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PREFERENCE ORDERS AND CONTINUOUS REPRESENTATIONS

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ABSTRACT. In this paper we prove some general theorems on the existence of continuous order-preserving functions on topological spaces with a continuous preorder. We use the concepts of network and netweight to prove new continuous representation theorems and we establish our main results for topological spaces that are countable unions of subspaces. Some results in the literature on path-connected, locally connected and separably connected spaces are shown to be consequences of the general theorems proved in the paper. Finally, we prove a continuous representation theorem for hereditarily separable spaces.

1. Introduction

Let (X, τ) be a topological space endowed with a total preorder $\preceq$. A real-valued function $f\colon (X, \preceq) \to \mathbb{R}$ on X is said to be *order-preserving* if $x \preceq y \iff f(x) \leq f(y)$. We say that X (or the total preorder $\preceq$) is *continuously representable* if there exists a τ-continuous order-preserving function f on X and we call the function f a *continuous representation* of X (or of the total preorder $\preceq$). In this paper we prove some general theorems on the existence of continuous order-preserving functions on topological spaces endowed with a continuous total preorder.

There are several approaches that have been used in the literature to deal with the representation problem such as the Euclidean distance approach, the topological approach and the set-theoretic approach. In this paper we use the topological approach to prove some general theorems on the existence of order-preserving functions.

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The representation problem is obviously of great interest from a purely mathematical point of view and it also arises in an essential manner in various problems in applied mathematics. For a discussion of the extensive literature on this subject we refer the reader to Bridges and Mehta [1] and Mehta [10].

The classical work on the existence of continuous representation theorems was done by Debreu [5, 6]. There were important earlier contributions by Cantor [4] and Eilenberg [8].

The theorem of Eilenberg [8] states that every connected and separable topological space endowed with a continuous total order has a continuous representation and the theorem of Debreu [5, 6] states that every second countable topological space endowed with a continuous total preorder has a continuous representation. In particular, second countability has the interesting feature that it is a hereditary property which implies, for example, that any continuous total preorder on an *arbitrary* subset of $\mathbb{R}^n$ has a continuous representation. It is precisely this kind of theorem that is very useful in many applications involving finite-dimensional spaces.

Later, other topological conditions for continuous representability were studied. One reason, among others, for such investigations is that second countability does not hold, in general, for Banach spaces and topological separability does not hold, in general, in infinite-dimensional spaces. Since Banach spaces and their generalizations are fundamental in mathematics and are used extensively in applications the limitation of these classical theorems becomes apparent and other criteria for continuous representability need to be found. The aim of this paper is to obtain other topological representation results which are of independent interest and which also generalize some existing theorems in the literature.

A natural generalization of the second countable spaces are the spaces with a countable network. A network in a topological space X is a family $\mathscr{N}$ of subsets of X such that every open set of X is the union of elements of $\mathscr{N}$. In this paper we use the concepts of network and netweight (the minimum of the cardinalities of the networks of X) to prove new continuous representation theorems. As far as we know, these concepts have not been used in the context of representation theory. In a topological space (X, τ) with countable netweight every continuous total preorder $\preceq$ has a continuous representation. The properties that make the concept of network interesting are that $(X, \tau_{\preceq})$ is second countable if and only if it has a countable netweight and the countability of netweight is preserved in weaker topologies. In particular, the concept of a network allows us to prove our main results for topological spaces that are countable unions of subspaces. In fact if $X = \bigcup_{n\in\mathbb{N}} X_n$, a topology on X has countable netweight if and only if every subspace topology on X_n has countable netweight. Note that a sufficient condition so that a subspace $(X_n, \tau_{\preceq}|X_n)$ has countable netweight is that it is representable and $\tau_{\preceq}|X_n = \tau_{\preceq|X_n}$. This is true, for example, if X_n is connected and separable.

In this context, the main result is Theorem 4 in Section 3 that includes and unifies the Debreu and Eilenberg Theorems.

Monteiro [11] proved that if (X, τ) is a path-connected topological space and $\preceq$ is a continuous total preorder that is countably bounded then X has a continuous representation. The simple idea which is the key to proving Monteiro's theorem is the fact that if X has a connected and separable subset which bounds the preorder then X has a continuous representation.

Candeal et. al. [2] proved in their paper that "path-connectedness" may be replaced by the weaker property "separably connected" in proving that countable boundedness implies the existence of a connected and separable subset F that bounds the preorder.

