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SIMPLEX-VALUED PROBABILITY

Roman Frič

Dedicated to Silvia Pulmannová

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ABSTRACT. We continue our study of generalized probability from the viewpoint of category theory. Assuming that each generalized probability measure is a morphism, we model basic probabilistic notions within the category cogenerated by its range. It is known that the closed unit interval $I = [0, 1]$, carrying a suitable difference structure, cogenerates the category ID in which the classical and fuzzy probability theories can be modeled. We study generalized probability theories modeled within two different categories cogenerated by a simplex $S_n = \left\{(x_1, x_2, \ldots, x_n) \in I^n : \sum_{i=1}^{n} x_i \leq 1\right\}$, carrying a suitable difference structure; since I and S_1 coincide, for $n = 1$ we get the fuzzy probability theory as a special case. In the first case, when the morphisms preserve the so-called pure elements, the resulting category S_nD, $n > 1$, and ID are isomorphic and the generalized probability theories modeled in ID and S_nD are "the same". In the second case, when the morphisms map each maximal element to a maximal element, the resulting categories WS_nD, $n > 1$, lead to different models of generalized probability theories. We define basic notions of the corresponding simplex-valued probability theories and mention some applications.

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1. Introduction

Categorical approach to probability theory leads to a better understanding of its mathematical foundations and it makes possible to compare different models in a natural way. For example, using particular results from [12], [6], [10], [13], [18], it has been proved in [14] that the fuzzy probability theory developed by

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S. Gudder (cf. [15]) and S. Bugajski (cf. [2], [3]) can be viewed as the minimal extension of the classical probability theory having quantum character.

By a categorical approach we mean a selection of a suitable concrete category $\mathscr{C}$ (objects are structured sets and morphisms are structure preserving maps, cf. [1]) in which objects model random events, morphisms model both probability measures (states) and random variables (dually observables), basic probabilistic constructions become "categorical", and "chasing diagrams" helps to understand and to prove relevant probabilistic theorems. If C is the range of the generalized probability measures, then a convenient choice of $\mathscr{C}$ is the category cogenerated by C: probability domains are subobjects of powers C^X of C (cf. [8], [18], [13], [14]).

The basic notions of any generalized probability theory are generalized random events, generalized probability measures called states, and generalized random variables. In the classical theory, random events are modeled by σ-fields of subsets of the set of all elementary events, each probability measure maps the random events into the interval $[0,1]$ and partially preserves operations on events, and each random variable is a measurable map of the elementary events into the real numbers $\mathbb{R}$. If $(\Omega, \mathbf{A}, P)$ is a classical probability space and a random variable f maps Ω into $\mathbb{R}$, then f "pushes forward" P into the distribution P_f, a probability measure on the real Borel sets defined by $P_f\big((-\infty, r)\big) = P\big(f^{\leftarrow}((-\infty, r))\big) = P\big(\{\omega \in \Omega :\ f(\omega) < r\}\big)$.

Fuzzy random events are modeled by the measurable functions $\mathscr{M}(\mathbf{A})$ on a classical measurable space $(\Omega, \mathbf{A})$ into the interval $[0,1]$, each fuzzy probability measure is the Lebesgue integral with respect to a probability measure on $\mathbf{A}$, and each fuzzy random variable is a "measurable" map of the set $\mathscr{P}(\mathbf{A})$ of all probability measures on $\mathbf{A}$ (as usually, we identify each elementary event ω and the corresponding degenerated point probability measure δ_ω) into the set $\mathscr{P}(\mathbf{B})$ of all probability measures on $\mathbf{B}$, where $(\Xi, \mathbf{B})$ is another classical measurable space. In a sense, $\mathscr{P}(\mathbf{A})$ becomes the set of all elementary fuzzy events. More precisely (cf. [2]):

Definition 1.1. Let $(\Omega, \mathbf{A})$, $(\Xi, \mathbf{B})$ be measurable spaces. Let T be a map of $\mathscr{P}(\mathbf{A})$ into $\mathscr{P}(\mathbf{B})$ such that, for each $B \in \mathbf{B}$, the assignment $\omega \mapsto \big(T(\delta_\omega)\big)(B)$ yields a measurable map of Ω into $[0,1]$ and

$$\big(T(P)\big)(B) = \int \big(T(\delta_\omega)\big)(B)\,\mathrm{d}P(\omega) \tag{1}$$

for all $P \in \mathscr{P}(\mathbf{A})$ and all $B \in \mathbf{B}$. Then T is said to be a fuzzy random variable (in the sense of Bugajski and Gudder, also a statistical map).

