VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-010-0033-7
Math. Slovaca **60** (2010), No. 5, 583–590

ON IDENTITIES IN ORTHOCOMPLEMENTED DIFFERENCE LATTICES

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To Sylvia Pulmannová with compliments and admiration

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. In this note we continue the investigation of algebraic properties of orthocomplemented (symmetric) difference lattices (ODLs) as initiated and previously studied by the authors. We take up a few identities that we came across in the previous considerations. We first see that some of them characterize, in a somewhat non-trivial manner, the ODLs that are Boolean. In the second part we select an identity peculiar for set-representable ODLs. This identity allows us to present another construction of an ODL that is not set-representable. We then give the construction a more general form and consider algebraic properties of the 'orthomodular support'.

1. Basic notions and preliminaries

Let us first recall the definition of ODL.

Definition 1.1. Let $L = (X, \wedge, \vee, ^{\perp}, 0, 1, \triangle)$, where $(X, \wedge, \vee, ^{\perp}, 0, 1)$ is an orthocomplemented lattice and $\triangle\colon X^2 \to X$ is a binary operation. Then L is said to be an *orthocomplemented difference lattice* (abbr., an ODL) if the following identities hold in L:

(D_1) $x \triangle (y \triangle z) = (x \triangle y) \triangle z$,

(D_2) $x \triangle 1 = x^{\perp}$, $1 \triangle x = x^{\perp}$,

(D_3) $x \triangle y \leq x \vee y$.

2000 Mathematics Subject Classification: Primary 06C15, 03G12, 81B10.
Keywords: orthomodular poset, quantum logic, symmetric difference, Boolean algebra.
The authors acknowledge the support of the research plan MSM 0021620839 that is financed by the Ministry of Education of the Czech Republic and the grant GAČR 201/07/1051 of the Czech Grant Agency.

Obviously, the class of all ODLs forms a variety. We will denote it by $\mathcal{ODL}$. (It should be noted that a certain version of symmetric difference has been dealt with in the area of orthomodular lattices — see [4, 5, 13]. Our approach essentially differs from the above quoted papers since we take the operation $\triangle$ as primitive.)

Let $L = (X, \wedge, \vee, {}^{\perp}, 0, 1, \triangle)$ be an ODL. Then the orthocomplemented lattice $(X, \wedge, \vee, {}^{\perp}, 0, 1)$ will be denoted by L_{supp} and called the *support* of L. Occasionally, we allow ourselves to harmlessly abuse the notation by identifying an ODL L with the couple $(L_{\text{supp}}, \triangle)$.

Let us list basic properties of ODLs as we shall use them in the sequel. Let us note that in this list (and in other results of preliminary nature like Thm. 1.3 and Thm. 2.5) this paper overlaps with [10]. Main novelties lie in Thm. 2.2 and in the proof technique of Thm. 2.8.

Proposition 1.2. *Let L be an ODL. Then the following statements hold true $(x, y \in L)$:*

(1) $x \triangle 0 = x$, $0 \triangle x = x$,
(2) $x \triangle x = 0$,
(3) $x \triangle y = y \triangle x$,
(4) $x \triangle y^{\perp} = x^{\perp} \triangle y = (x \triangle y)^{\perp}$,
(5) $x^{\perp} \triangle y^{\perp} = x \triangle y$,
(6) $x \triangle y = 0 \Leftrightarrow x = y$,
(7) $(x \wedge y^{\perp}) \vee (y \wedge x^{\perp}) \leq x \triangle y \leq (x \vee y) \wedge (x \wedge y)^{\perp}$.

Proof. Let us first observe that the property (D$_2$) yields $1 \triangle 1 = 1^{\perp} = 0$. Let us verify the properties (1)–(7). Suppose that $x \in L$.

(1) $x \triangle 0 = x \triangle (1 \triangle 1) = (x \triangle 1) \triangle 1 = x^{\perp} \triangle 1 = (x^{\perp})^{\perp} = x$. Further, $0 \triangle x = (1 \triangle 1) \triangle x = 1 \triangle (1 \triangle x) = 1 \triangle x^{\perp} = (x^{\perp})^{\perp} = x$.

