbf{R}^{\emptyset}(A_i, f)\,.$$

Let (A, f) be a connected monounary algebra, $x \in A$ be an arbitrary element. Denote $A^x = \bigcup\limits_{n\in\mathbb{N}_0} f^{-n}(x) \cup \mathbb{N}$ (we may assume that the sets $\mathbb{N}$ and $\bigcup\limits_{n\in\mathbb{N}_0} f^{-n}(x)$ are disjoint) and for $y \in A^x$ we set $\tilde{f}(y) = f(y)$ if $y \in \bigcup\limits_{n\in\mathbb{N}} f^{-n}(x)$, $\tilde{f}(x) = 1$ and $\tilde{f}(y) = y + 1$ for $y \in \mathbb{N}$.

Note that $s_f(x) = s_{\tilde{f}}(x)$ for all $x \in \bigcup\limits_{n\in\mathbb{N}_0} f^{-n}(x)$.

Lemma 3.3. *If (A, f) is a connected monounary algebra, $B \in R(A, f)$, then*

$$\langle \mathbf{B} \rangle \cong \prod_{x\in f^{-1}(B)-B} \mathbf{R}^{\emptyset}(A^x, \tilde{f})\,.$$

Proof. Suppose that $B \subseteq C \in R(A, f)$ and let $x \in f^{-1}(B) - B$ be such that $x \in C$. Denote $B^x = \bigcup\limits_{n\in\mathbb{N}_0} f^{-n}(x) \cup B$. Then (B^x, f) is a subalgebra of (A, f) and $C \cap B^x$ is a retract of (B^x, f). A mapping $\varphi\colon \langle B\rangle \to \prod\limits_{x\in f^{-1}(B)-B} \langle B\rangle \cap R(B^x, f)$, where $\varphi(C)(x) = C \cap B^x$ is an isotone bijection with an isotone inverse. The statement follows from the fact that for all $x \in f^{-1}(B) - B$ the lattice $(\langle B\rangle \cap R(B^x, f), \subseteq)$ is isomorphic to $\mathbf{R}^{\emptyset}(A^x, \tilde{f})$. $\square$

Lemma 3.2 and Lemma 3.3 illustrate that if we are interested in lattice properties inherited by direct product it is sufficient to investigate this properties for $\mathbf{R}^{\emptyset}(A, f)$, where (A, f) is a connected monounary algebra.

4. Least and minimal retracts

In this section we will deal with minimal retracts and we prove necessary and sufficient conditions under which the system of all retracts contains a least element.

Suppose that $B \subseteq D$ are subsets of A, where D is a retract of (A, f). It is not difficult to verify that B is a retract of (A, f) if and only if B is a retract of (D, f). Thus the minimal retracts of $\mathbf{R}(A, f)$ consist of the algebras having no proper retract. Connected monounary algebras having no proper retract are either cycles or isomorphic to $(\mathbb{Z}, \mathrm{suc})$ or to $(\mathbb{N}, \mathrm{suc})$. Let us notice that in [6, 1.6] it was proved that a connected monounary algebra (A, f) contains the copy of $(\mathbb{N}, \mathrm{suc})$ as a retract if and only if (A, f) is bounded. According to remarks (i) and (ii) bellow Theorem 2.1 we obtain the following lemma.

Lemma 4.1. *Let (A, f) be a connected monounary algebra. The system $\mathbf{R}(A, f)$ contains no minimal element if and only if (A, f) is unbounded.*

We say that a lattice L is *atomic* if L has the least element 0, and for every $a \in L$, $a \neq 0$, there is an atom $p \leq a$.

Lemma 4.2. *Let (A, f) be a connected monounary algebra. The lattice $\mathbf{R}^{\emptyset}(A, f)$ is atomic if and only if (A, f) is not unbounded.*

Proof. If $\mathbf{R}^{\emptyset}(A, f)$ is atomic, then there is at least one atom and thus (A, f) is not unbounded, due to Lemma 4.1.

Conversely, suppose that (A, f) is not unbounded. In this case either there is an element $x \in A$ with $s_f(x) = \infty$ or (A, f) is bounded. First consider that there is $x \in A$, with $s_f(x) = \infty$. If (A, f) possesses a cycle C then obviously C is a retract of (A, f) and C is contained in each subalgebra, hence in each retract of (A, f). If $A^{\mathbb{Z}} \neq \emptyset$ then according to (iv) bellow Theorem 2.1 for every $R \in R(A, f)$ it holds that $R^{\mathbb{Z}} \neq \emptyset$. Hence any $X \in R^{\mathbb{Z}}$ is a minimal retract with $X \subseteq R$.

Finally, suppose that (A, f) is bounded. In this case (A, f) contains no cycle and $A^{\mathbb{Z}} = \emptyset$. By way of contradiction, assume that there exists $R \in R(A, f)$, with $M \nsubseteq R$ for all minimal retracts M. According to Lemma 4.1 the algebra (R, f) is unbounded. Since $R \subseteq A$ and (A, f) is connected we obtain that (A, f) is unbounded, which is a contradiction. □

Now we will find necessary and sufficient conditions under which $\mathbf{R}(A, f)$ contains a least element.

Lemma 4.3. *Let (A, f) be a connected monounary algebra. The system $\mathbf{R}(A, f)$ contains a least element if and only if one of the following statements holds:*

(i) *the algebra (A, f) contains a cycle C,*

(ii) *the algebra (A, f) contains precisely one subalgebra isomorphic to $(\mathbb{Z}, \mathrm{suc})$,*

(iii) *the algebra (A, f) contains the unique retract which is isomorphic to $(\mathbb{N}, \mathrm{suc})$.*

Proof. It is evident that at most one of this condition is valid. Suppose that $\mathbf{R}(A, f)$ contains a least element. Since a least retract contains no proper retract, we obtain that (A, f) contains either a cycle or retracts isomorphic to $(\mathbb{Z}, \mathrm{suc})$ or to $(\mathbb{N}, \mathrm{suc})$. If the algebra (A, f) contains at least two subalgebras isomorphic to $(\mathbb{Z}, \mathrm{suc})$, then these subalgebras are minimal retracts of $\mathbf{R}(A, f)$, therefore the system $\mathbf{R}(A, f)$ contains no least element. Similarly in the case if (A, f) contains at least two retracts isomorphic to $(\mathbb{N}, \mathrm{suc})$.