In order to generalize these results of Monteiro and Candeal et al., and other similar results, we introduce the idea of a $\Gamma_{\preceq}$-structure on a preordered topological space. We prove that the separable connectedness may be replaced by the $\Gamma_{\preceq}$-structure in proving some continuous representation theorems.

Finally, we prove a continuous representation theorem for a certain class of hereditarily separable spaces which are more general than second countable spaces.

2. Preliminaries

A *preorder* $\preceq$ on an arbitrary set X is a binary relation on X which is reflexive and transitive. An anti-symmetric preorder is said to be an *order*. A *total preorder* $\preceq$ on a set X is a preorder such that if $x, y \in X$ then $[x \preceq y] \vee [y \preceq x]$. If $\preceq$ is a preorder on X, then as usual we denote the associated *asymmetric* relation by $\prec$ and the associated *equivalence* relation by $\sim$ and these are defined, respectively, by $[x \prec y \iff (x \preceq y) \wedge \neg(y \preceq x)]$ and $[x \sim y \iff (x \preceq y) \wedge (y \preceq x)]$. If $\preceq$ is a preorder on a set X, then we will refer to the pair $(X, \preceq)$ as a preordered set.

Let $(X, \preceq)$ be a totally preordered set. We say that a subset F of X bounds a subset E of X if for every $x \in E$ there are $a, b \in F$ such that $a \preceq x \preceq b$. If $E = \{a\}$, we simply say that F bounds a. If $E = X$, we also say that F bounds $\preceq$. The preorder $\preceq$ is said to be countably bounded if there is a countable subset F that bounds $\preceq$.

Let $(X, \preceq)$ be a totally preordered set. A subset F of X is said to be *order-convex* if $a, b \in F$ with $a \prec b$ and $a \prec c \prec b$ imply that $c \in F$.

Let $(X, \preceq)$ be a totally preordered set. The family of all sets of the form $L(x) = \{a \in X : \ a \prec x\}$ and $G(x) = \{a \in X : \ x \prec a\}$, where $x \in X$, is a subbasis for a topology on X. This topology denoted by $\tau_{\preceq}$ is called the *order topology* on X. The pair $(X, \tau_{\preceq})$ is called a *totally preordered topological space.*

If $(X, \preceq)$ is a totally preordered set and τ is a topology on X, then the preorder $\preceq$ is said to be τ-continuous on X if for each $x \in X$ the sets $\{a \in X : \ x \prec a\}$ and $\{a \in X : \ a \prec x\}$ are τ-open in X, that is τ is finer than $\tau_{\preceq}$. In the following $(X, \tau, \preceq)$ will denote a topological space with a total continuous preorder.

If $(X, \preceq)$ is a preordered set then a real-valued function f on X is said to be

(i) *increasing* if for every $x, y \in X$, $[x \preceq y \implies f(x) \leq f(y)]$

(ii) *order-preserving* if f is increasing and $[x \prec y \implies f(x) < f(y)]$.

We recall that if $(X, \preceq)$ is a totally preordered set, then (ii) is equivalent to

(iii) for every $x, y \in X$, $[x \preceq y \iff f(x) \leq f(y)]$.

A total preorder $\preceq$ on a set X, or the totally preordered set $(X, \preceq)$, is said to be *representable* if there is an order-preserving function $f \colon (X, \preceq) \to \mathbb{R}$ that is, a real-valued function f on X such that $x \preceq y \iff f(x) \leq f(y)$. A total preorder $\preceq$ on a topological space (X, τ) is said to be *continuously representable* if there exists a real-valued τ-continuous order-preserving function on X. In this case, we say that $(X, \tau, \preceq)$ or X has a *continuous representation.*

If the total preorder is interpreted as a preference relation on a set of alternatives then the order preserving function is called a *utility function* and we say that the utility function represents the preference relation.

For further discussion and elaboration of these ideas the reader is referred to Bridges and Mehta [1] and Mehta [10].

Let $(X, \preceq)$ be a totally preordered set and let X' be the set of equivalence classes with respect to the relation $\sim$. If $x \in X$ we denote the equivalence class of x by $[x]$. The preorder $\preceq$ on X induces a natural order $\preceq'$ on X' defined by $[x] \preceq' [y] \iff x \preceq y$.