Observe that if f is a classical measurable map of Ω into Ξ, then the distribution D_f of f (sending a probability P into $P_f = P \circ f^{\leftarrow}$) is a fuzzy random variable. Indeed, $\big(D_f(\delta_\omega)\big)(B) = 1$ iff $f(\omega) \in B$ and (1) means $D_f(P) = P \circ f^{\leftarrow}$, $P \in \mathscr{P}(\mathbf{A})$.

In fact, this means that (identifying a set A and its characteristic function χ_A) the classical probability theory can be studied within the fuzzy probability theory

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(the classical random events coincide with the crisp, i.e., $\{0,1\}$-valued fuzzy events). Moreover, this makes our goal to discuss basic probability notions from the viewpoint of category theory more simple and natural.

Observe that the fuzzy random events carry a suitable algebraic structure (MV-algebra, more precisely generated Łukasiewicz tribe) and there is a one-to-one correspondence between the fuzzy random variables, sending $\mathscr{P}(\mathbf{A})$ into $\mathscr{P}(\mathbf{B})$, and dual maps from the fuzzy events $\mathscr{M}(\mathbf{B})$ into the fuzzy events $\mathscr{M}(\mathbf{A})$ partially preserving the algebraic structure of fuzzy events. Such maps are called observables and they play a more fundamental role in the fuzzy probability theory than in the classical one.

Now, we recall basic facts about the category ID of D-posets of fuzzy sets and sequentially continuous D-homomorphisms, a category in which probability theory "goes well" (it is the category cogenerated by I, the interval $[0,1]$ carrying a suitable difference structure). In particular, σ-fields of sets and generated Łukasiewicz tribes form full subcategories of ID and observables, probability measures and states become morphisms.

D-posets have been introduced in [17] in order to model events in quantum probability (cf. [23]). They generalize MV-algebras and other probability domains ([22], [12]). It is known that D-posets are equivalent to effect algebras introduced in [7]. More recent results about D-posets and effect algebras can be found in [4], [5], [9], and [20].

Remind that a D-poset is a partially ordered set Y with the greatest element 1, the least element 0, and a partial binary operation $\ominus$ called *difference*, such that $a \ominus b$ is defined iff $b \leq a$, and the following axioms are assumed:

(D1) $a \ominus 0_Y = a$ for each $a \in Y$;

(D2) If $c \leq b \leq a$, then $a \ominus b \leq a \ominus c$ and $(a \ominus c) \ominus (a \ominus b) = b \ominus c$.

The closed interval I carrying the natural order and difference is a canonical example of a D-poset. Fundamental to applications ([10], [11], [19]) are *D-posets of fuzzy sets* ([16]), i.e. systems $\mathscr{X} \subseteq I^X$ carrying the pointwise partial order, containing the top and bottom elements (constant functions 1_X and 0_X) of I^X, and closed with respect to the partial operation difference defined pointwise: for $g \leq f$, $(f \ominus g)(x) = f(x) - g(x)$, $x \in X$ (we always assume that $\mathscr{X}$ is reduced, i.e., points of X are separated by functions of $\mathscr{X}$). If X is a singleton, then I^X will be condensed to I. Denote ID the category having D-posets of fuzzy sets carrying the pointwise convergence of sequences as objects and having sequentially continuous D-homomorphisms as morphisms. Objects of ID are subobjects of the powers I^X.