Now suppose that one of this condition holds. If an algebra (A, f) contains a cycle C, then obviously C is the least element of $\mathbf{R}(A, f)$.

In the case that (A, f) contains precisely one copy of $(\mathbb{Z}, \mathrm{suc})$ as a subalgebra, then this is the only atom in $\mathbf{R}^{\emptyset}(A, f)$ and since $\mathbf{R}^{\emptyset}(A, f)$ is atomic we obtain that this atom is the least element of $\mathbf{R}(A, f)$. Similarly in the case that (A, f) contains unique retract isomorphic to $(\mathbb{N}, \mathrm{suc})$. $\square$

Further assume that (A, f) contains more than one connected component. Let A_i, $i \in I$, denote the system of all connected components of (A, f). Let C be a cycle of (A, f). We remark the definition of minimal cycle. The cycle C is called *minimal* if whenever D is a cycle of (A, f) such that $|D|$ divides $|C|$, then $|D| = |C|$, i.e., $|C|$ is the minimal element of the set $\{|D| : \ D \text{ is a cycle of } (A, f)\}$ according to divisibility.

Lemma 4.4. *Let (A, f) be a monounary algebra with at least two connected components. The system $\mathbf{R}(A, f)$ contains the least element if and only if one of the following statement is valid:*

(i) *there exists a connected component containing a cycle and for each minimal cycles $C, D \subseteq A$, $|C| = |D|$ implies $C = D$,*

(ii) *there is no connected component containing a cycle and (A, f) contains precisely one subalgebra isomorphic to $(\mathbb{Z}, \mathrm{suc})$.*

P r o o f. Suppose that there exists a connected component of (A, f) containing a cycle. Let M denote the system of all minimal cycles of (A, f). Then for every A_i, $i \in I$, there exists a cycle $C \in M$ and a homomorphism of A_i into C. Let (A, f) satisfies the condition (i), that is for all $n \in \mathbb{N}$ there exists at most one $C \in M$ with $|C| = n$. Denote $O = \bigcup\limits_{C \in M} C$. Obviously, O is a retract and it is contained in each retract of (A, f).

Conversely, let $n \in \mathbb{N}$ be such that there exist at least two minimal cycles of cardinality n. Denote these cycles by C_1 and C_2. There exists $M' \subseteq M$ satisfying the condition: $|C| \neq n$ for each $C \in M'$ and

$$(\forall D \in M)\Big(|D| \neq n \implies (\exists D' \in M')\big(|D| = |D'|\big)\Big) \qquad \text{and}$$
$$(\forall C, D \in M')\big(|C| \neq |D|\big).$$

Then $O_1 = M' \cup C_1$ and $O_2 = M' \cup C_2$ are two distinct minimal retracts of (A, f), hence $\mathbf{R}(A, f)$ does not contain any least element.

Consider the case (ii). If (A, f) contains no cycle and only one copy of $(\mathbb{Z}, \mathrm{suc})$ then every component can be homomorphically mapped into the component containing a copy of $(\mathbb{Z}, \mathrm{suc})$. On the other side, this component cannot be homomorphically mapped into any other component, since there are no elements of degree ∞. Thus the subalgebra isomorphic to $(\mathbb{Z}, \mathrm{suc})$ forms the least retract of $\mathbf{R}(A, f)$.

Now suppose that (A,f) contains more than one copy of $(\mathbb{Z},\mathrm{suc})$. These copies of $(\mathbb{Z},\mathrm{suc})$ are minimal retracts of (A,f), thus $\mathbf{R}(A,f)$ contains no least element. Further assume that (A,f) contains no copy of $(\mathbb{Z},\mathrm{suc})$. Then each component of (A,f) is either bounded or unbounded. Every bounded component can be homomorphically mapped into any other component. First suppose that there exists an unbounded component and let A_i, $i \in I'$ be a system of all unbounded components. Let R be an arbitrary retract of (A,f) with $R \subseteq \bigcup\limits_{i\in I'} A_i$. Since no A_i, $i \in I'$, contains a minimal retract, there exists $R' \in R(A,f)$ with $R' \subset R$. If the system of all unbounded components is empty, then $R(A,f)$ contains at least two retracts isomorphic to $(\mathbb{N},\mathrm{suc})$ and these retracts are minimal in $\mathbf{R}(A,f)$. $\square$

5. Concepts related to semimodularity

The aim of this section is to investigate concept of semimodularity, M-symmetry, and Mac Lane's condition in retract lattices of monounary algebras.

A lattice L is called *semimodular* if it satisfies the *Upper Covering Condition*, that is for $a,b \in L$,

$$a \prec b \implies \big(a \vee c \prec b \vee c \ \text{ or } \ a \vee c = b \vee c\big)\,.$$

Let L be a lattice. A pair of elements (a,b), $a,b \in L$ is called *modular*, in notation $a\,M\,b$, if

$$x \leq b \implies \big(x \vee (a \wedge b) = (x \vee a) \wedge b\big)\,.$$

If (a,b) is not a modular pair, then we write $a\bar{M}b$.

The lattice L is called *M-symmetric* if $a\ M\ b$ implies that $b\ M\ a$, for any $a,b \in L$.

A lattice L satisfies *Mac Lane's condition* (Mac_1) if, for any $x,y,z \in L$ such that $y \wedge z < x < z < y \vee x$, there exists an element $t \in L$ such that $y \wedge z < t \leq y$ and $x = (x \vee t) \wedge z$.

A lattice L satisfies *Mac Lane's condition* (Mac_2) if, for any $x,y,z \in L$ such that $y \wedge z < x < z < y \vee z$, there exists an element $t \in L$ such that $y \wedge z < t \leq y$ and $x = (x \vee t) \wedge z$.

