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ON ALGEBRAS OF MULTIDIMENSIONAL PROBABILITIES

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Dedicated to Professor Sylvia Pulmannová on the occasion of her 70th birthday

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. The probability $p(s)$ of the occurrence of an event pertaining to a physical system which is observed in different states s determines a function p from the set S of states of the system to $[0,1]$. The function p is called a multidimensional probability or numerical event. Sets of numerical events which are structured either by partially ordering the functions p and considering orthocomplementation or by introducing operations $+$ and $\cdot$ in order to generalize the notion of Boolean rings representing classical event fields are studied with the goal to relate the algebraic operations $+$ and $\cdot$ to the sum and product of real functions and thus to distinguish between classical and quantum mechanical behaviour of the physical system. Necessary and sufficient conditions for this are derived, as well for the case that the functions p can assume any value between 0 and 1 as for the special cases that the values of p are restricted to two or three different outcomes.

1. Introduction

Let S be the set of states a physical system can accept during a certain experiment. The probabilities $p(s)$ of the occurrence of an event obtained by observing the physical system for the different states $s \in S$ determines a function from S to $[0,1]$, called a numerical event or multidimensional probability (cf. [1, 2]). For example, one could think of finding the value of an observable within a given set of reals for different states $s \in S$.

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Studying the system with regard to the occurrence of different events leads to a set P of functions from S to $[0,1]$ which are partially ordered by the order $\leq$ of functions. Denoting the constant functions with values 0 and 1 by 0 and 1, respectively, it is natural to assume that

(1) $0 \in P$ and

(2) $p' := 1 - p \in P$ for all $p \in P$.

For $p, q, r \in P$ we write $p \perp q$ if the functions p and q are *orthogonal*, i.e. $p(s) \leq q'(s)$ for all $s \in S$, and call (p, q, r) an *orthogonal triple* if $p \perp q \perp r \perp p$. If one assumes that in addition to (1) and (2) it holds that

(3) $p, q, r \in P$ and (p, q, r) an orthogonal triple imply $p + q + r \in P$

(with $+$ the addition in $\mathbb{R}$) then $(P, \leq, ')$ is called an *algebra of S-probabilities* or *algebra of multidimensional probabilities* (cf. [1, 2]).

An algebra of S-probabilities is an orthomodular poset with respect to $\leq$ and $'$ which admits a full set of states, and vice versa, any orthomodular poset which admits a full set of states is isomorphic to an algebra of S-probabilities (cf. [7]). — For the definition of an orthomodular poset and states confer e.g. the monograph by Pták and Pulmannová [8].

Moreover, we point out that an algebra of S-probabilities allows to distinguish a classical mechanical behaviour from a quantum mechanical one, namely, a system is classical if and only if P is a Boolean algebra (cf. [1, 2]).

In the case of a Boolean algebra and in many other cases the algebra of S-probabilities is not only a poset but a lattice, as the following examples show.

Examples 1.1.

(a) Let $\mathscr{H}$ be a Hilbert space, S the set of one-dimensional subspaces of $\mathscr{H}$, and for every $s \in S$ let a_s be a fixed unit vector in s. Denoting the set of orthogonal projectors of $\mathscr{H}$ by $\mathscr{P}(\mathscr{H})$ and writing $\langle \cdot, \cdot \rangle$ for the inner product in $\mathscr{H}$ the set of functions $\big\{ s \to \langle Q a_s, a_s \rangle : \ Q \in \mathscr{P}(\mathscr{H}) \big\}$ gives rise to an algebra of S-probabilities which is an orthomodular lattice.

(b) Let $S = \{1, \dots, n\}$, $n \in \mathbb{N}$, and $k \,|\, n$. For $A \subseteq S$ denote by I_A the characteristic function of A, i.e. $I_A \in \{0,1\}^S$,

$$I_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \in S \setminus A. \end{cases}$$

Then $P = \{ I_A : \ A \subseteq S, \ k \text{ divides } |A| \}$ is an algebra of S-probabilities which is a lattice if and only if $k = 1$ (in this case P is the Boolean algebra $\{0,1\}^S$) or $n/k \leq 2$: If $n = k$ then $P = \{0, 1\}$, and if $n = 2k$, then P is the orthomodular lattice $\mathrm{MO}\big(\binom{n}{n/2}/2\big)$ with 0 and 1 and $\binom{n}{n/2}$ pairwise incomparable elements in between. In the case $n/k \geq 3$ and $k > 1$ it can be seen easily that for two sets $A_1, A_2 \subseteq S$ with $|A_1| = |A_2| = k$ and $|A_1 + A_2| = 2$, where $+$ denotes the symmetric difference of sets, the supremum $I_{A_1} \vee I_{A_2}$ does not exist.