2. Category S_nD

The category S_nD has been introduced and basic probability notions within S_nD have been defined in [13]. In the resulting S_nD-probability we have n-component probability domains in which each event represents a body of competing

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components and the range of a state represents a simplex S_n of n-tuples of possible "rewards" — the sum of the rewards is a number from $[0,1]$. For $n=1$ we get fuzzy events and the corresponding fuzzy probability theory. For $n=2$ we get IF-events, i.e., pairs (μ,ν) of fuzzy sets $\mu,\nu\in[0,1]^X$ such that $\mu(x)+\nu(x)\leq 1$ for all $x\in X$, but we order our pairs (events) coordinatewise. Hence the structure of IF-events (where $(\mu_1,\nu_1)\leq(\mu_2,\nu_2)$ whenever $\mu_1\leq\mu_2$ and $\nu_2\leq\nu_1$) is different and, consequently, the resulting IF-probability theory models a different principle ([21]).

For $n\in\{1,2,\dots\}$ denote $S_n=\Big\{(x_1,x_2,\dots,x_n)\in I^n:\ \sum\limits_{i=1}^{n}x_i\leq 1\Big\}$ carrying the coordinatewise partial order, difference and sequential convergence. Let X be a nonempty set and let S_n^X be the set of all maps of X into S_n; if X is a singleton $\{a\}$, then $S_n^{\{a\}}$ will be condensed to S_n. Let $\mathbf{f}\in S_n^X$. Then there are n maps $f_1,f_2,\dots,f_n$ of X into I such that for each $x\in X$ we have $\mathbf{f}(x)=(f_1(x),f_2(x),\dots,f_n(x))$; we shall write $\mathbf{f}=(f_1,f_2,\dots,f_n)$. In what follows, S_n^X carries the coordinatewise partial order ($\mathbf{g}\leq\mathbf{f}$ iff $g_i\leq f_i$ for all i, $1\leq i\leq n$), the coordinatewise partial difference (for $\mathbf{g}\leq\mathbf{f}$ define $\mathbf{f}\ominus\mathbf{g}=(f_1\ominus g_1,f_2\ominus g_2,\dots,f_n\ominus g_n)$), and the coordinatewise sequential convergence inherited from S_n. Elements $(f_1,f_2,\dots,f_n)\in S_n^X$ such that $\sum\limits_{i=1}^{n}f_i(x)=1$, $x\in X$, are maximal. If for some index i, $1\leq i\leq n$, we have $f_j(x)=0$ for all $j\neq i$ and all $x\in X$, then $(f_1,f_2,\dots,f_n)$ is said to be *pure*; denote $\mathbf{p_i}$ the corresponding *maximal pure* element of S_n^X. Clearly, if for all i, $1\leq i\leq n$, the functions f_i are constant zero functions, then $(f_1(x),f_2(x),\dots,f_n(x))$ is the least element of S_n^X; it is called the *bottom* element and denoted by $\mathbf{b}$. To avoid complicated notation, if no confusion can arise, then the bottom elements, resp. the ith maximal pure elements, will be denoted by the same symbol $\mathbf{b}$, resp. $\mathbf{p_i}$, $1\leq i\leq n$, independently of the ground set X.

Let X be a nonempty set. We are interested in subsets $\mathscr{X}\subseteq S_n^X$ closed with respect to the difference, containing the bottom element and all maximal pure elements of S_n^X. For $n=1$ we get D-posets of fuzzy sets and for $n>1$ we get a structure which generalizes fuzzy events to higher dimensions.

Let $\mathscr{B}_1,\mathscr{B}_2,\dots,\mathscr{B}_n\subseteq I^X$ be (reduced) ID-posets. Define $S(\mathscr{B}_1,\mathscr{B}_2,\dots,\mathscr{B}_n)$ to be the set of all $(f_1,f_2,\dots,f_n)\in S_n^X$ such that $f_i\in\mathscr{B}_i$, $1\leq i\leq n$. If there exists an ID-poset $\mathscr{B}\subseteq I^X$ such that $\mathscr{B}=\mathscr{B}_i$, $1\leq i\leq n$, then $S(\mathscr{B}_1,\mathscr{B}_2,\dots,\mathscr{B}_n)$ is condensed to $S_n(\mathscr{B})$. In applications we consider only the later case.