In [13, Lemma 3.1.1], it is proved that condition (Mac_1) holds in a lattice if and only if condition (Mac_2).

If a lattice satisfies (Mac_1) or, equivalently, (Mac_2) we briefly say that the lattice satisfies *Mac Lane's condition* (Mac).

We remark the relationship between modularity, M-symmetry, (Mac) and semimodularity. It is obvious that a lattice L is modular if and only if $a\,M\,b$ for all $a,b \in L$, thus modularity implies M-symmetry. Similarly, if L is modular,

$x, y, z \in L$ satisfy the assumption of (Mac$_2$) and we put $t = y$, we obtain that $x = (x \vee t) \wedge z$. Hence modularity implies (Mac), too. Concepts of M-symmetry and (Mac) are independent in the sense that there exists a lattice satisfying the condition of M-symmetry but not (Mac) and vice versa. On the other hand both M-symmetry and (Mac) imply semimodularity. We note that this three conditions coincide for example in the class of lattices of finite length.

It is not difficult to verify that all this properties are inherited by direct product, thus we mainly investigate the lattice $\mathbf{R}^{\emptyset}(A, f)$ for a connected monounary algebra (A, f).

5.1. Covering relation in $\mathbf{R}^{\emptyset}(A, f)$.

First we investigate the covering relation in $\mathbf{R}^{\emptyset}(A, f)$ for a connected case.

If $\emptyset \neq B \subseteq A$, then we denote $f \restriction B$ the partial operation on B such that $\operatorname{dom}(f \restriction B) = \big\{b \in B \cap \operatorname{dom}(f) : \ f(b) \in B\big\}$ and if $b \in \operatorname{dom}(f \restriction B)$, then $(f \restriction B)(b) = f(b)$. Then $(B, f \restriction B)$ is called a *relative subalgebra* of (A, f). If there is no danger of confusion we briefly write (B, f) instead of $(B, f \restriction B)$.

Lemma 5.1. *Let (A, f) be a connected monounary algebra, $B, C \in R(A, f)$, $\emptyset \neq B \subseteq C$. Then C covers B in $\mathbf{R}(A, f)$ if and only if the relative subalgebra $(C - B, f)$ is isomorphic to $\big((-\infty, 0\rangle, \mathrm{suc}\big)$ or $\big(\langle 0, n\rangle, \mathrm{suc}\big)$ for some $n \in \mathbb{N}_0$.*

Proof. Suppose that the relative subalgebra $(C - B, f)$ is isomorphic to $\big((-\infty, 0\rangle, \mathrm{suc}\big)$ or $\big(\langle 0, n\rangle, \mathrm{suc}\big)$ for some $n \in \mathbb{N}_0$. Let $B, C, D \in R(A, f)$ be such that $\emptyset \neq B \subseteq D \subseteq C$ and $B \neq D$. Then D contains an element from $C - B$, since $\emptyset \neq D - B \subseteq C - B$. The set D is a retract, so for all $x \in C - B$ we obtain: if $x \in D$, then $f^n(x) \in D$, for all $n \in \mathbb{N}_0$. Further let $x \in (C - B) \cap D$ be such that $f^{-1}(x) \neq \emptyset$. With respect to the assumptions there exists the unique element $y \in f^{-1}(x) \cap C$. The set D is a retract, therefore $D \cap f^{-1}(x) \neq \emptyset$ and $y \in D$, since $D \subseteq C$. So we obtain that $D = C$.

Conversely, suppose that the relative subalgebra $(C - B, f)$ is not isomorphic to $\big((-\infty, 0\rangle, \mathrm{suc}\big)$ or to $\big(\langle 0, n\rangle, \mathrm{suc}\big)$.

a) There exists $x \in C - B$ such that $\big|f^{-1}(x) \cap C\big| \geq 2$. Let D_x be a subset $\emptyset \neq D_x \subseteq f^{-1}(x) \cap C$, such that $\big\{s_f(y) : \ y \in D_x\big\}$ is cofinal in $\big\{s_f(y) : \ y \in f^{-1}(x)\big\}$ and $D_x \neq f^{-1}(x) \cap C$. Note that it is always possible to choose such subset D_x because $\big|f^{-1}(x) \cap C\big| \geq 2$.

Denote

$$D = \Big(C - \bigcup_{n \in \mathbb{N}} f^{-n}(x)\Big) \cup \Big(C \cap \bigcup_{y \in D_x} \bigcup_{m \in \mathbb{N}_0} f^{-m}(y)\Big).$$

Since $x \notin B$ and $D_x \neq f^{-1}(x) \cap C$ we obtain that $B \subset D \subset C$. Now we will verify that D is a retract of (A, f). It is easy to see that (D, f) is a subalgebra of (A,f). Using the properties of an inverse image we have

$$f^{-1}(D) = \Big(f^{-1}(C) - \bigcup_{n\in\mathbb{N}} f^{-(n+1)}(x)\Big) \cup \Big(f^{-1}(C) \cap \bigcup_{y\in D_x} \bigcup_{m\in\mathbb{N}_0} f^{-(m+1)}(y)\Big).$$

Suppose that $u \in f^{-1}(D)$. For $u \in \Big(f^{-1}(C) - \bigcup_{n\in\mathbb{N}} f^{-(n+1)}(x)\Big)$ and $u \notin f^{-1}(x)$ there exists $z \in C$ with $f(u) = f(z)$, $s_f(u) \le s_f(z)$ and obviously $z \in D$. Since $\{s_f(y) :\ y \in D_x\}$ is cofinal in $\{s_f(y) :\ y \in f^{-1}(x)\}$, for $u \in f^{-1}(x)$ there exists $y \in D_x$ with $s_f(u) \le s_f(y)$. Finally, let $u \in f^{-1}(C) \cap f^{-(m+1)}(y)$ for $y \in D_x$, $m \in \mathbb{N}_0$. The set C is a retract thus there is $z \in C$ satisfying $f(u) = f(z)$, $s_f u \le s_f(z)$ and $z \in f^{-m}(y)$, hence $z \in D$.