If (X, τ) is a topological space with a total preorder $\preceq$, the quotient set X' can be equipped with the quotient topology τ'. We note that $\preceq$ is continuous with respect to τ if and only if $\preceq'$ is continuous with respect to τ'.

The following well known result relates continuous representations on X and X'.

Proposition 1. *Let (X, τ) be a topological space, $\preceq$ a continuous total preorder on X, (X', τ') the corresponding quotient space with total order $\preceq'$ and $\pi \colon X \to X'$ the quotient map. If $f \colon X \to \mathbb{R}$ and $g \colon X' \to \mathbb{R}$ are functions such that $g \circ \pi = f$, then f is a continuous representation if and only if g is a continuous representation.*

3. Countable netweight and representation theorems

In this section we introduce the concepts of network and netweight of a topological space and use them to prove new continuous representation theorems. To the best of our knowledge, these concepts have not been used in the literature on representation theorems.

Definition 1. A family $\mathscr{N}$ of subsets of a topological space X is called a *network* for X if every non empty open subset of X is union of elements of $\mathscr{N}$.

As usual, we define by

$$nw(X) = \min\{|\mathscr{N}| : \ \mathscr{N} \text{ is a network for } X\} + \omega.$$

and

$$w(X) = \min\{|\mathscr{B}| : \ \mathscr{B} \text{ is a base for } X\} + \omega$$

the *network weight* (or *net weight*) and the *weight* of X respectively.

We recall that if $(X, \tau_{\preceq})$ is a totally ordered topological space ([9, 3.12.4.(d), p. 222]) then $nw(X) = w(X)$.

The same result can be proved when X is a totally preordered topological space.

Proposition 2. *Let $(X, \tau_{\preceq})$ be a totally preordered topological space and (X', τ') the corresponding ordered quotient space. Then $w(X) = w(X')$, $nw(X) = nw(X')$ and $nw(X) = w(X)$.*

Proof. It is easy to see that every open set in X is saturated with respect to the quotient map so the quotient map π is open. Hence, if $\mathscr{B}$ is a basis (network) for X then $\{\pi(B) : \ B \in \mathscr{B})\}$ is a basis (network) for X' and if $\mathscr{B}'$ is a basis (network) for X' then $\{\pi^{-1}(B') : \ B' \in \mathscr{B}'\}$ is a basis (network) for X. □

Using the concept of network we prove some continuous representation theorems for topological spaces that are countable unions of subspaces. Note that if $X = \bigcup_{i\in\mathbb{N}} X_i$, $(X, \tau_{\preceq})$ has countable netweight if and only if every subspace $(X_i, \tau_{\preceq}|X_i)$ has countable netweight.

Theorem 1. *Let (X, τ) be a topological space and $\preceq$ be a continuous total preorder on X and let $X = \bigcup_{i\in\mathbb{N}} X_i$. Then X has a continuous representation if and only if $\tau_{\preceq}|X_i$ has countable netweight (or countable weight) for every $i \in \mathbb{N}$.*

Proof.

$\Longrightarrow$: The second-countability of $\tau_{\preceq}$ follows from [1, Theorem 3.2.9]. Hence $\tau_{\preceq}|X_i$ is second countable for every $i \in \mathbb{N}$ and, a fortiori, has countable netweight.

$\Longleftarrow$: If $\mathscr{B}_i$ is a countable network of $\tau_{\preceq}|X_i$ for every $i \in \mathbb{N}$, then $\bigcup_i \mathscr{B}_i$ is a countable network of $(X, \tau_{\preceq})$. So $(X, \tau_{\preceq})$ has countable netweight and hence it is second countable. Now, the existence of a continuous representation follows from [1, Theorem 3.2.9]. $\square$

The following corollary gives a sufficient condition so that a subspace $(X_i, \tau_{\preceq}|X_i)$ has a countable netweight.

Corollary 1. *Let (X, τ) be a topological space and $\preceq$ be a continuous total preorder on X. Let $X = \bigcup_{i \in \mathbb{N}} X_i$, where $(X_i, \tau|X_i, \preceq|X_i)$ is representable for every $i \in \mathbb{N}$. If $\tau_{\preceq}|X_i = \tau_{\preceq|X_i}$ for every $i \in \mathbb{N}$, then X has a continuous representation.*

Remark 1. We note that if $X = \bigcup_{i \in \mathbb{N}} X_i$, is a space with topology τ and continuous total preorder $\preceq$, then the representability of $(X_i, \tau|X_i, \preceq|X_i)$ for every $i \in \mathbb{N}$ does not imply, in general, the representability of X.