Definition 2.1. Let X be a nonempty set. Let $\mathscr{X}$ be a subset of S_n^X, carrying the coordinatewise order, the coordinatewise convergence and closed with respect to the inherited difference. Assume that $\mathscr{X}$ contains the bottom element and all maximal pure elements. Then $(\mathscr{X},\leq,\ominus,\mathbf{b},\mathbf{p_1},\dots,\mathbf{p_n})$ is said to be an *S_nD-domain*. If there is a (reduced) ID-poset $\mathscr{B}\subseteq I^X$ such that $\mathscr{X}=S_n(\mathscr{B})$, then $(\mathscr{X},\leq,\ominus,\mathbf{b},\mathbf{p_1},\dots,\mathbf{p_n})$ is said to be a *simple S_nD-domain* and $\mathscr{B}$ is said to be the *base* of $\mathscr{X}$.

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If no confusion can arise, then $(\mathscr{X}, \leq, \ominus, \mathbf{b}, \mathbf{p_1}, \ldots, \mathbf{p_n})$ will be reduced to $\mathscr{X}$. *In what follows, all S_nD-domains are assumed to be simple.* Clearly, the categories ID and S_1D coincide and each S_n^X is a simple S_nD-domain.

Definition 2.2. Let $\mathbf{h}$ be a map of a simple S_nD-domain $\mathscr{X}$ into a simple S_nD-domain $\mathscr{Y}$ such that

(i) $\mathbf{h}(\mathbf{v}) \leq \mathbf{h}(\mathbf{u})$ whenever $\mathbf{u}, \mathbf{v} \in \mathscr{X}$ and $\mathbf{v} \leq \mathbf{u}$, and then $\mathbf{h}(\mathbf{u} \ominus \mathbf{v}) = \mathbf{h}(\mathbf{u}) \ominus \mathbf{h}(\mathbf{v})$;

(ii) $\mathbf{h}$ maps the bottom element of $\mathscr{X}$ to the bottom element of $\mathscr{Y}$ and the ith maximal pure element of $\mathscr{X}$ to the ith maximal pure element of $\mathscr{Y}$, for all i, $1 \leq i \leq n$.

Then $\mathbf{h}$ is said to be an *S_nD-homomorphism.* A sequentially continuous S_nD-homomorphism of $\mathscr{X}$ into $\mathscr{Y}$ is said to be an *S_nD-observable.* A sequentially continuous S_nD-homomorphism of $\mathscr{X}$ into S_n is said to be an *S_n-valued state* or, simply, a *state.*

3. ID and S_nD are isomorphic

Let $\mathscr{A} \subseteq I^X$ be a D-poset of fuzzy sets and let $S_n(\mathscr{A})$ be the corresponding simple S_nD-domain. It is known that $\mathscr{A}$ carries the dual effect algebra structure based on the partial operation $\oplus$ dual to the difference: for $f, g \in \mathscr{A}$, $f \oplus g$ is defined whenever there exists $h \in \mathscr{A}$ such that $f = h \ominus g$ and then $f \oplus g = h$ (cf. [4]). Using the same symbol $\oplus$, we generalize the partial effect operation to $S_n(\mathscr{A})$ in a natural way: for $\mathbf{f}, \mathbf{g} \in S_n(\mathscr{A})$, $\mathbf{f} \oplus \mathbf{g}$ is defined whenever there exists $\mathbf{h} \in S_n(\mathscr{A})$ such that $\mathbf{f} = \mathbf{h} \ominus \mathbf{g}$ and then $\mathbf{h} = \mathbf{f} \oplus \mathbf{g}$. It is easy to see that if $\mathbf{f} = (f_1, f_2, \ldots, f_n) \in S_n(\mathscr{A})$, then $\sum_{i=1}^{n} f_i(x) \leq 1$, $x \in X$, implies that $f_1 \oplus f_2$, $f_1 \oplus f_2 \oplus f_3$, $\ldots$, $f_1 \oplus \cdots \oplus f_n$ are well-defined in $\mathscr{A}$ and if h is a D-homomorphism of $\mathscr{A}$ into a D-poset of fuzzy sets $\mathscr{B} \subseteq I^Y$, then $\big(h(f_1), h(f_2), \ldots, h(f_n)\big) \in S_n(\mathscr{B})$. Indeed, h is an "effect homomorphism" and $\mathbf{f} = (f_1, f_2, \ldots, f_n) \in S_n(\mathscr{A})$ implies $f_1 \oplus \cdots \oplus f_n \in \mathscr{A}$, $h(f_1 \oplus \cdots \oplus f_n) = h(f_1) \oplus \cdots \oplus h(f_n) \in \mathscr{B}$, and hence $\big(h(f_1), h(f_2), \ldots, h(f_n)\big) \in S_n(\mathscr{B})$.