b) $C - B$ contains at least two components isomorphic with $\big((-\infty, 0\rangle, \mathrm{suc}\big)$ or $\big(\langle 0, n\rangle, \mathrm{suc}\big)$. Denote one of this components by D'. Then $D = B \cup D'$ is a retract of (A,f) satisfying $B \subset D \subset C$. □

Remark 5.2. Since the retracts of a connected monounary algebra (A,f) which cover $\emptyset$ in $\mathbf{R}^{\emptyset}(A,f)$ are minimal, for $B \in R(A,f)$ we obtain: B covers $\emptyset$ if and only if (B,f) is isomorphic to $(\mathbb{N}, \mathrm{suc})$ or to $(\mathbb{Z}, \mathrm{suc})$ or (B,f) is a cycle.

As a consequence of Lemma 5.1 we obtain the following theorem.

Theorem 5.3. *If (A,f) is a connected monounary algebra, then $\mathbf{R}^{\emptyset}(A,f)$ forms a semimodular lattice.*

Proof. Let $B, C, D \in R^{\emptyset}(A,f)$ and suppose that $B \prec C$. We have

$$(C \cup D) - (B \cup D) \subseteq C - B,$$

hence if $B \cup D \ne C \cup D$ then $B \cup D \prec C \cup D$. □

Since direct products of semimodular lattices are again semimodular, together with the fact (3.2) we obtain the following result:

Corollary 5.4. *Let (A,f) be a monounary algebra, $B \in R(A,f)$. Then $\langle \mathbf{B})$ forms a semimodular lattice.*

5.2. M-symmetry

Next we will deal with M-symmetry. First we prove the following technical lemma.

Lemma 5.5. *Let (A,f) be a connected monounary algebra, $B, C \in R(A,f)$, $B \wedge C = \emptyset$ and $c \in (B \cap C)^0$. Then there exists a retract $X \subset C$ with $c \notin X$ such that $X \cup [c]$ is a retract of (A,f).*

Proof. Since $B \wedge C = \emptyset$ we get that for any subalgebra $S \subseteq B \cap C$, S is not a retract of (A, f). The subalgebra $[c]$ is not a retract, therefore there is an element $y \in f^{-1}([c])$, $y \notin [c]$, with $s_f(y) > s_f(c')$, where c' is the unique element of $[c]$ with $f(y) = f(c')$. Let $n_0 \in \mathbb{N}_0$ be the least number such that there is an element $y \in A$, $f(y) = f^{(n_0+1)}(c)$ and $s_f(y) > s_f(f^{n_0}(c))$. Since $[c] \subseteq C$ and C is a retract, for all $y' \in f^{-1}(f^{(n_0+1)}(c))$ there exists $z \in C$ with $f(z) = f(y') = f^{(n_0+1)}(c)$, $s_f(z) \geq s_f(y')$.

Denote $X = C - \bigcup\limits_{n=0}^{\infty} f^{-n}(f^{n_0}(c))$. Obviously $X \subset C$ and we show that X is a retract of (A, f). Suppose that $y \in f^{-1}(X)$. If $f(y) \neq f^{(n_0+1)}(c)$ then there is an element satisfying the condition of Theorem 2.1 since C is a retract. In the case $f(y) = f^{(n_0+1)}(c)$ we can take the corresponding element z. From the choice of n_0 it is evident that $X \cup [c]$ is a retract of (A, f). □

A lattice with 0 is called $\perp$-*symmetric* if it satisfies the condition

$$\big(a\,M\,b \ \ \& \ \ a \wedge b = 0\big) \implies b\,M\,a\,.$$

The relation aMb holds in a lattice if and only if it holds in $[a \wedge b, a \vee b]$ (see [13]). It is not difficult to verify the following two facts:

(i) A lattice L is M-symmetric if and only if $\langle x)$ is $\perp$-symmetric for all $x \in L$.

(ii) A direct product of $\perp$-symmetric lattices is $\perp$-symmetric.

According to Lemma 3.3 and the previous facts (i) and (ii) to prove that $\mathbf{R}^{\emptyset}(A, f)$ is M-symmetric it is sufficient to show that $\mathbf{R}^{\emptyset}(A, f)$ is $\perp$-symmetric, for any connected monounary algebra (A, f). This immediately follows from the following theorem.

Theorem 5.6. *Let (A, f) be a connected monounary algebra, $B, C \in R(A, f)$ and $B \wedge C = \emptyset$. Then $B\,M\,C$ if and only if $(B \cap C)^0 = \emptyset$.*

Proof. Since $B \wedge C = \emptyset$, there exists no proper retract $R \subseteq B \cap C$. Therefore $B^{\mathbb{Z}} \cap C^{\mathbb{Z}} = \emptyset$, i.e., there is no common copy of $(\mathbb{Z}, \mathrm{suc})$ in B and C. Suppose that $(B \cap C)^0 = \emptyset$. Let X be a retract such that $\emptyset \subset X \subseteq C$. Then we have to prove:

$$X \cup (B \wedge C) = X = (X \cup B) \wedge C\,.$$

To establish the second equality, by way of contradiction assume that $X \subset (X \cup B) \wedge C$. Since B and C contain no common copy of $(\mathbb{Z}, \mathrm{suc})$ and $B \wedge C \subseteq B \cap C$, using Lemma 2.3 we obtain that $\big((X \cup B) \wedge C\big)^{\mathbb{Z}} = X^{\mathbb{Z}}$. Thus there is an element $c \in A^0$ with $c \in (X \cup B) \wedge C$ and $c \notin X$. Since $c \in (X \cup B) \wedge C \subseteq (X \cup B) \cap C = X \cup (B \cap C)$ we have that $c \in B \cap C$ which is a contradiction.