In fact, let $X = \mathbb{R} \times \{0, 1\}$ with the lexicographic order $\preceq$ and the topology $\tau_{\preceq}$. If we set $X_1 = \mathbb{R} \times \{0\}$ and $X_2 = \mathbb{R} \times \{1\}$, the subspaces X_1 and X_2 have both the Sorgenfrey topology (finer than the ordinary topology) and so they are both representable but X is not representable since the Sorgenfrey topology is not second countable.

Remark 2. In the setting of the Corollary 1, the condition $\tau_{\preceq}|X_i = \tau_{\preceq|X_i}$ for every $i \in \mathbb{N}$ is sufficient but not necessary to have a continuous representation of X. One can consider a subset $A \subset \mathbb{R}$ such that $\tau_{\preceq}|A \neq \tau_{\preceq|A}$.

It is known that if A is a order-convex subspace of $(X, \tau, \preceq)$, then $\tau_{\preceq}|A = \tau_{\preceq|A}$. Hence the following theorems hold.

Theorem 2. *Let (X, τ) be a topological space and $\preceq$ be a continuous total preorder on X. Let $X = \bigcup_{i \in \mathbb{N}} X_i$, where $(X_i, \tau|X_i, \preceq|X_i)$ is representable for every $i \in \mathbb{N}$. If X_i is a order-convex subspace of $(X, \tau, \preceq)$ for every $i \in \mathbb{N}$, then X has a continuous representation.*

As a direct consequence of Theorem 2, we obtain the following known generalization of the classical theorem of Eilenberg [8] on the existence of a continuous order-preserving function for a continuous total order on a connected and separable space.

Theorem 3. *Let (X, τ) be a topological space and $\preceq$ be a continuous total preorder on X. Let $X = \bigcup_{i \in \mathbb{N}} X_i$ where X_i is a connected separable subspace of (X, τ) for every $i \in \mathbb{N}$, then X has a continuous representation.*

Proof. Note that, for every $i \in \mathbb{N}$, we have $\tau|X_i \supset \tau_{\preceq}|X_i \supset \tau_{\preceq|X_i}$ and so $\preceq|X_i$ is continuous with respect to $\tau|X_i$. Then, by Eilenberg's Theorem, $(X_i, \tau|X_i, \preceq|X_i)$ is representable for every $i \in \mathbb{N}$. Moreover every X_i is order-convex since it is connected. □

We now make some observations about the meaning, significance and consequences of Theorem 3.

Remark 3. Theorem 3 generalizes [3, Theorem 1] by Candeal et al. To see this let Y be locally connected, and separable with countable dense subset D. Since every connected component C_α of Y is open, then $C_\alpha \cap D \neq \emptyset$ for every α. So the family of all (pairwise disjoint) connected components of Y is countable. Moreover, each connected component is separable (since it is an open subset in a separable space). Of course, a space X that satisfies the hypotheses of Theorem 3 is, in general, not locally connected.

Our Theorem 3 also holds if "C_n connected and separable" is replaced by "C_n locally connected and separable". In fact, if C_n is locally connected and separable, then C_n is a countable union of its connected (separable) components, as we have already observed. So $X = \bigcup\limits_{n \in \mathbb{N}} C_n$ is also a countable union of connected, separable subspaces, because connected and separable subspaces of C_n are obviously also connected and separable subspaces of X.

Remark 4. Theorem 3 should also be compared with the result of Candeal et. al. [3, p. 709, statement b] in which a particular topology (the sum or "final topology") is used on the free union of the connected (or locally connected) and separable spaces. In this connection it is interesting to observe that if a topological space (X, τ) is the union of pairwise disjoint open subsets X_n then the topology τ coincides with the sum topology of the subspaces X_n ([9, p. 74], or [7, p. 127]). However, in general, this need *not* be the case. In particular, in Remark 3 above the sets C'_n are separable but they may not be open sets.

The following Theorem 4 includes the Eilenberg and Debreu Theorems on the existence of a continuous representation.