(c) The set $P = \{0, 1, x, 1-x, |2x-1|, 1-|2x-1|\} \subseteq [0,1]^{[0,1]}$ forms an algebra of S-probabilities with $S = [0,1]$. P is a lattice where the elements $x, 1-x, |2x-1|, 1-|2x-1|$ are pairwise incomparable and $x \vee |2x-1| = 1 \neq \max\{x, |2x-1|\}$. This example can be generalized as follows: Any set $P \subseteq [0,1]^S$ satisfying (1) and (2) where $(P \setminus \{0,1\}, \leq)$ is an antichain forms an algebra of S-probabilities.

If $(P, \leq)$ is a lattice then it is also possible to consider P from the point of view of generalized event fields, structures that rely on operations $+$ and $\cdot$ such that $+$ is directly related to the addition of functions in $\mathbb{R}$. If P pertains to a classical physical system the Boolean ring which corresponds to the Boolean algebra P is a so-called generalized event field which, in general, is defined as follows. To begin with we have to explain what a generalized Boolean quasiring is.

Definition 1.2. (cf. [3, 4, 5, 6]) An algebra $(R, +, \cdot)$ of type $(2,2)$ is called a *generalized Boolean quasiring* (GBQR) if there exist $0, 1 \in R$ such that (1)–(8) hold for all $x, y, z \in R$:

(1)	$x + y = y + x$	(2)	$0 + x = x$
(3)	$(xy)z = x(yz)$	(4)	$xy = yx$
(5)	$xx = x$	(6)	$x0 = 0$
(7)	$x1 = x$	(8)	$1 + (1 + xy)(1 + x) = x$

(The elements 0 and 1 of a GBQR are uniquely defined.)

Omitting (1) and considering $+$ as a partial operation $\oplus$ on R defined on $\{0,1\} \times R$ one obtains a so-called *partial GBQR* (pGBQR). A pGBQR $(R, \oplus, \cdot)$ can be extended to a GBQR $(R, +, \cdot)$ by defining

$$0 + x = x + 0 = x,$$
$$1 + x = x + 1 = 1 \oplus x$$

for all $x \in R$ and arbitrarily setting up $x + y = y + x$ for all $x, y \in R \setminus \{0,1\}$. In particular, one can extend $\oplus$ by taking for $+$ one of the two operations $+_1$ or $+_2$ given by

$$x +_1 y = 1 \oplus (1 \oplus x(1 \oplus y))(1 \oplus (1 \oplus x)y),$$
$$x +_2 y = (1 \oplus (1 \oplus x)(1 \oplus y))(1 \oplus xy)$$

which both coincide with the symmetric difference within Boolean algebras.

pGBQRs $(R, \oplus, \cdot)$ and hence the GBQRs $(R, +, \cdot)$ obtained by extending $\oplus$ in an arbitrary but unique way are in one-to-one correspondence with bounded lattices $(L, \vee, \wedge, {}^*, 0, 1)$ with an antitone involution * by means of

$$x \vee y = 1 \oplus (1 \oplus x)(1 \oplus y), \quad x \wedge y = xy, \quad x^* = 1 \oplus x,$$

and

$$0 \oplus x = x, \quad 1 \oplus x = x^*, \quad xy = x \wedge y.$$

Let $(R(L), +, \cdot)$ be the GBQR corresponding to a bounded lattice with antitone involution $(L, \vee, \wedge, {}^*, 0, 1)$. In $R(L)$ the relation $\leq$ shall always refer to the order of the lattice L, and the same should apply to the orthogonality relation $\perp$ ($x \perp y$ if and only if $x \leq y^*$).

Now we can explain the notion of a generalized event field.