Conversely assume that there is an element $c \in (B \cap C)^0$. We show that $B \bar{M} C$. According to Lemma 5.5 there exists a retract X with $\emptyset \subset X \subset C$ with $c \notin X$. Hence we obtain:

$$X \cup (B \wedge C) = X \subset X \cup [c] \subseteq (X \cup B) \wedge C \ ,$$

since $X \cup [c]$ is a retract and $X \cup [c] \subseteq (X \cup B) \cap C$. $\square$

Corollary 5.7. *Let (A, f) be a connected monounary algebra. Then the lattice $\mathbf{R}^{\emptyset}(A, f)$ is M-symmetric.*

A direct product of M-symmetric lattices is M-symmetric, thus together with Lemma 3.2 we obtain the following corollary.

Corollary 5.8. *Let (A, f) be a monounary algebra, $B \in R(A, f)$. Then $\langle \mathbf{B})$ is M-symmetric.*

5.3. Mac Lane's condition

Further we will deal with Mac Lane's condition. It turns out that the retract lattice $\mathbf{R}^{\emptyset}(A, f)$ of a connected monounary algebra (A, f) does not fulfil the condition (Mac) in general.

Lemma 5.9. *Let (A, f) be a connected monounary algebra. Suppose that there is a triple $X, Y, Z \in R(A, f)$ such that $\emptyset = Y \wedge Z \subset X \subset Z \subset Y \cup Z$ and X, Y, Z does not fulfil* (Mac$_2$)*. Then (A, f) is unbounded and $(Z^0 - X^0) \cap T^0$ is infinite for all T with $\emptyset \subset T \subseteq Y$.*

P r o o f. Since $Y \wedge Z \subset Z \subset Y \cup Z$, the retracts Y, Z are incomparable. Assume that there exists $\emptyset \subset T \subseteq Y$ such that $(Z \cap T)^0 = \emptyset$. According to 5.6, $T \; M \; Z$ and we obtain that $(X \cup T) \wedge Z = X \cup (T \wedge Z) = X \cup \emptyset = X$.

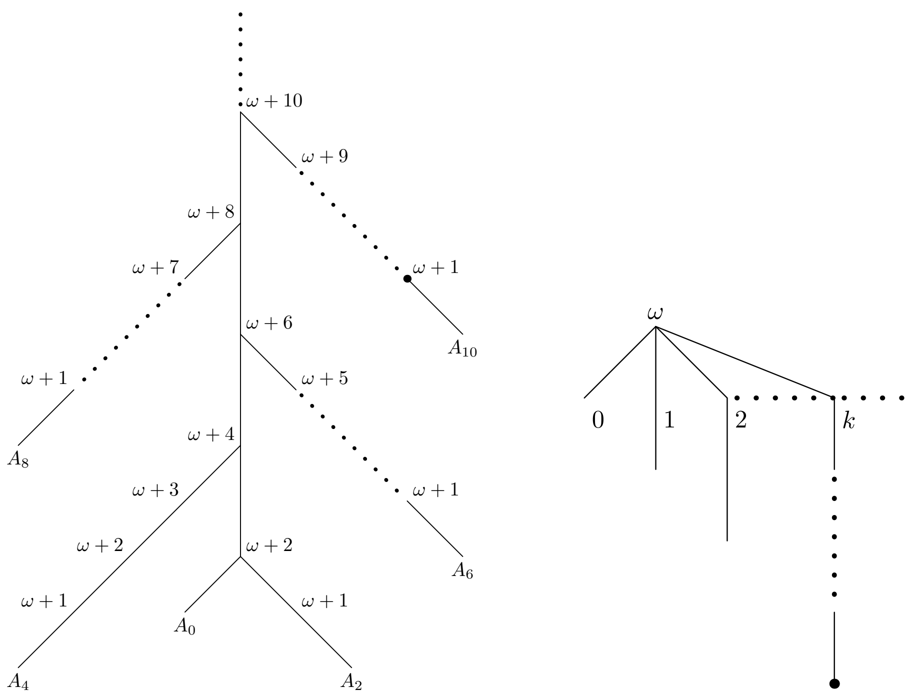
If $\mathbf{R}^{\emptyset}(A, f)$ is atomic, then any atom $T \subseteq Y$ satisfies $(Z \cap T)^0 = \emptyset$. Hence, the algebra (A, f) is unbounded.

Now suppose that there is $T \in R(A, f)$, $T \subseteq Y$ with $Z^0 \cap T^0 \subseteq X^0$. Since (A, f) is unbounded, $A^{\mathbb{Z}} = \emptyset$ and thus for any $R \in R(A, f)$, $R = [R^0]$. Then $\big((X \cup T) \wedge Z\big)^0 \subseteq \big((X \cup T) \cap Z\big)^0 = X^0$. On the other side, $X \subseteq (X \cup T) \wedge Z$, hence we obtain that $X = (X \cup T) \wedge Z$.

Finally, suppose that there is $\emptyset \subset T \subseteq Y$ with $\big|(Z^0 - X^0) \cap T^0\big| = n$, for some $n \in \mathbb{N}$. Then using Lemma 5.5 in at most n steps we find a retract $\emptyset \subset T_1 \subseteq T$ with $T_1^0 \cap Z^0 \subseteq X^0$. Consequently, $\big|(Z^0 - X^0) \cap T^0\big| \geq \aleph_0$, for all $T \in R(A, f)$, $\emptyset \subset T \subseteq Y$. $\square$

Example 5.10. An example of a connected monounary algebra (A, f), such that $\mathbf{R}^{\emptyset}(A, f)$ does not fulfil (Mac).

We give only a schematic description of an algebra (A, f). For $n \in \mathbb{N}_0$ let (A_{2n}, f_{2n}) be the partial monounary algebra isomorphic to the algebra on the figure at the right side. The numbers on figures denote degrees of the corresponding elements. Define an algebra (A, f) in such a way as the left figure illustrate. For each algebra (A_{2n}, f_{2n}), we denote the only element of degree ω by the symbol ω^{2n}, and similarly we denote by symbols $0^{2n}, 1^{2n}, \ldots$ the elements in the set $f^{-1}(\omega^{2n})$. Suppose that R is a retract of (A, f).