Theorem 3.1. *Let $\mathscr{X} = S_n(\mathscr{A}) \subseteq S_n^X$ and $\mathscr{Y} = S_n(\mathscr{B}) \subseteq S_n^Y$ be simple S_nD-domains.*

(i) *Let h be a D-homomorphism of $\mathscr{A}$ into $\mathscr{B}$. For $\mathbf{f} = (f_1, f_2, \ldots, f_n) \in \mathscr{X}$ put $\mathbf{h}(\mathbf{f}) = \big(h(f_1), h(f_2), \ldots, h(f_n)\big) \in \mathscr{Y}$ and denote $\mathbf{h}$ the resulting map of $\mathscr{X}$ into $\mathscr{Y}$. Then $\mathbf{h}$ is an S_nD-homomorphism.*

(ii) *Let $\mathbf{h}$ be an S_nD-homomorphism of $\mathscr{X}$ into $\mathscr{Y}$. Then there exists a unique D-homomorphism h of $\mathscr{A}$ into $\mathscr{B}$ such that for each $\mathbf{f} = (f_1, f_2, \ldots, f_n) \in \mathscr{X}$ we have $\mathbf{h}(\mathbf{f}) = \big(h(f_1), h(f_2), \ldots, h(f_n)\big)$.*

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P r o o f. The proof of (i) is straightforward and it is omitted.

(ii) Given $\mathbf{g} = (g_1, g_2, \dots, g_n) \in S_n^Z$, for each k, $1 \le k \le n$, define $\mathrm{red}_k(\mathbf{g}) = (h_1, h_2, \dots, h_n)$, where $h_k = g_k$ and $h_j = 0_Z$ otherwise.