Definition 1.3. (cf. [1]) A GBQR $(R, +, \cdot)$ is called a *generalized event field* if the following condition (T) is satisfied:

(T) If the elements x_1, x_2, x_3 have the property that $x_i \perp x_j$ for $i \neq j$, $i, j \in \{1, 2, 3\}$, in which case we will say that (x_1, x_2, x_3) is an *orthogonal triple*, then $(x_i + x_j) \perp x_k$ for $\{i, j, k\} = \{1, 2, 3\}$.

Examples 1.4.

(a) As can be seen easily, any "classical" σ-algebra of events corresponds to a generalized event field if we take the symmetric difference for $+$. In this case $+ = +_1 = +_2$.

(b) Any GBQR $(R, +, \cdot)$ with $x +_1 y \leq x + y \leq x +_2 y$ for all $x, y \in R$ (in which case we write $+_1 \leq + \leq +_2$) satisfies (T). This is due to the fact that for $+_1 \leq + \leq +_2$ the orthogonality of x and y implies $x + y = x \vee y$.

(c) $(R(P), +, \cdot)$ with $+_1 \leq + \leq +_2$ for any algebra P of S-probabilities which is a lattice. Condition (3) of algebras of S-probabilities guarantees that (T) is fulfilled.

In this paper we study generalized event fields $(R(P), +_1, \cdot)$ where P is an algebra of S-probabilities and characterize these event fields among the generalized event fields $(R(L), +, \cdot)$, where L is an arbitrary ortholattice. This way we also obtain a characterization for an ortholattice to be an orthomodular lattice admitting a full set of states. Moreover, we give necessary and sufficient conditions for $(R(P), +, \cdot)$ to be a classical field of events, i.e. a Boolean algebra. Finally, we study the special case that one deals with a small set of values.

2. Preliminary remarks

Let $P \subseteq [0,1]^S$. In the following we investigate relations between the axioms/properties of algebras of S-probabilities. Here is the list of properties we will study:

(1) $0 \in P$.

(2) For all $f \in P$ we have $f' = 1 - f \in P$.

(3) For all $f, g, h \in P$, if (f, g, h) is an orthogonal triple then $f + g + h \in P$.

(4) For all $f, g \in P$, if $f \perp g$ then $f + g \in P$.

(5) For all $f, g, h \in P$, if (f, g, h) is an orthogonal triple then $f + g + h \leq 1$.

(6) $P \subseteq \{0,1\}^S$.

(7) For all $f, g \in P$, $fg \in P$.

(8) $(P, \leq)$ is a Boolean algebra.

(9) $P \setminus \{0\}$ contains no function f with $f \leq 1/2$.

Lemma 2.1. *The following implications between properties* (1)–(9) *hold:*

(a) (1),(3) $\Longrightarrow$ (4)   (b) (3) $\Longrightarrow$ (5)   (c) (4),(5) $\Longrightarrow$ (3)
(d) (4),(6) $\Longrightarrow$ (3)   (e) (1),(2),(6),(7) $\Longrightarrow$ (8)   (f) (4),(5) $\Longrightarrow$ (9)

Proof.

(a) If $f \perp g$ then $(f, g, 0)$ is an orthogonal triple and by (3) we have $f + g = f + g + 0 \in P$.

(b) is obvious.

(c) If (f, g, h) is an orthogonal triple then by (4), $f + g \in P$ and by (5), $(f + g) \perp h$. Employing (4) again yields $f + g + h \in P$.

(d) If (f, g, h) is an orthogonal triple then (6) means that f, g, h are characteristic functions on pairwise disjoint subsets S_f, S_g, S_h of S, thus $f + g$ which is in P by (4) is the characteristic function on $S_f \cup S_g$ and $(f + g) \perp h$, and by applying (4) to this orthogonal pair we obtain $f + g + h \in P$.

(e) (6) means that P consists of characteristic functions of certain subsets of S. By (1) this set $\bar{P}$ of subsets of S contains the empty set, by (2), $\bar{P}$ is closed under (set-theoretical) complements, and by (7) $\bar{P}$ is closed under (set-theoretical) intersection. Hence $\bar{P}$ is also closed under union of sets and so $\bar{P}$ and therefore also P is a Boolean subalgebra of $\{0,1\}^S$.