Theorem 4. *Let (X, τ) be a topological space and $\preceq$ be a continuous total preorder on X. If $X = \bigcup\limits_{i \in \mathbb{N}} X_i$ where, for every $i \in \mathbb{N}$, X_i is connected and separable or has countable netweight (in particular countable weight) then X has a continuous representation.*

Proof. As in the proof of Theorem 3, if X_i is connected and separable, then $(X_i, \tau|X_i, \preceq|X_i)$ is representable. Then the topology $\tau_{\preceq}|X_i = \tau_{\preceq|X_i}$ is second countable. Otherwise $(X_i, \tau|X_i)$ has countable netweight, hence $(X_i, \tau_{\preceq}|X_i)$ has countable netweight, too. Hence, the conclusion follows from Theorem 1. □

We remark that every space that satisfies the hypotheses of Theorem 4 is separable. Now we give an example of a continuously representable topological space that is not second countable or connected or a countable union of connected and separable spaces but which satisfies the conditions of Theorem 4.

Example 1. In $\mathbb{R}$ consider the topology τ generated by the following basis

$$\mathscr{B} = \big\{[q, z) \setminus \{s_n\}_{n\in\mathbb{N}} : \ q \in \mathbb{Q}, \ \{s_n\}_{n\in\mathbb{N}} \subset \mathbb{Q}, \ s_n \longrightarrow q\big\}.$$

Note that, for every $q \in \mathbb{Q}$ and for every $z \in \mathbb{R}$ the intervals (q, z) are in $\mathscr{B}$ (also the intervals $[q, z)$ are in $\mathscr{B}$) and so all the reals open intervals are in τ. Therefore, τ is strictly finer than the ordinary topology.

It is easy to see that $(\mathbb{R}, \tau)$ is totally disconnected (hence not countable union of connected subspaces).

Now, we will prove that $(\mathbb{R}, \tau)$ is not second countable. In fact, we will show that every rational point q does not have a local countable base. On the contrary, suppose that $\{U_n\}_{n\in\mathbb{N}}$ is a countable local base for q. Without loss of generality, we can suppose that $\{U_n\}_{n\in\mathbb{N}}$ is a decreasing family of sets, $U_n = [q, z_n)\setminus S_n \in \mathscr{B}$ and $z_n \longrightarrow q$. Now, we pick a rational number $a_n \in (q, z_n) \setminus S_n$ for every n. Then, if we put $U = [q, z) \setminus S$ with $S = \{a_n\}_{n\in\mathbb{N}}$ and $z > q$, it is easy to verify that $U \in \mathscr{B}$ and $U_n \nsubseteq U$ for every n. Hence $\{U_n\}_{n\in\mathbb{N}}$ cannot be a local base for q.

Finally, the subspace $\mathbb{R} \setminus \mathbb{Q}$ with the topology induced by τ is just the space of the irrationals numbers endowed with the ordinary topology.

Remark 5. We note that if (X, τ) is a topological space with countable netweight then every continuous total preorder on X is representable. But this condition is not necessary in order to every continuous total preorder on X be representable. For instance the product space $X = [0, 1]^{\mathbb{R}}$ is connected and separable and has weight c. Since X is also compact then $nw(X) = w(X) = c$.

We note an easy topological consequence of [1, Theorem 4, Proposition 1.6.11].

Corollary 2. *Let $(X, \tau_{\preceq})$ be a topological ordered space. If $X = \bigcup_{i\in\mathbb{N}} X_i$ where, for every $i \in \mathbb{N}$, X_i is connected and separable or has countable netweight then X is second countable.*

4. Representation theorems for topological preordered spaces with $\Gamma_{\preceq}$-structure

Now, we introduce the concept of a $\Gamma_{\preceq}$-structure on a topological space with a continuous total preorder $\preceq$. We will prove that the main theorem of Candeal et. al. [2], on separably connected space, can be also generalized to spaces with a

$\Gamma_{\preceq}$-structure. We recall that a topological space is said to be *separably connected* if for every $a, b \in X$ there is a connected and separable subset $C \subset X$ such that $a, b, \in C$.

Definition 2. Let X be a topological space with a total preorder $\preceq$ and let $\mathscr{F}$ be the family of finite non-empty subsets of X. A $\Gamma_{\preceq}$-structure on X is a family $\{\Gamma_A\}_{A \in \mathscr{F}}$ of separable and connected subsets of X such that

1. If $A \subset B$ and Γ_A, Γ_B are in a same connected component of X, then $\Gamma_A \cap \Gamma_B \neq \emptyset$.
2. If $\bigcup\limits_{n \in \mathbb{N}} A_n$ bounds $\preceq$, where $A_n \in \mathscr{F}$, then for every $x \in X$ there is a connected component C of X such that $F = \bigcup\limits_{n \in \mathbb{N}} \{\Gamma_{A_n} : \Gamma_{A_n} \subset C\}$ bounds x (in particular $\bigcup\limits_{n \in \mathbb{N}} \Gamma_{A_n}$ bounds $\preceq$).