Let $\mathbf{f} = (f_1, f_2, \dots, f_n) \in S_n(\mathscr{A})$ and let $\mathbf{h}(\mathbf{f}) = (u_1, u_2, \dots, u_n) \in S_n(\mathscr{B})$. Since $\mathbf{h}(\mathbf{f} \ominus \mathrm{red}_n(\mathbf{f})) = \mathbf{h}(\mathbf{f}) \ominus \mathbf{h}(\mathrm{red}_n(\mathbf{f})) = \mathbf{h}\big((f_1, f_2, \dots, f_{n-1}, 0_X)\big) \in S_n(\mathscr{B})$ and $\mathbf{h}$ preserves order, necessarily there are elements $v_k \in S_n(\mathscr{B})$, $1 \le k \le n$, such that $\mathbf{h}\big((f_1, f_2, \dots, f_{n-1}, 0_X)\big) = (v_1, v_2, \dots, v_{n-1}, 0_Y) \in S_n(\mathscr{B})$ and $\mathbf{h}\big(\mathrm{red}_n(\mathbf{f})\big) = (0_Y, 0_Y, \dots, 0_Y, v_n)$. Hence $\mathbf{h}\big(\mathrm{red}_n(\mathbf{f})\big) = (0_Y, 0_Y, \dots, 0_Y, u_n) = \mathrm{red}_n(u_1, u_2, \dots, u_n)$ and $\mathbf{h}\big((f_1, f_2, \dots, f_{n-1}, 0_X)\big) = (u_1, u_2, \dots, u_{n-1}, 0)$, i.e., $u_i = v_i$ for all i, $1 \le i \le n$. Inductively, $\mathbf{h}\big(\mathrm{red}_k(\mathbf{f})\big) = \mathrm{red}_k(u_1, u_2, \dots, u_n)$, $1 \le k \le n$. For each k, $1 \le k \le n$, define $\mathscr{X}_k = \big\{(g_1, g_2, \dots, g_n) \in S_n(\mathscr{A}) : \ g_l = 0_X \text{ for all } l \ne k, \ 1 \le l \le n\big\}$. Then $\mathbf{h}$ on $\mathscr{X}_k$ can be identified with an S_nD-homomorphism h_k on $\mathscr{A}$ into $\mathscr{B}$ and $\mathbf{h}(\mathbf{f}) = \big(h_1(f_1), h_2(f_2), \dots, h_n(f_n)\big)$. Now, it suffices to prove that $h_i = h_j$ for all $i \ne j$, $1 \le i \le n$, $1 \le j \le n$. Contrariwise, suppose that there exists $f \in \mathscr{A}$ and $i < j$ such that $u = h_i(f) < h_j(f) = v$. Define $\mathbf{g} = (g_1, g_2, \dots, g_n) \in S_n(\mathscr{A})$ as follows: $g_i = 1_X \ominus f$, $g_j = f$, and $g_k = 0_X$ otherwise. Then $\mathbf{h}(\mathbf{g}) = (w_1, w_2, \dots, w_n) \in S_n(\mathscr{B})$, where $w_i = h_i(1_X \ominus f) = 1 - u$, $w_j = h_j(f) = v$, and $w_k = 0_Y$ otherwise. Then $\sum_{i=1}^{n} w_i = 1_Y - u + v > 1_Y$, a contradiction. This completes the proof. □

Denote S_nD the category of simple S_nD-domains and sequentially continuous S_nD-homomorphisms.

Theorem 3.2. *The categories ID and S_nD are isomorphic.*

P r o o f. The assertion is an obvious corollary of the previous theorem and the one-to-one correspondence between simple S_nD-domains $S_n(\mathscr{A}) \subseteq S_n^X$ and D-posets of fuzzy sets $\mathscr{A} \subseteq I^X$ as their bases. □

Consequently, even though I (the same as S_1) and S_n, $n > 1$, are quite different cogenerators, the ID-probability and S_nD-probability are "the same" (for all n). In the next section we propose a possible solution of this "paradox".

4. Categories WS_nD

Definition 4.1. Let $\mathbf{h}$ be a map of a simple S_nD-domain $S_n(\mathscr{A})$ into a simple S_nD-domain $S_n(\mathscr{B})$ such that

(i) $\mathbf{h}$ preserves the order and difference;

(ii) $\mathbf{h}$ maps each maximal element of $S_n(\mathscr{A})$ to a maximal element of $S_n(\mathscr{B})$.

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Then $\mathbf{h}$ is said to be a *weak S_nD-homomorphism*. A sequentially continuous weak S_nD-homomorphism is said to be a *weak S_nD-observable*. A sequentially continuous weak S_nD-homomorphism of $S_n(\mathscr{A})$ into S_n is said to be a *weak S_n-valued state* or, simply, a *weak state*.

Denote WS_nD the category of all simple S_nD-domains and all sequentially continuous weak S_nD-homomorphisms. Clearly, each S_nD-homomorphism is a weak S_nD-homomorphisms.

Let $\mathscr{X} = S_n(\mathscr{A}) \subseteq S_n^X$ and $\mathscr{Y} = S_n(\mathscr{B}) \subseteq S_n^Y$ be simple S_nD-domains and let h be a sequentially continuous D-homomorphism of $\mathscr{A}$ into $\mathscr{B}$. Unlike in S_nD, in WS_nD there are many possibilities how to define a morphism of $\mathscr{X}$ into $\mathscr{Y}$ starting from the morphism h of $\mathscr{A}$ into $\mathscr{B}$. Of course, $\mathbf{h}\big((f_1, f_2, \dots, f_n)\big) = \big(h(f_1), h(f_2), \dots, h(f_n)\big)$, $(f_1, f_2, \dots, f_n) \in \mathscr{X}$, is a trivial one.