(2) Let us first show that $x^{\perp} \triangle x^{\perp} = x \triangle x$. We consecutively obtain $x^{\perp} \triangle x^{\perp} = (x \triangle 1) \triangle (1 \triangle x) = (x \triangle (1 \triangle 1)) \triangle x = (x \triangle 0) \triangle x = x \triangle x$. Moreover, we have $x \triangle x \leq x$ as well as $x \triangle x = x^{\perp} \triangle x^{\perp} \leq x^{\perp}$. This implies that $x \triangle x \leq x \wedge x^{\perp} = 0$.

(3) If $y \in L$, then $x \triangle y = (x \triangle y) \triangle 0 = (x \triangle y) \triangle [(y \triangle x) \triangle (y \triangle x)] = x \triangle (y \triangle y) \triangle x \triangle (y \triangle x) = x \triangle 0 \triangle x \triangle (y \triangle x) = x \triangle x \triangle (y \triangle x) = 0 \triangle (y \triangle x) = y \triangle x$.

(4) $x \triangle y^{\perp} = x \triangle (y \triangle 1) = (x \triangle y) \triangle 1 = (x \triangle y)^{\perp}$. The equality $x^{\perp} \triangle y = (x \triangle y)^{\perp}$ follows from $x \triangle y^{\perp} = (x \triangle y)^{\perp}$ by applying the equality (3).

(5) Using (4) we obtain $x^{\perp} \triangle y^{\perp} = (x^{\perp} \triangle y)^{\perp} = (x \triangle y)^{\perp\perp} = x \triangle y$.

(6) If $x = y$, then $x \triangle y = 0$ by the condition (2). Conversely, suppose that $x \triangle y = 0$. Then $x = x \triangle 0 = x \triangle (y \triangle y) = (x \triangle y) \triangle y = 0 \triangle y = 0$.

(7) The property (D$_3$) together with the properties (4), (5) imply that $x \triangle y \leq x \vee y$, $x \triangle y \leq x^{\perp} \vee y^{\perp} = (x \wedge y)^{\perp}$, $x \wedge y^{\perp} \leq x \triangle y$, $x^{\perp} \wedge y \leq x \triangle y$. $\square$

The following observation links ODLs with orthomodular lattices (OMLs) and, in turn, with quantum logics (for a link of quantum logics with theoretical physics, see [3, 6, 8]).

Theorem 1.3. *Let L be an ODL. Then its support L_{supp} is an OML.*

Proof. Assume that $x, y \in L$ and $x \le y$, $y \wedge x^{\perp} = 0$. Let us prove that $x = y$. Since $x \le y$, we conclude that $(x \wedge y^{\perp}) \vee (y \wedge x^{\perp}) = y \wedge x^{\perp} = 0$ and $(x \vee y) \wedge (x \wedge y)^{\perp} = y \wedge x^{\perp} = 0$. By Prop. 1.2.(6),(7), we infer that $x \triangle y = 0$ and therefore $x = y$. □

In view of the above proposition, all notions of OMLs can be referred to in ODLs, too. In particular, we may say that two elements x, y in an ODL commute (in symbols, $x\ C\ y$) if they commute in L_{supp} (for the notion of commutativity in OMLs, see [1, 9, 14]).

The following proposition shows that for the commutative pairs the operation $\triangle$ in L can be recovered from L_{supp}.

Proposition 1.4. *Let L be an ODL. Let $x, y \in L$ with $x\ C\ y$. Then $x \triangle y = (x \wedge y^{\perp}) \vee (y \wedge x^{\perp}) = (x \vee y) \wedge (x \wedge y)^{\perp}$.*

Proof. According to Prop. 1.2.(7), we have the inequalities $(x \wedge y^{\perp}) \vee (y \wedge x^{\perp}) \le x \triangle y \le (x \vee y) \wedge (x \wedge y)^{\perp}$. Since the elements x, y commute, the left-hand side of the previous inequality coincides with the right-hand side and therefore $x \triangle y = (x \wedge y^{\perp}) \vee (y \wedge x^{\perp}) = (x \vee y) \wedge (x \wedge y)^{\perp}$. □

Let us note that each Boolean algebra can be viewed as an ODL (more general ODLs will be met later, see also [10, 12]).

Proposition 1.5. *Let B be a BA. Then there exists exactly one mapping $\triangle\colon \dot{B} \times \dot{B} \to \dot{B}$ which fulfils all the conditions* (D$_1$), (D$_2$) *and* (D$_3$) *of* Def. 1.1.