We note that, if there is a connected $C \subset X$ such that $\Gamma_A \subset C$ for every $A \in \mathscr{F}$, then the properties 1 and 2 in the previous definition become:

1. If $A \subset B$ then $\Gamma_A \cap \Gamma_B \neq \emptyset$
2. If $\bigcup\limits_{n \in \mathbb{N}} A_n$ bounds $\preceq$ then $\bigcup\limits_{n \in \mathbb{N}} \Gamma_{A_n}$ also bounds $\preceq$.

This is the case, for example, when X is connected.

We observe that a separably connected space X with a preorder $\preceq$ admits a $\Gamma_{\preceq}$-structure. In fact the following result holds:

Theorem 5. *Let X be a topological space. Then X is separably connected if and only if X has a $\Gamma_{\preceq}$-structure for every total preorder $\preceq$ defined on X.*

P r o o f. Suppose X to be separably connected and let $\preceq$ be a preorder defined on X. Let $A \in \mathscr{F}$. If $|A| = 1$, we set $\Gamma_A = A$. Otherwise, if a is a fixed element of A, we set

$$\Gamma_A = \bigcup_{x \in A \setminus \{a\}} C_x$$

where C_x is a separable and connected subspace of X containing a and x. Note that every Γ_A is separable and connected. Moreover, since $\Gamma_A \supset A$ for every (finite) set A, the family $\{\Gamma_A\}_{A \in \mathscr{F}}$ satisfies the conditions 1 and 2 of Definition 2.

Conversely, suppose that X is a topological space such that, for every preorder $\preceq$ defined on X, there is a corresponding $\Gamma_{\preceq}$-structure. Let $a, b \in X$ and let $Y = X \setminus \{a, b\}$. If $\preceq'$ is a fixed ordering on Y, consider the ordering $\preceq$ on X that extends $\preceq'$ such that for every $y \in Y$ one has $a \prec y \prec b$. Let $\{\Gamma_A\}_{A \in \mathscr{F}}$ be a $\Gamma_{\preceq}$-structure on X. Then, if $A = \{a, b\}$, Γ_A is separable and connected. Moreover, since A bounds X also Γ_A bounds X. This forces $a, b \in \Gamma_A$ and we conclude that X is separably connected. $\square$

It is easy to give examples of a continuously representable total preorder on a space X with a $\Gamma_{\preceq}$-structure such that X is not separably connected.

Example 2. Let

$$Y = \{(x,y) \in \mathbb{R}^2 : \ x^2 + y^2 = 1, \ \ y \geq 0\}$$

and

$$Z = \{(x,0) \in \mathbb{R}^2 : \ -1 < x < 1\}.$$

Consider Y equipped with the Euclidean topology and Z with the discrete topology. Let X be the topological sum of Y and Z. Moreover, we define the following total preorder on X:

$$(a_1, b_1) \preceq (a_2, b_2) \iff a_1 \leq a_2$$

where $\leq$ is the usual ordering of $\mathbb{R}$. Then X is not separably connected (in fact X is not connected) and $\Gamma_A = Y$ for every $A \in \mathscr{F}$ defines a $\Gamma_{\preceq}$-structure on X.

Theorem 6. *Let X be a topological space with a continuous total preorder $\preceq$ and let $\{\Gamma_A\}$ be a $\Gamma_{\preceq}$-structure on X. The following are equivalent:*

1. *X has a continuous representation.*
2. *$\preceq$ is countably bounded.*
3. *There is a countable family $\{F_k\}_{k\in\mathbb{N}}$ of connected and separable subsets of X such that for every $x \in X$ there exists F_k that bounds x.*
4. *The quotient space X' is a countable union of connected and separable subspaces.*

Proof.

1 $\implies$ 2: It is well known.