(f) Suppose there is $f \in P \setminus \{0\}$ with $f \leq 1/2$ and let n be the maximum among all $k \in \mathbb{N}$ such that $kf \leq 1$. Then $n \geq 2$ and $kf \perp f$ for $k = 1, \dots, n-1$ and by (4) and induction on k we infer $kf \in P$ for $k = 1, \dots, n$. Moreover, $((n-1)f, f, f)$ is an orthogonal triple in P and (5) yields $(n+1)f \leq 1$ which is a contradiction to the choice of n. $\square$

As an easy consequence of Lemma 2.1 one can see that an algebra of S-probabilities can be defined by a different set of axioms.

Proposition 2.2. *$P \subseteq [0,1]^S$ is an algebra of S-probabilities if and only if P satisfies* (1), (2), (4) *and* (5).

The following result provides a characterization for the classicality of an algebra of S-probabilities in terms of properties of its atoms.

Proposition 2.3. *An orthomodular poset $\mathscr{P} = (P, \leq, ', 0, 1)$, in particular an algebra of S-probabilities, is a finite Boolean algebra if and only if* (i), (ii) *and* (iii) *hold:*

(i) *$(P, \leq)$ has finitely many atoms.*

(ii) *The atoms of $(P, \leq)$ are pairwise orthogonal.*

(iii) *$(P, \leq)$ is atomic, i.e. for every $b \in P \setminus \{0\}$ there exists an atom a of P with $a \leq b$.*

P r o o f. Let A denote the set of all atoms of $\mathscr{P}$.

Obviously (i)–(iii) are necessary for $\mathscr{P}$ being a finite Boolean algebra.

Now assume $\mathscr{P}$ to satisfy (i)–(iii). Because of (i), A is finite. Let f denote the mapping $a \mapsto \{x \in A : \ x \leq a\}$ from P to 2^A and g the mapping $B \mapsto \bigvee B$ from 2^A to P. Then f and g are well-defined and order-preserving. Let $a \in P$. Then $g(f(a)) = \bigvee\{x \in A : \ x \leq a\} \leq a$. Because of the orthomodularity of $(P, \leq)$, $g(f(a)) < a$ would imply $a \wedge (g(f(a)))' > 0$. Because of (iii) there would exist $b \in A$ with $b \leq a \wedge (g(f(a)))'$. But then we would have $b \leq g(f(a))$ and $b \leq (g(f(a)))'$, i.e., $b \leq g(f(a)) \wedge (g(f(a)))' = 0$ contradicting $b \in A$. Therefore $g(f(a)) = a$. Now let C be a subset of A. Then $f(g(C)) = \big\{x \in A : x \leq \bigvee C\big\} \supseteq C$. Now $f(g(C)) \neq C$ would imply the existence of some $c \in A \setminus C$ with $c \leq \bigvee C$. According to (ii) this would imply $c \leq d'$ for all $d \in C$ and hence $c \leq \bigwedge\{d' : \ d \in C\} = \big(\bigvee C\big)'$ which together with $c \leq \bigvee C$ would yield $c \leq \big(\bigvee C\big) \wedge \big(\bigvee C\big)' = 0$ contradicting $c \in A$. This shows $f(g(C)) = C$. Hence f and g are mutually inverse isomorphisms between $(P, \leq)$ and $(2^A, \subseteq)$ and since the latter is a finite Boolean algebra the same is true for $\mathscr{P}$. $\square$

3. GBQRs of S-probabilities

We call a GBQR $(R(L), +, \cdot)$ such that L is an algebra of S-probabilities a *GBQR of S-probabilities*, i.e. L is an orthomodular lattice which admits a full set of states. However, in the following we only assume that L is an ortholattice. We recall that a *state* of L is a function $m\colon L \to [0,1]$ such that $m(0) = 0$, $m(1) = 1$ and $a \perp b$ implies $m(a \vee b) = m(a) + m(b)$ (with $+$ in $\mathbb{R}$), from which we infer $m(a') = 1 - m(a) =: m'(a)$ and, moreover, that $a \leq b$ implies $m(a) \leq m(b)$ for $a, b \in L$. If for a set M of states of L the converse is also true, i.e. $m(a) \leq m(b)$ for all $m \in M$ implies $a \leq b$, then M is called *full*.