We obtain the following facts:

(i) If $R \cap A_{2n} \neq \emptyset$ for some $n \in \mathbb{N}_0$, then $R \cap A_{2m} \neq \emptyset$ for all $m \in \mathbb{N}_0$, $m \geq n$.

(ii) If $R \cap A_{2n} \neq \emptyset$, then $R \cap f^{-1}(\omega^{2n})$ is an infinite subset of $\{0^{2n}, 1^{2n}, \ldots\}$. Note, that this set determines the set $R \cap A_{2n}$ uniquely.

Define retracts X, Y, Z using subsets of A_{2n} as follows: For each $n \in \mathbb{N}_0$ put $X \cap (A_{2n} \cap f^{-1}(\omega^{2n})) = \{(2l+1)^{2n} : l \in \mathbb{N}_0\}$, i.e., “odd numbers”, $Y \cap (A_{2n} \cap f^{-1}(\omega^{2n})) = \{(2l)^{2n} : l \in \mathbb{N}_0\}$, i.e., “even” numbers. Further $Z \cap (A_{4n} \cap f^{-1}(\omega^{4n})) = \{l^{4n} : l \in \mathbb{N}_0\}$ and $Z \cap (A_{4n+2} \cap f^{-1}(\omega^{4n+2})) = \{(2l+1)^{4n+2} : l \in \mathbb{N}_0\}$.

It is not difficult to verify that: $\emptyset = Y \wedge Z \subset X \subset Z \subset Y \cup Z = A$.

Let $\emptyset \neq T \subseteq Y$ be an arbitrary retract. Let $n_0 \in \mathbb{N}_0$ be the least integer such that $T \cap A_{4n_0} \neq \emptyset$. Since $T \subseteq Y$ we obtain that $T \cap (A_{4n_0} \cap f^{-1}(\omega^{4n_0}))$

is an infinite subset of the set $\{(2l)^{2n} : \ l \in \mathbb{N}_0\}$. Hence $X \cup (T \cap A_{4n_0})$ is a retract of (A, f) and $X \cup (T \cap A_{4n_0}) \subseteq (X \cup T) \cap Z$, thus we obtain that $X \subset X \cup (T \cap A_{4n_0}) \subseteq (X \cup T) \wedge Z$. Consequently, the algebra (A, f) does not satisfy the condition (Mac$_2$).

There are two equivalent weaker conditions (E$_6$) or (E$_7$), respectively.

(E$_6$) The relations $y \wedge z < x < z < y \vee z$ imply the existence of an element t such that $y \wedge z < t \leq y$ and $(x \vee t) \wedge z < z$.

(E$_7$) The relations $y \wedge z < x < z < y \vee x$ imply the existence of an element t such that $y \wedge z < t \leq y$ and $(x \vee t) \wedge z < z$.

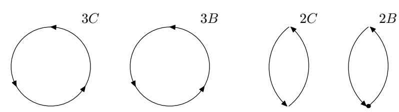
Conditions (E$_6$) and (E$_7$) are equivalent in any lattice. Obviously, (Mac$_1$) implies (E$_7$) and (Mac$_2$) implies (E$_6$). We note that the condition (E$_6$) implies semimodularity.

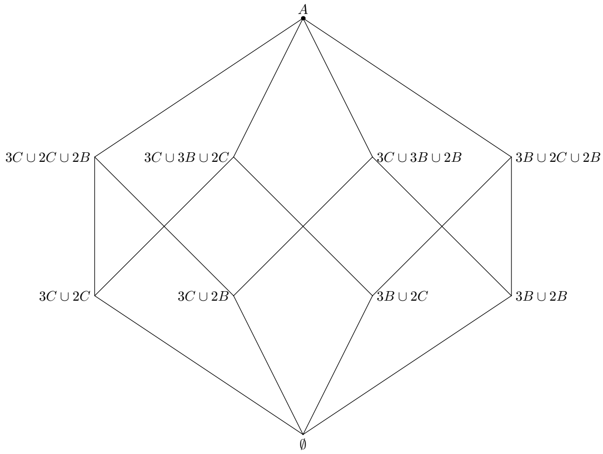
Theorem 5.11. *Let (A, f) be a connected monounary algebra. Then the lattice $\mathbf{R}^{\emptyset}(A, f)$ satisfies Mac Lane's condition* (E$_6$).

P r o o f. Since the condition (E$_6$) is inherited by direct products, according to Lemma 3.3 it is sufficient to show that for any connected (A, f), $\mathbf{R}^{\emptyset}(A, f)$ satisfies the condition (E$_6$) for all $X, Y, Z \in R(A, f)$ with $\emptyset = Y \wedge Z \subset X \subset Z \subset Y \cup Z$. According to Lemma 5.9, we may assume that (A, f) is unbounded and $(Z^0 - X^0) \cap Y^0 \neq \emptyset$. Let $z \in Z^0 - X^0 \cap Y^0$ be an arbitrary element. Due to Lemma 5.5 there exists a retract $T \subseteq Y$ such that $z \notin T$. Since $z \notin (X \cup T) \cap Z \supseteq (X \cup T) \wedge Z$, we obtain that $(X \cup T) \wedge Z \subset Z$. □

At the end of this section we show, that the lattice $\mathbf{R}^{\emptyset}(A, f)$ need not to be semimodular in general (non-connected case).

Example 5.12. A monounary algebra (A, f) consisting of two cycles of length 3 and two cycles of length 2 and the corresponding lattice of retracts $\mathbf{R}^{\emptyset}(A, f)$.





6. Modular lattices

In this section we will deal with modular lattices. First we prove a criterion for a retract lattice to be modular.