Let $t(1, 2, \dots, n) = \big(t(1), t(2), \dots, t(n)\big)$ be a permutation of the indices of $\mathbf{f} = (f_1, f_2, \dots, f_n) \in \mathscr{X}$. Define $\mathbf{h}_t(\mathbf{f}) = \big(h(f_{t(1)}), h(f_{t(2)}), \dots, h(f_{t(n)})\big)$ and denote $\mathbf{h}_t$ the corresponding map of $\mathscr{X}$ into $\mathscr{Y}$.

For $\mathbf{f} = (f_1, f_2, \dots, f_n) \in \mathscr{X}$, define $\mathbf{h}_1(\mathbf{f}) = \big(h(f_1) \oplus \dots \oplus h(f_n), 0_Y, \dots, 0_Y\big)$ and denote $\mathbf{h}_1$ the corresponding map of $\mathscr{X}$ into $\mathscr{Y}$. Analogously, define maps $\mathbf{h}_k$, $1 \le k \le n$, where the kth coordinate of $\mathbf{h}_1(\mathbf{f})$ is equal to $h(f_1) \oplus \dots \oplus h(f_n)$ and all other coordinates are equal to 0_Y.

For $\mathbf{f} = (f_1, f_2, \dots, f_n) \in \mathscr{X}$, define $\mathbf{h}_a(\mathbf{f}) = (g_1, g_2, \dots, g_n) \in \mathscr{Y}$ as follows: $g_i = \big(h(f_1) \oplus \dots \oplus h(f_n)\big)n^{-1}$, $1 \le i \le n$. Clearly, $\mathbf{h}_a(\mathbf{f}) \in \mathscr{Y}$. Denote $\mathbf{h}_a$ the corresponding map of $\mathscr{X}$ into $\mathscr{Y}$.

The proof of the next assertion is straightforward and it is omitted.

Lemma 4.2. *The maps $\mathbf{h}_t$, $\mathbf{h}_a$, and $\mathbf{h}_k$, $1 \le k \le n$, are sequentially continuous weak S_nD-homomorphisms of $\mathscr{X}$ into $\mathscr{Y}$.*

If Y is a singleton, then h maps $\mathscr{X}$ into S_n and $\mathbf{h}_t$, $\mathbf{h}_a$, and $\mathbf{h}_k$, $1 \le k \le n$, are weak states.

Problem. Let $\mathbf{h}$ be a sequentially continuous weak S_nD-homomorphism of a simple S_nD-domain $S_n(\mathscr{A}) \subseteq S_n^X$ into a simple S_nD-domain $S_n(\mathscr{B}) \subseteq S_n^Y$. Does there exist a sequentially continuous D-homomorphism h of $\mathscr{A}$ into $\mathscr{B}$ and a simple rule defining $\mathbf{h}$ in terms of h?

At the end, we hint some situations which could be modeled within WS_nD. For $n = 1$, S_nD and WS_nD are the same as ID. The interpretation of a state is roughly the same as in the classical probability theory, but the domain of the state can contain also fuzzy events. For $n > 1$, the interpretation of a weak state in WS_nD defined via a state on the base of its simple S_nD-domain should involve some strategy (policy) how to "redistribute" the state along n coordinates. For example, if t is the neutral permutation, then the weak state $\mathbf{h}_t$ on $S_n(\mathscr{A})$ "measures" all coordinates via the same state h on $\mathscr{A}$. The weak state $\mathbf{h}_a$ on $S_n(\mathscr{A})$ averages all coordinates. Observe that the fact that weak observables and weak states in general do not preserve pure elements leads to notions depending on the number of coordinates. Hence for $n < m$ the cogenerators S_n and S_m lead to different generalized probability theories.