Proof. To prove the existence, take for $\triangle$ the standard symmetric difference in Boolean algebras. In other words, let us set $x \triangle y = (x \wedge y^{\perp}) \vee (y \wedge x^{\perp})$. The properties (D$_1$), (D$_2$) and (D$_3$) of Def. 1.1 are then obviously fulfilled.

Let us prove the uniqueness of $\triangle$. Let $\triangle_1 \colon \dot{B} \times \dot{B} \to \dot{B}$ be a mapping that fulfils conditions (D$_1$), (D$_2$) and (D$_3$). Thus, the couple $(B, \triangle_1)$ is an ODL. If $x, y \in B$, then $x\ C\ y$, and therefore $x \triangle_1 y = (x \wedge y^{\perp}) \vee (y \wedge x^{\perp}) = x \triangle y$ (Prop. 1.4). □

2. Results

In view of Prop. 1.5 we can (and shall) understand any Boolean algebra as an ODL with the uniquely defined operation $\triangle$. A natural question arises how to characterize Boolean algebras (= Boolean ODLs) among ODLs in terms of the operation $\triangle$. The departure point is the following result (observe that what we claim is that a strengthening of the condition (D$_3$) makes the ODL in question Boolean).

Proposition 2.1. *Let L be an ODL. Then L is a Boolean algebra exactly when the formula $x \triangle y \le x \vee (y \wedge x^{\perp})$ is valid in L.*

Proof. If L is a Boolean algebra, then for any pair of elements $x, y \in L$ we have $x \vee (y \wedge x^{\perp}) = x \vee y \ge x \triangle y$. Conversely, let L fulfil the above formula. In order to prove that L is Boolean, let us use [9, p. 31]. Consider elements $x, y \in L$ with $x \wedge y = 0$. According to our assumption, $x^{\perp} \triangle y \le x^{\perp} \vee (y \wedge (x^{\perp})^{\perp}) = x^{\perp} \vee (y \wedge x) = x^{\perp}$. Since $x^{\perp} \le x^{\perp}$, we see in view of the condition (D_3) that we have $x^{\perp} \triangle (x^{\perp} \triangle y) \le x^{\perp}$. But $x^{\perp} \triangle (x^{\perp} \triangle y) = y$. Therefore $y \le x^{\perp}$ and we find that L is Boolean. □

The identity of Prop. 2.1 inspires one to consider other natural identities with the potential to be 'Boolean'. The following result summarizes this effort. In a certain sense it provides a definition of Boolean algebra in terms of 'abstract symmetric difference'.

Theorem 2.2. *Let L be an ODL. Then L is a Boolean algebra exactly when L fulfils any of the following four identities:*

(a) $(x \vee z) \triangle (y \vee z) \le x \triangle y$,
(b) $x \triangle (x \vee y) \le x \triangle y$,
(c) $x \vee y = x \triangle y \triangle (x \wedge y)$,
(d) $x \triangle y \triangle (x \vee y) \le x \triangle y \triangle (x \wedge y)$,

Proof. Evidently, if L is Boolean, then all identities (a)–(d) hold true. In proving the vice versa part, we first prove that (a) $\Longrightarrow$ (b) and (b) $\Longrightarrow$ L is Boolean. Let us suppose the condition (a). By setting $z = x$, we obtain $x \triangle (y \vee x) \le x \triangle y$, which is the condition (b). Assuming the condition (b) and taking into account $x \, C \, x \vee y$, we see that $x \triangle y \ge x \triangle (x \vee y) = (x \vee y) \wedge x^{\perp}$. Then $(x \triangle y)^{\perp} \le ((x \vee y) \wedge x^{\perp})^{\perp}$. It follows that $(x \triangle y)^{\perp} = x \triangle y^{\perp} \le x \vee (y^{\perp} \wedge x^{\perp})$. Writing y instead of $y^{\perp}$, we obtain $x \triangle y \le x \vee (y \wedge x^{\perp})$ which is the identity of Prop. 2.1.