Theorem 6.1. *Let (A, f) be a connected monounary algebra. The lattice $\mathbf{R}^{\emptyset}(A, f)$ is modular if and only if $(B \wedge C)^0 = (B \cap C)^0$ for all $B, C \in R^{\emptyset}(A, f)$.*

Proof. We show that BMC if and only if $(B\wedge C)^0 = (B\cap C)^0$. For $R \in R(A, f)$ and $y \in A$ denote by $\tilde{R}_y$ a subset of A^y, $\tilde{R}_y = \big(R \cap \bigcup_{n=0}^{\infty} f^{-n}(y)\big) \cup \bigcup_{n=1}^{\infty} \tilde{f}^n(y)$. It is not difficult to verify that $\tilde{R}$ is a retract of $(A^y, \tilde{f})$. Suppose that $B\,\bar{M}\,C$. According to Lemma 3.3, $\langle B \wedge C) \cong \prod_{x\in f^{-1}(B\wedge C)-(B\wedge C)} \mathbf{R}^{\emptyset}(A^x, \tilde{f})$. Hence there exists $y \in f^{-1}(B \wedge C) - (B \wedge C)$, such that $\tilde{B}_y\,\bar{M}\,\tilde{C}_y$ in $\mathbf{R}^{\emptyset}(A^y, \tilde{f})$.Obviously $\tilde{B}_y \wedge \tilde{C}_y = \emptyset$. Due to Theorem 5.6 there exists $c \in (\tilde{B}_y \cap \tilde{C}_y)^0$ and we obtain that $c \in (B \cap C)^0$ and $c \notin (B \wedge C)^0$.

Now suppose that $B\,M\,C$. We show that $(B \cap C)^0 \subseteq (B \wedge C)^0$. Due to Theorem 5.6 we may assume that $B \wedge C \neq \emptyset$. By way of contradiction suppose that there exists $c \in (B \cap C)^0$ with $c \notin (B \wedge C)^0$. Let y be an element such that $y \notin B \wedge C$, $f(y) \in B \wedge C$ and there exists $n \in \mathbb{N}_0$ with $f^n(c) = y$. Since $\tilde{B}_y \wedge \tilde{C}_y = \emptyset$ and $c \in \tilde{B}_y \cap \tilde{C}_y$ due to 5.6 we obtain that $\tilde{B}_y\,\bar{M}\,\tilde{C}_y$. Hence there is a retract $\tilde{X}_y$ of $(A^y, \tilde{f})$ such that $\tilde{X}_y \cup (\tilde{B}_y \wedge \tilde{C}_y) = \tilde{X}_y \subset (\tilde{X}_y \cup \tilde{B}_y) \wedge \tilde{C}_y$. Put $X = \big(\tilde{X}_y \cap \bigcup_{n=0}^{\infty} f^{-n}(y)\big) \cup (B \wedge C)$. Obviously X is a retract of (A, f). Using Lemma 3.3 we obtain that $X \cup (B \wedge C) = X \subset (X \cup B) \wedge C$, thus $B\,\bar{M}\,C$ which is a contradiction. $\square$

Corollary 6.2. *Let (A, f) be a connected monounary algebra. If $\mathbf{R}^{\emptyset}(A, f)$ is modular then $\mathbf{R}^{\emptyset}(A, f)$ is distributive.*

Proof. First assume that (A, f) possesses no cycle. Using Theorem 6.1 we show that $\mathbf{R}^{\emptyset}(A, f)$ is isomorphic to a sublattice of $\left(\mathscr{P}(A^0) \times \mathscr{P}(A^{\mathbb{Z}}), \cup, \cap\right)$. Each retract R of the algebra (A, f) is equal to a support of subalgebra generated by a set $R^0 \cup \bigcup R^{\mathbb{Z}}$. Denote $L = \left\{(R^0, R^{\mathbb{Z}}) : \ R \in R^{\emptyset}(A, f)\right\}$. Obviously for all $B, C \in R^{\emptyset}(A, f)$ hold $B \subseteq C$ if and only if $B^0 \subseteq C^0$ and $B^{\mathbb{Z}} \subseteq C^{\mathbb{Z}}$. Hence a mapping $\varphi\colon \mathbf{R}^{\emptyset}(A, f) \to L$, $\varphi(R) = (R^0, R^{\mathbb{Z}})$ is isotone with isotone inverse. Since $(B \cup C)^0 = B^0 \cup C^0$, $(B \cup C)^{\mathbb{Z}} = B^{\mathbb{Z}} \cup C^{\mathbb{Z}}$ and $(B \wedge C)^0 = (B \cap C)^0 = B^0 \cap C^0$, $(B \wedge C)^{\mathbb{Z}} = (B \cap C)^{\mathbb{Z}} = B^{\mathbb{Z}} \cap C^{\mathbb{Z}}$, we obtain that L is a sublattice of $\mathscr{P}(A^0) \times \mathscr{P}(A^{\mathbb{Z}})$.

If (A, f) contains a cycle C, then according to the previous part of the proof and Lemma 3.3, a lattice $\mathbf{R}(A, f)$ is a direct product of distributive lattices $\mathbf{R}^{\emptyset}(A^x, \tilde{f})$, $x \in f^{-1}(C) - C$. Hence $\mathbf{R}^{\emptyset}(A, f)$ is distributive, too. □

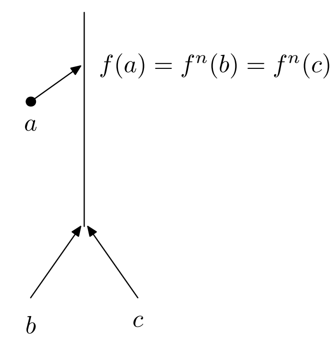
Finally we give a description of connected monounary algebras with modular retract lattices via forbidden configuration of their elements.