2 $\implies$ 3: Assume that $A = \{x_n\}_n$ bounds X and put $A_i = \{x_1, x_2, \dots, x_i\}$. If $\{C_\alpha\}_{\alpha\in A}$ is the family of connected components of X, we set

$$F_\alpha = \bigcup_{i\in\mathbb{N}} \{\Gamma_{A_i} : \ \Gamma_{A_i} \subset C_\alpha\}$$

for every $\alpha \in A$. Then the family $\{F_\alpha : \ \ F_\alpha \neq \emptyset\}$ is countable and every F_α, by property 1 of Γ-structure, is connected and separable. The remainder of the proof follows by property 2.

3 $\implies$ 4: Let $\pi\colon X \to X'$ be the quotient map, we will prove that $X' = \bigcup_{k\in\mathbb{N}} \pi(F_k)$. Let $x \in X$ and let F_k be a connected and separable subset of X that bounds x. The sets $A = \{a \in F_k : \ \ a \preceq x\}$ and $B = \{b \in F_k : \ \ x \preceq b\}$ are non-empty closed subsets of F_k, because F_k bounds x and $\preceq$ is continuous. Since $A \cup B = F_k$ and F_k is connected, then $A \cap B \neq \emptyset$. Hence, there is $y \in A \cap B$ such that $\pi(x) = \pi(y)$. It follows that $X' = \pi(X) = \bigcup\{\pi(F_k) : \ \ k \in \mathbb{N}\}$ where every $\pi(F_k)$ is connected and separable.

$4 \implies 1$: This follows from Theorem 4. $\square$

From the proof of Theorem 6 we also deduce the following result which generalizes a theorem of Monteiro [11] on the existence of a continuous representation.

Theorem 7. *Let $X = \bigcup_{n\in\mathbb{N}} X_n$ be a topological space and let $\preceq$ be a continuous total preorder on X. Assume that there is a family $\{F_k\}_{k\in\mathbb{N}}$ of connected and separable subsets of X such that for every $x \in X$ there is F_k that bounds x. Then X has a continuous representation.*

In the next proposition we extend a result of Candeal, Herves, Indurain [2] to the case when X is the countable union of spaces with a Γ-structure.

Theorem 8. *Let $X = \bigcup_{n\in\mathbb{N}} X_n$ be a topological space and let $\preceq$ be a continuous total preorder on X. Suppose that for every n, X_n has a $\Gamma_{\preceq|X_n}$-structure. Then $\preceq|X_n$ is countably bounded for every n if and only X has a continuous representation.*

Proof.

$\implies$: Let $\preceq|X_n$ be countably bounded for every n. The subspace topology on X_n, which it inherits as a subspace of $(X, \tau_\preceq)$ is, in general, finer than the preorder topology induced by $\preceq|X_n$. Therefore, $\preceq|X_n$ is $\tau|X_n$-continuous, because $\preceq$ is τ-continuous. By applying Theorem 6 to every X_n we get a family $\{F_k\}_{k\in\mathbb{N}}$ of connected and separable subsets of X such that for every X_n and for every $x \in X_n$ there is $F_k \subset X_n$ that bounds x. The existence of a continuous representation follows from Theorem 7.

$\impliedby$: Obvious. $\square$

Remark 6. Theorem 8 does not hold if the assumption on countable boundedness of every $\preceq|X_n$ is replaced by countable boundedness of $\preceq$. This is proved in the following example.

Let (Y, τ') be a separably connected space with a continuous total preorder $\preceq'$ not countably bounded. If $a, b \notin Y$, $a \neq b$, let (X, τ) be the topological extension of Y, where $X = Y \cup \{a\} \cup \{b\}$ and τ is the topology generated by $\tau' \cup \big\{\{a\}, \{b\}\big\}$. Moreover, consider on X the total preorder defined by $\preceq|Y = \preceq'$ and $a \preceq y \preceq b$ for every $y \in Y$. Theorem 5 provides a $\Gamma_{\preceq|X_n}$-structure on X_n, $n = 1, 2, 3$, where $X_1 = Y$, $X_2 = \{a\}$, $X_3 = \{b\}$. Now it is easy to check that the preorder $\preceq$ is continuous with respect to τ and, of course, countably bounded, but if X has a continuous representation then so does Y.

5. A representation theorem for hereditarily separable spaces

We now prove a representation theorem for a certain class of hereditarily separable spaces. We first prove some simple properties of connected components of topological preordered spaces.