To complete the the proof, let us verify (c) $\Longrightarrow$ (d) and (d) $\Longrightarrow$ L is Boolean. Let us suppose the condition (c). Then $x \triangle y \triangle (x \vee y) = x \triangle y \triangle (x \triangle y \triangle (x \wedge y))$ and therefore $x \triangle y \triangle (x \vee y) = x \wedge y$. It follows that $x \triangle y \triangle (x \vee y) = x \wedge y \le x \vee y = x \triangle y \triangle (x \wedge y)$ and so we have derived the condition (d). Finally, having the condition (d), it is sufficient to show that any pair $x, y \in L$ with $x \wedge y = 0$ satisfies $x \le y^{\perp}$. But then $x \triangle y \triangle (x \vee y) \le x \triangle y$. Since $x \triangle y \le x \triangle y$, we utilize ($D_3$) to obtain $x \triangle y \triangle (x \triangle y \triangle (x \vee y)) \le x \triangle y$. As $x \triangle y \triangle (x \triangle y \triangle (x \vee y)) = x \vee y$, we infer that $x \vee y \le x \triangle y$. But $x \triangle y \le x \vee y$ and therefore $x \triangle y = x \vee y$. This implies that $y \le x \triangle y$ and therefore $x \triangle y^{\perp} \le y^{\perp}$. Since $y^{\perp} \le y^{\perp}$, we utilize (D_3) to obtain $(x \triangle y^{\perp}) \triangle y^{\perp} \le y^{\perp}$. But $(x \triangle y^{\perp}) \triangle y^{\perp} = x \triangle (y^{\perp} \triangle y^{\perp}) = x$ and the proof is complete. □

In the next considerations we take up 'nearly Boolean ODLs' — the ODLs that are set-representable. We will find out that there is a formula which allows us to see that not all ODLs are nearly Boolean. With the help of Boolean algebras we will first introduce a certain class of ODLs. We will utilize it in the crucial example of the next section. Prior to that, let us fix some notation. Let B be a non-trivial Boolean algebra and let $\mathcal{B}$ be a system of subalgebras of B. Let us say that $\mathcal{B}$ is a *disjoint system of subalgebras* of B if for all $B_1, B_2 \in \mathcal{B}$ with $B_1 \neq B_2$ we have $B_1 \cap B_2 = \{0,1\}$, and neither of the inclusions $B_1 \subseteq B_2$ and $B_2 \subseteq B_1$ is valid. Moreover, if $\bigcup \mathcal{B} = B$, then the system $\mathcal{B}$ is said to be a *partition* of the algebra B.

Let B be a Boolean algebra and let $\mathcal{B}$ be a disjoint system of subalgebras of B. Let us construct an OML, K, and the mapping $\triangle_K \colon K^2 \to K$ as follows:

In the first step we construct a partition $\mathcal{B}'$ of B determined by the following requirement: If $\mathcal{B}$ is a partition of B, then we set $\mathcal{B}' = \mathcal{B}$. Otherwise, we add to $\mathcal{B}$ all necessary four-element subalgebras of B such that the resulting system $\mathcal{B}'$ is a partition of B. In the second step we take for K the horizontal sum of the system $\mathcal{B}'$ (the horizontal sum alias the $\{0,1\}$-pasting is a standard construction in OMLs, see [9, 14]). And finally, if $x, y \in K$, let us set $x \triangle_K y = x \triangle_B y$ (note that K and B live on the same set).

The couple $(K, \triangle_K)$ so obtained will be denoted by $L^{\mathcal{B}}$.

Proposition 2.3. *The algebra $L^{\mathcal{B}}$ is an ODL.*

Proof. Conditions (D_1) and (D_2) are obvious. Let us verify condition (D_3). Let $x, y \in B$. If there is $B_1 \in \mathcal{B}$ such that $x, y \in B_1$, then $x \vee_K y = x \vee_B y$. As a result, $x \triangle y = x \triangle_B y \leq x \vee_B y = x \vee_K y$. If there is no B_1 such that $x, y \in B_1$, then $x \vee_K y = 1$. The inequality $x \triangle y \leq x \vee_K y$ is obvious and the proof is done. □

Let B be a Boolean algebra, $|B| \geq 4$. Let us take the finest partition of B, $\mathcal{B}$. Thus, the elements of $\mathcal{B}$ consist of all four-element subalgebras of B. Let us consider the algebra $L^{\mathcal{B}}$. Obviously, the OML $L^{\mathcal{B}}_{\mathrm{supp}}$ coincides with the familiar MO_κ for an apropppriate cardinal number κ (in fact, if B is finite, it is easily seen that $\kappa = 2^n - 1$ for some $n \in \mathbb{N}$). We will allow ourselves to denote the ODL $L^{\mathcal{B}}$ by MO_κ, too.