Theorem 6.3. *Let (A, f) be a connected monounary algebra possessing no cycle with $s_f(x) \in \omega \cup \{\infty\}$ for all $x \in A$. Then $\mathbf{R}^{\emptyset}(A, f)$ is not modular if and only if there exist $a, b, c \in A$ satisfying*

(i) $f(b) = f(c)$ *and* $s_f(b) = s_f(c) = \max\left\{s_f(x) : \ x \in f^{-1}(f(b))\right\}$,

(ii) *there is $n \in \mathbb{N}$ such that* $f(a) = f^n(b) = f^n(c)$ *and* $s_f(a) < s_f(f^{n-1}(b))$,

(iii) $s_f(x) \le s_f(f^{m-1}(b))$ *for all* $x \in f^{-1}(f^m(b))$, $m = 1, \dots, n$.



Proof. First assume that $a, b, c \in A$ satisfy the conditions of the theorem. Denote $M = A - \bigcup\limits_{i=0}^{\infty} f^{-i}(f(a))$. Put

$$B = M \cup \bigcup_{i=0}^{n} f^i(b) \cup \bigcup_{i=1}^{\infty} f^{-i}(b) \cup \bigcup_{i=0}^{\infty} f^{-i}(a)$$

and

$$C = M \cup \bigcup_{i=0}^{n} f^i(c) \cup \bigcup_{i=1}^{\infty} f^{-i}(c) \cup \bigcup_{i=0}^{\infty} f^{-i}(a)\,.$$

According to the condition (iii), B, C are retracts of (A, f).

We show that $B\bar{M}C$. Due to the condition (ii), $s_f(a) \in \omega$, thus there is an element $a' \in \bigcup\limits_{i=0}^{\infty} f^{-i}(a)$ with $a' \in A^0$. Obviously, $a' \in (B \cap C)^0$. We show that $a' \notin (B \wedge C)^0$. If $a' \in B \wedge C$, then also $f(a) \in B \wedge C$. Now we have $s_f(a) < s_f(f^{n-1}(b))$ (condition (ii)) and $(B \cap C) \cap f^{-1}(f(a)) \subseteq \{a, f^{n-1}(b)\}$ (we note that $f^{n-1}(b)$ does not belong to this set only if $n = 1$). If $n > 1$, then $f^{n-1}(b) \in B \wedge C$. Since $\bigcup\limits_{i=0}^{\infty} f^{-i}(f^{n-1}(b)) \cap B \cap C = \bigcup\limits_{i=1}^{n-1} f^i(b)$, we obtain that $f(b) \in B \wedge C$. The case $n = 1$ is similar. This yields a contradiction since $f^{-1}(f(b)) \cap B \cap C = \emptyset$. According to Theorem 6.1, $B\bar{M}C$, hence $\mathbf{R}^{\emptyset}(A, f)$ is not modular.

Conversely, assume that $\mathbf{R}^{\emptyset}(A, f)$ is not modular. Hence there exist $B, C \in R(A, f)$ such that $B\bar{M}C$. Let $a' \in (B \cap C)^0$ be such that $a' \notin (B \wedge C)^0$. Since $[a']$ is not a retract, there is $i \in \mathbb{N}_0$ such that $s_f(f^i(a')) < s_f(z)$ for some $z \in f^{-1}(f^{i+1}(a'))$, i.e., $\big(f^i(a'), (f^{i+1}(a')\big)$ forms a gap. For the element $f^{i+1}(a')$ it holds that $s_f(f^{i+1}(a')) \in \omega \cup \{\infty\}$, thus the algebra $(A^{f^{i+1}(a')}, \tilde{f})$ is not unbounded. Suppose that $\mathbf{R}^{\emptyset}(A^{f^{i+1}(a')}, \tilde{f})$ contains at least two atoms N_1, N_2. Let $b \in N_1$, $c \in N_2$ be such elements that $c \notin N_1$, $b \notin N_2$ and $\tilde{f}(b) = \tilde{f}(c)$. Obviously, also $f(b) = f(c)$. Denote by $a = f^i(a')$ and by n the least number with $f^n(b) = f(a)$. The sets N_1 and N_2 are retracts of $(A^{f^{i+1}(a')}, \tilde{f})$ thus $s_f(x) \leq s_f(f^{m-1}(b))$ for all $x \in f^{-1}(f^m(b))$ and $m = 1, \dots, n$ and we obtain that a triple a, b, c satisfying the conditions of the lemma.

To complete the proof, by way of contradiction assume, that there is no $i \in \mathbb{N}_0$ such that $\big(f^i(a'), f^{i+1}(a')\big)$ forms a gap and $\mathbf{R}^{\emptyset}(A^{f^{i+1}(a')}, \tilde{f})$ contains at least two atoms. Denote $I = \big\{i \in \mathbb{N}_0 : \ \big(f^i(a'), f^{i+1}(a')\big) \text{ forms a gap}\big\}$. For $i \in I$ let N_i denotes the only atom of $\mathbf{R}^{\emptyset}(A^{f^{i+1}(a')}, \tilde{f})$. Put $Y_i = N_i \cap \bigcup\limits_{m=0}^{\infty} f^{-m}(f^{i+1}(a'))$. Since $a' \in B \cap C$ and $\big(f^i(a'), f^{i+1}(a')\big)$ forms a gap we obtain that $Y_i \subseteq B \cap C$. The set $R = [a'] \cup \bigcup\limits_{i \in I} Y_i$ is a retract. The contradiction follows from the fact that $a' \in R \subseteq B \wedge C$. $\square$

Remark 6.4. If an algebra (A, f) contains an element of degree ω, it is not difficult to verify that the lattice $\mathbf{R}^{\emptyset}(A, f)$ is not modular. Hence for all connected algebras which possess no cycle, $\mathbf{R}^{\emptyset}(A, f)$ is modular if and only if for all $x \in A$, $s_f(x) \in \omega \cup \{\infty\}$ and no triple of elements satisfies the conditions of Theorem 6.3.

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* *Institute of Mathematics*
P. J. Šafárik University
Jesenná 5
SK–041 54 Košice
SLOVAKIA
E-mail: danica.studenovska@upjs.sk

** *Mathematical Institute*
Slovak Academy of Science
Grešákova 6
SK–040 01 Košice
SLOVAKIA
E-mail: pocs@saske.sk