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REFERENCES

[1] ADÁMEK, J.: *Theory of Mathematical Structures*, Reidel, Dordrecht, 1983.
[2] BUGAJSKI, S.: *Statistical maps I. Basic properties*, Math. Slovaca **51** (2001), 321–342.
[3] BUGAJSKI, S.: *Statistical maps II. Operational random variables*, Math. Slovaca **51** (2001), 343–361.
[4] DVUREČENSKIJ, A.—PULMANNOVÁ, S.: *New Trends in Quantum Structures*, Kluwer Academic Publ./Ister Science, Dordrecht/Bratislava, 2000.
[5] CHOVANEC, F.—KÔPKA, F.: *D-posets*. In: Handbook of Quantum Logic and Quantum Structures: Quantum Structures (K. Engesser, D. M. Gabbay, D. Lehmann, eds.), Elsevier, Amsterdam, 2007, pp. 367–428.
[6] CHOVANEC, F.—FRIČ, R.: *States as morphisms* Internat. J. Theoret. Phys. (To appear). Published online DOI10.1007/s10773-009-0234-4.
[7] FOULIS, D. J.—BENNETT, M. K.: *Effect algebras and unsharp quantum logics*, Found. Phys. **24** (1994), 1331–1352.
[8] FRIČ, R.: *Convergence and duality*, Appl. Categ. Structures **10** (2002), 257–266.
[9] FRIČ, R.: *Duality for generalized events*, Math. Slovaca **54** (2005), 49–60.
[10] FRIČ, R.: *Remarks on statistical maps and fuzzy (operational) random variables*, Tatra Mt. Math. Publ. **30** (2005), 21–34.
[11] FRIČ, R.: *Statistical maps: a categorical approach*, Math. Slovaca **57** (2007), 41–57.
[12] FRIČ, R.: *Extension of domains of states*, Soft Comput. **13** (2009), 63–70.
[13] FRIČ, R.—PAPČO, M.: *On probability domains* Internat. J. Theoret. Phys. (To appear). Published online DOI10.1007/s10773-009-0162-3.
[14] FRIČ, R.—PAPČO, M.: *A categorical approach to probability* (Submitted)
[15] GUDDER, S.: *Fuzzy probability theory*, Demonstratio Math. **31** (1998), 235–254.
[16] FRIČ, R.—PAPČO, M.: *A categorical approach to probability theory*, Studia Logica **94** (2010), 215–230.
[17] KÔPKA, F.—CHOVANEC, F.: *D-posets*, Math. Slovaca **44** (1994), 21–34.
[18] PAPČO, M.: *On measurable spaces and measurable maps*, Tatra Mt. Math. Publ. **28** (2004), 125–140.
[19] PAPČO, M.: *On fuzzy random variables: examples and generalizations*, Tatra Mt. Math. Publ. **30** (2005), 175–185.
[20] PAPČO, M.: *On effect algebras*, Soft Comput. **12** (2007), 26–35.
[21] RIEČAN, B.: *Probability theory on IF events*. In: Algebraic and Proof-Theoretic Aspects of Non-classical Logics. Papers in Honour of Daniele Mundici on the occasion of his 60th Birthday (S. Aguzzoli et al., eds.), Lecture Notes in Comput. Sci. 4460, Springer, Berlin, 2007, pp. 290–308.
[22] RIEČAN, B.—MUNDICI, D.: *Probability on MV-algebras*. In: Handbook of Measure Theory, Vol. II (E. Pap, ed.), North-Holland, Amsterdam, 2002, pp. 869–910.
[23] RIEČAN, B.—NEUBRUNN, T.: *Integral, Measure, and Ordering*, Kluwer Acad. Publ., Dordrecht-Boston-London, 1997.

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Mathematical Institute
Slovak Academy of Sciences
Grešákova 6
SK–040 01 Košice
SLOVAK REPUBLIC

Catholic University in Ružomberok
Nám. A. Hlinku 60
Hrabovská cesta
SK–034 01 Ružomberok
SLOVAK REPUBLIC

E-mail: fric@saske.sk

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