Let us return to the ODLs that are set-representable. They form a variety ([10]) and in view of the Stone set representation for Boolean algebras they could be seen as nearly Boolean. Though the name itself suggests their definition, let us recall it in more formal terms.

Let X be a set and let $\mathcal{D}$ a family of subsets of X such that

(1) $X \in \mathcal{D}$,

(2) the family $\mathcal{D}$ forms a lattice with respect to the inclusion relation, and

(3) $\mathcal{D}$ is closed under the formation of the set symmetric difference.

Obviously, $\mathcal{D}$ constitutes an ODL. Let us call it concrete. If L is an ODL that is isomorphic with a concrete one, then L is said to be *set-representable* (abbr., a SRODL). Let us denote by $\mathcal{SRODL}$ the class of all such ODLs.

The set-representable ODLs can be characterized in terms of certain evaluations. Let $\oplus$ stand for the addition modulo 2 on the set $\{0,1\}$ (i.e., $0 \oplus 0 = 1 \oplus 1 = 0$, $0 \oplus 1 = 1 \oplus 0 = 1$).

Definition 2.4. Let L be an ODL and let $e : L \to \{0,1\}$. Then e is said to be an *ODL-evaluation* (abbr., *evaluation*) on L if the following properties are fulfilled for any $x, y \in L$:

(E_1) $e(1_L) = 1$,

(E_2) $x \le y \implies e(x) \le e(y)$,

(E_3) $e(x \bigtriangleup y) = e(x) \oplus e(y)$.

Let $\mathcal{E}(L)$ be the set of all ODL-evaluations on L. The following result provides a characterization of $\mathcal{SRODL}$ in terms of $\mathcal{E}(L)$. The proof is straightforward ([10]) and we will omit it.

Theorem 2.5. *Let L be an ODL. Then L is a SRODL if and only if*

$$\big(\forall a, b \in L\big)\Big(a \not\le b \implies \big(\exists e \in \mathcal{E}(L)\big)\big(e(a) = 1 \ \& \ e(b) = 0\big)\Big).$$

The variety of SRODLs is rather large. For instance, the ODLs MO_κ, for $\kappa = 2^n - 1$ or κ infinite, are SRODLs. We will see that in general $L^{\mathcal{B}}$ does not have to be a SRODL (Though $L^{\mathcal{B}}_{\mathrm{supp}}$ is always a set-representable OML!). It is the objective of this section to show this - it will be established as a consequence of a certain identity valid in SRODLs.

Let us start off with the following result that concerns the intrinsic property of SRODLs. It could be viewed, in a sense, as a contribution to a general research plan indicated in [7].

Theorem 2.6. *Every SRODL L satisfies the following formula:*

$$(\forall x, y, z_1, z_2 \in L)\big(x \perp y \implies (x \bigtriangleup z_1) \wedge (y \bigtriangleup z_2) \le z_1 \vee z_2\big).$$

Proof. Let us suppose that there are elements $x, y, z_1, z_2 \in L$ with $x \perp y$ but $(x \bigtriangleup z_1) \wedge (y \bigtriangleup z_2) \not\le z_1 \vee z_2$. As L is set-representable, there is an ODL-evaluation e such that $e((x \bigtriangleup z_1) \wedge (y \bigtriangleup z_2)) = 1$, $e(z_1 \vee z_2) = 0$. Since $z_1, z_2 \le z_1 \vee z_2$ it has to be $e(z_1) = e(z_2) = 0$. By the same reasoning, $e(x \bigtriangleup z_1) = e(y \bigtriangleup z_2) = 1$. Because $1 = e(x \bigtriangleup z_1) = e(x) \oplus e(z_1)$ and $e(z_1) = 0$, we have $e(x) = 1$. Analogously, $e(y) = 1$. But this is absurd in view of the orthogonality of elements x and y. □

Let us note that the previous result allows us to formulate the following identity valid in $\mathcal{SRODL}$.

Proposition 2.7. *Let L be an ODL. Then the formula of* Thm. 2.6 *holds in L exactly when the following identity holds in L:*

$$(x \bigtriangleup z_1) \wedge ((x^\perp \wedge y) \bigtriangleup z_2) \le z_1 \vee z_2 \ .$$

Proof. It is sufficient to take into account that $x \perp y$ is equivalent with $y = x^\perp \wedge y$. □

The identity of Thm. 2.6 allows us to prove the following result.