If for $a, b \in L$ with $a \neq b$ there always exists an $m \in M$ with $m(a) \neq m(b)$ then M is called *point-separating*. For every fixed $a \in L$ a function f_a from M to $[0,1]$ is defined by $f_a(m) = m(a)$ for all $m \in M$. In some instances we will denote the function f_a also by $f(a)$ for notational reasons. In particular, we will use the form $f(a)$ when a is a more involved term and no evaluation of $f(a)$ in a certain state $m \in M$ is considered at the same time, so that the function $f(a)$ never can be mixed up with the value $f_a(m)$. Let L_M denote the set of all functions f_a, $a \in L$, assumed to be ordered by $\leq$ and endowed with the operation $'$ defined by

$$f'_a(m) = 1 - f_a(m) = f_{a'}(m).$$

If the supremum of two functions f_a and f_b exists in L_M we will denote it by $f_a \vee f_b$ also indicating this way that the supremum exists. In order to omit extensive use of brackets we agree that $\vee$ and $\wedge$ bind stronger than $+$.

Theorem 3.1. *Let L be an ortholattice with a point-separating set M of states. Then $(R(L), +_1, \cdot)$ is a GBQR of S-probabilities if and only if $f_a \perp f_b$ implies $f(a +_1 b) = f(a \wedge b' +_1 b) = f_a \vee f_b$ for all $a, b \in L$.*

Proof. Let $(R(L), +_1, \cdot)$ be a GBQR of S-probabilities. Then for M we choose a full set of states and therefore $m(a) \perp m(b)$ for all $m \in M$ implies $a \perp b$, from which we infer $f(a +_1 b) = f(a \wedge b' +_1 b) = f_a + f_b = f_a \vee f_b$ because, as shown in [7], if $f_a \perp f_b$ then $f_a \vee f_b$ exists and equals $f_a + f_b$.

Conversely, assume $f_a \perp f_b$ entails $f(a +_1 b) = f(a \wedge b' +_1 b) = f_a \vee f_b$. If for $a, b \in L$ we have $m(a +_1 b) = m(a \wedge b' +_1 b)$ for all $m \in M$ then

$$m(a \wedge b') + m(a' \wedge b) = m(a \wedge b' \wedge b') + m((a' \vee b) \wedge b) = m(a \wedge b') + m(b)$$

which implies that $m(a' \wedge b) = m(b)$, and with interchanged roles of a and b that $m(b' \wedge a) = m(a)$, which means that $f(a +_1 b) = f_a + f_b = f_a \vee f_b$.

Now, if $f_a \perp f_b \perp f_c \perp f_a$ for $a, b, c \in L$ then $f_a + f_b \le f'_c = 1 - f_c$ and we obtain $f_a + f_b + f_c \le 1$ from which we can conclude $f_a \vee f_b \vee f_c \in L_M$ proving $(L_M, \le, ')$ to be an algebra of S-probabilities. Since M is point-separating the mapping $a \mapsto f_a$ is a bijection, and because $a \le b$ implies $m(a) \le m(b)$ (as pointed out above for ortholattices L), L is order-isomorphic to L_M and hence isomorphic to L_M as an ortholattice. □

Corollary 3.2. *An ortholattice with a point-separating set M of states is orthomodular and admits a full set of states if and only if for all $a, b \in L$ the relation $f_a \perp f_b$ implies $f((a \wedge b') \vee (a' \wedge b)) = f((a \wedge b') \vee b) = f_a \vee f_b$.*

If a GBQR of S-probabilities is a Boolean ring we call it *classical* (in accordance with the fact that a classical physical phenomenon can be characterized by the fact that the corresponding algebra of S-probabilities is Boolean).

Theorem 3.3. *Let L be an ortholattice with a point-separating set M of states. Then $(R(L), +_1, \cdot)$ is a classical GBQR of S-probabilities if and only if*

$$f(a \wedge b') + f(a' \wedge b) + f(a \wedge b) = f(a \vee b) = f_a \vee f_b$$

for all $a, b \in L$.