Lemma 1. *Every non-trivial connected subspace of $(X, \tau_{\preceq})$, where $\preceq$ is a total order, has non-empty interior.*

Proof. Let Y be a connected subspace of X and let $a, b \in Y$, $a \neq b$. Suppose that $a < b$. Then the order interval (a, b) is non-empty and $(a, b) \subset Y$. In fact, if $c \in (a, b) \setminus Y$, then $\big((-\infty, c) \cap Y, (c, +\infty) \cap Y\big)$ would be a disconnection of Y. □

The proof of Lemma 1 does not hold if $\preceq$ is a preorder. In fact, if $c \in (a, b) \setminus Y$, there may be $c' \in Y$ with $c' \sim c$ and $\big((-\infty, c) \cap Y, (c, +\infty) \cap Y\big)$ would not be a disconnection of Y.

Of course, the following (more general) result holds as well.

Lemma 2. *Let (X, τ) be a topological space and let $\preceq$ be a continuous total order on X. Then every non-trivial connected subspace of X has non-empty interior.*

The property stated in Lemmas 1 and 2 is characteristic of the order topologies.

Example 3. Let $A = \bigcup_{k \in \mathbb{N}} (2k, 2k+1) \subset \mathbb{R}$ and let $X = A^{\mathbb{N}}$. Then X is second-countable because it is a countable product of second-countable spaces. Moreover every $C \subset X$ is a connected (separable) component of X if and only if $C = \prod_{n \in \mathbb{N}} C_n$ where C_n is a connected component of A ([9, Theorem 6.1.21, p. 356]). So, the cardinality of the connected components of X is uncountable. Moreover every connected component of X has empty interior. In fact, every basic open of X is of the form $\prod_{n \in \mathbb{N}} B_n$, where $B_n = A$ for all but finitely many n and so $B \nsubseteq C$ for every connected component C of X. Of course, by Lemma 2, there is no total order on X that is continuous with respect to the topology defined on X.

The following result is now an easy consequence.

Proposition 3. *Let (X, τ) be a separable (hereditarily separable) space with countably many trivial connected components and let $\preceq$ be a continuous total order on X. Then X is a countable union of connected (separable and connected) subspaces.*

Note that a separable space with a continuous total order is not in general a countable union of connected subspaces. In fact, the space of the irrational numbers is an example of a second countable topological ordered space that is the union of uncountably many trivial components.

We next consider the hereditarily separable spaces from the representation point of view. These spaces are important because they are generalizations of second countable spaces.

Theorem 9. *Let (X, τ) be a topological space and let $\preceq$ be a continuous total preorder on X such that (X', τ') is hereditarily separable with countably many trivial connected components. Then X has a continuous representation.*

Note that if (X, τ) is hereditarily separable then (X', τ') also has the same property.

We now give an example of a topological space X that is not separable such that the quotient space X' is not second countable but hereditarily separable and with countably many trivial connected components.

Example 4. In $X = \mathbb{R} \times \mathbb{R}$ consider the topology τ generated by the basis consisting of sets of the form $T \times \{a\}$ where $a \in \mathbb{R}$ and T is an open real interval or $T = [k + \frac{1}{h}, z) \setminus \{s_n\}_{n\in\mathbb{N}}$ for $k \in \mathbb{Z}$, $h \in \mathbb{N}^+$, $z \in \mathbb{R}$, $\{s_n\}_{n\in\mathbb{N}} \subset \mathbb{Q}$, $s_n \longrightarrow k + \frac{1}{h}$.

Let define on X the total preorder

$$(x, a) \preceq (y, b) \iff x \le y.$$

We can prove that (X, τ) is not separable and that (X', τ') is not second countable (see Example 1) but hereditarily separable. Indeed, let $S \subset X'$ and let $D \subset S$ be a countable dense subset with respect to the usual topology on the reals. Then it is easy to verify that $D' = D \cup \left(\{k + \frac{1}{h} : \ k \in \mathbb{Z},\ \ h \in \mathbb{N}^+\} \cap S\right)$ is a countable dense subset of S with respect to the topology τ'. Moreover X' has countably many trivial connected components. In fact, the family of connected components is exactly $\left\{\left[k + \frac{1}{h+1}, k + \frac{1}{h}\right) : \ k \in \mathbb{Z},\ \ h \in \mathbb{N}^+\right\} \cup \left\{\{k\} : \ k \in \mathbb{Z}\right\}$ since the induced topology on $\left(k + \frac{1}{h+1}, k + \frac{1}{h}\right)$ is just the Euclidean topology. Therefore, X satisfies all the hypotheses of Theorem 9.

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