Theorem 2.8. *There is a Boolean algebra B and a disjoint system $\mathcal{B}$ of subalgebras of B such that $L^{\mathcal{B}}$ is not set-representable ODL.*

Proof. Take $B = \exp\{1,2,3,4,5\}$. Let us make use of the following notation. Set $0_B = \emptyset$ and $1_B = \{1,2,3,4,5\}$. Let us denote by $\underline{n_1 \dots n_k}$, where $n_1 < \cdots \cdots < n_k$ and $k \leq 5$, the element $a \subseteq \{1,2,3,4,5\}$ such that $a = \{n_1, \dots, n_k\}$. For any $a \subseteq \{1,2,3,4,5\}$, let us write $a^\perp = \{1,2,3,4,5\} \setminus a$. Thus, for instance, $\underline{12}^\perp = \{3,4,5\}$.

Let us go on with the construction. Consider the following subalgebras of B:

$B_1 = \{0_B, \underline{12}, \underline{3}, \underline{45}, \underline{12}^\perp, \underline{3}^\perp, \underline{45}^\perp, 1_B\}$,

$B_2 = \{0_B, \underline{15}, \underline{2}, \underline{34}, \underline{15}^\perp, \underline{2}^\perp, \underline{34}^\perp, 1_B\}$,

$B_3 = \{0_B, \underline{13}, \underline{24}, \underline{5}, \underline{13}^\perp, \underline{24}^\perp, \underline{5}^\perp, 1_B\}$.

Let us set $\mathcal{B} = \{B_1, B_2, B_3\}$. It is easily seen that $\mathcal{B}$ is a disjoint system of subalgebras of B. Consider $L^{\mathcal{B}}$ and test this ODL for the formula of Thm. 2.6. Set $x = \underline{12}$, $y = \underline{3}$, $z_1 = \underline{34}$, $z_2 = \underline{234}$ $(= \underline{15}^\perp)$. Then $x \perp y$, $x \triangle z_1 = \underline{1234}$ $(= \underline{5}^\perp)$ and $y \triangle z_2 = \underline{24}$. We see that both elements $x \triangle z_1$ and $y \triangle z_2$ lie in B_3, and $(x \triangle z_1) \wedge (y \triangle z_2) = \underline{24}$. But $z_1 \vee z_2 = \underline{234} \not\geq \underline{24}$. It follows that $L^{\mathcal{B}}$ does not satisfy the formula of Thm. 2.6 and therefore $L^{\mathcal{B}}$ is not a set-representable ODL. ☐

The following fact given by the previous construction (one takes the ODL $L^{\mathcal{B}}$ exhibited above) could be of a mild separate interest.

Observation 2.9. *There is a non set-representable ODL, L, such that L_{supp} is a non-modular set-representable OML.*

The above construction of $L^{\mathcal{B}}$ allows one not only to find an ODL with rather surprising properties but also to show that a certain class of OMLs (the horizontal sums of Boolean algebras) are embeddable into ODLs. The following proposition clarifies this situation in general.

Proposition 2.10. *Let L be an OML obtained as a horizontal sum of Boolen algebras. Then L is OML-embeddable into an ODL.*

Proof. Let L be a horizontal sum of Boolean algebras B_α, $\alpha \in I$. As known (see e.g. [15]), there exists a Boolean algebra, B, such that each B_α $(\alpha \in I)$ is a subalgebra of B and, moreover, if $\alpha_1 \neq \alpha_2$ then $B_{\alpha_1} \cap B_{\alpha_2} = \{0,1\}$. As a result, the system B_α, $\alpha \in I$ constitutes a disjoint system of subalgebras of B. It is clear that L is embeddable into $L^{\mathcal{B}}$ and this completes the proof. ☐

The horizontal sums of Boolean algebras constitute an important class of OMLs, [2]. It would be therefore desirable, in connection with the interplay between OMLs and ODLs, to answer the following questions. We will formulate them in the conclusion of this paper.

1. Could any horizontal sum of Boolean algebras be OML-embedded in a set-representable ODL?
2. If L_{supp} is a set-representable and modular OML, does the ODL L have to be set-representable?

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Received 26. 5. 2009
Accepted 9. 10. 2009

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