P r o o f. If $R(L)$ is classical then $R(L)$ is a Boolean ring, hence $+_1 = +_2$ which means

$$(a \wedge b') \vee (a' \wedge b) = (a \vee b) \wedge (a' \vee b') = ((a' \wedge b') \vee (a \wedge b))'.$$

Therefore

$$\begin{aligned} m(a \wedge b') + m(a' \wedge b) &= (m(a' \wedge b') + m(a \wedge b))' \\ &= 1 - ((1 - m(a \vee b)) + m(a \wedge b)) \\ &= m(a \vee b) - m(a \wedge b) \end{aligned}$$

for all $a, b \in L$ and $m \in M$. Hence

$$f(a \wedge b') + f(a' \wedge b) + f(a \wedge b) = f(a \vee b) = f_a \vee f_b.$$

(That $f(a \vee b) = f_a \vee f_b$ is due to the fact that L_M is isomorphic to L.)

Conversely, if $f(a \wedge b') + f(a' \wedge b) + f(a \wedge b) = f(a \vee b) = f_a \vee f_b$ then the mapping $a \mapsto f_a$, $a \in L$, is an ortholattice isomorphism of L onto L_M and therefore $f_a \perp f_b \perp f_c \perp f_a$ for $a, b, c \in L$ implies

$$f_a + f_b = f_a \wedge f'_b + f'_a \wedge f_b + f_a \wedge f_b = f(a \vee b) - f_a \vee f_b,$$

from which we conclude $f_a + f_b + f_c = f_a \vee f_b \vee f_c \in L_M$. Thus it follows that L_M is an algebra of S-probabilities and L is orthomodular.

The condition $f(a \wedge b') + f(a' \wedge b) + f(a \wedge b) = f(a \vee b)$ is equivalent to the condition $m(a +_1 b) = m(a +_2 b)$ for all $a, b \in L$ and all $m \in M$. Because L is isomorphic to L_M we therefore obtain $+_1 = +_2$ which within orthomodular lattices implies that the lattice has to be Boolean, as can be easily verified. □

4. GBQRs of S-probabilities with a small number of values

Let L be a set of functions from S in $\{0,1\}$, $0 \in L$ and $f' = 1 - f \in L$ for $f \in L$. First we prove some basic properties of L in order to obtain results about algebras of 2-valued S-probabilities.

Lemma 4.1. *If $f, g \in \{0,1\}^S$ then*

(i) $fg = \min\{f, g\}$,

(ii) $f + g - fg = \max\{f, g\}$.

Proof. $\min\{a, b\} = ab$ and $\max\{a, b\} = 1 - \min\{1-a, 1-b\} = 1 - (1-a)(1-b)$ $= a + b - ab$ for all $a, b \in \{0,1\}$. □

Lemma 4.2. *If $P \subseteq \{0,1\}^S$ and P is closed with respect to $'$ then the following are equivalent:*

(i) *$fg \in P$ for all $f, g \in P$,*

(ii) *$\min\{f, g\} \in P$ for all $f, g \in P$,*

(iii) *$f + g - fg \in P$ for all $f, g \in P$,*

(iv) *$\max\{f, g\} \in P$ for all $f, g \in P$.*

Proof.

(i) $\Longrightarrow$ (ii): $\min\{f, g\} = fg$ for all $f, g \in P$.

(ii) $\Longrightarrow$ (iii): $f + g - fg = \big(\min\{f', g'\}\big)'$ for all $f, g \in P$.

(iii) $\Longrightarrow$ (iv): $\max\{f, g\} = f + g - fg$ for all $f, g \in P$.

(iv) $\Longrightarrow$ (i): $fg = \big(\max\{f', g'\}\big)'$ for all $f, g \in P$. □

Lemma 4.3. *If $L \subseteq \{0,1\}^S$, $0 \in L$ and $f', fg \in L$ for all $f, g \in L$ then $(L, \leq, ')$ is a Boolean algebra.*

Proof. $(L, \leq, ')$ is a distributive lattice with an antitone involution. Since $\max\{f, f'\} = 1$ for all $f \in L$, $(L, \leq, ')$ is Boolean. □

Lemma 4.4. *Suppose $L \subseteq \{0,1\}^S$, $0 \in L$ and L is closed with respect to $'$. If $f, g \in L$ and $f \perp g$ imply $f + g \in L$ then L is an algebra of S-probabilities.*

P r o o f. This follows from (d) of Lemma 2.1. $\square$

Lemma 4.5. *If $L \subseteq \{0,1\}^S$ is an algebra of S-probabilities and $fg \in L$ for all $f, g \in L$ then $(L, \leq, ')$ is an Boolean algebra.*

P r o o f. This follows from (e) of Lemma 2.1. $\square$

Theorem 4.6. *Let $(L, \leq, ')$ be an algebra of S-probabilities in $\{0,1\}$ which is an ortholattice. If $f +_1 g = f + g - 2fg$ then $(R(L), +_1, \cdot)$ is a classical GBQR of S-probabilities.*

P r o o f. For arbitrary functions $f_i, g_i \in [0,1]^S$, $i = 1,2$, we have the following elementary property:

$$\big((\forall i \in \{1,2\})(f_i \leq g_i) \;\;\&\;\; f_1 + f_2 = g_1 + g_2\big) \implies (\forall i \in \{1,2\})(f_i = g_i). \tag{4.1}$$

Our assumption $f +_1 g = f + g - 2fg$ means $(f \cap g') \cup (f' \cap g) = f(1-g) + g(1-f)$, and since $(f \cap g') \perp (f' \cap g)$ this can be written in the form

$$(f \cap g') + (f' \cap g) = f(1-g) + g(1-f). \tag{4.2}$$

The algebra of S-probabilities L is a subset of $\{0,1\}^S$, hence

$$f \cap g' = \inf_L \{f, g'\} \leq \inf_{\{0,1\}^S} \{f, g'\} = f \cdot g' = f(1-g), \quad \text{and} \quad f' \cap g \leq g(1-f). \tag{4.3}$$

Now we can apply (4.1) to the elements involved in (4.2) and (4.3) and obtain $f \cap g' = f(1-g) \in L$. Therefore we also have $f \cap g = fg \in L$ and Lemma 4.5 yields the result. $\square$

The remaining part of this section is devoted to algebras of S-probabilities with values in $\{0, 1/2, 1\}$.

Lemma 4.7. *If $f, g \in \{0, 1/2, 1\}^S$ then*

(i) $(1/2)\lceil 2fg \rceil = \min\{f, g\}$,

(ii) $(1/2)\lfloor 2(f + g - fg) \rfloor = \max\{f, g\}$.

P r o o f.

$$\min\{a, b\} = (1/2)\lceil 2ab \rceil$$

and

$$\begin{aligned}\max\{a,b\} &= 1-\min\{1-a,1-b\} = 1-(1/2)\lceil 2(1-a)(1-b)\rceil \\ &= (1/2)(2-\lceil 2(1-a)(1-b)\rceil) = (1/2)\lfloor 2-2(1-a)(1-b)\rfloor \\ &= (1/2)\lfloor 2(a+b-ab)\rfloor\end{aligned}$$

for all $a,b\in\{0,1/2,1\}$. $\square$

Lemma 4.8. *If $P\subseteq\{0,1/2,1\}^S$ and P is closed with respect to $'$ then the following are equivalent:*

(i) $(1/2)\lceil 2fg\rceil\in P$ *for all* $f,g\in P$,

(ii) $\min\{f,g\}\in P$ *for all* $f,g\in P$,

(iii) $(1/2)\lfloor 2(f+g-fg)\rfloor\in P$ *for all* $f,g\in P$,

(iv) $\max\{f,g\}\in P$ *for all* $f,g\in P$.

Proof.

(i) $\Longrightarrow$ (ii): $\min\{f,g\}=(1/2)\lceil 2fg\rceil$ for all $f,g\in P$.

(ii) $\Longrightarrow$ (iii): $(1/2)\lfloor 2(f+g-fg)\rfloor=\big(\min\{f',g'\}\big)'$ for all $f,g\in P$.

(iii) $\Longrightarrow$ (iv): $\max\{f,g\}=(1/2)\lfloor 2(f+g-fg)\rfloor$ for all $f,g\in P$.

(iv) $\Longrightarrow$ (i): $(1/2)\lceil 2fg\rceil=\big(\max\{f',g'\}\big)'$ for all $f,g\in P$. $\square$

Lemma 4.9. *If $P\subseteq\{0,1/2,1\}^S$ and $f',(1/2)\lceil 2fg\rceil\in P$ for all $f,g\in P$ then $(P,\le,')$ is a distributive lattice with an antitone involution. If, moreover, $0\in P$ then $(P,\le)$ is a Boolean algebra if and only if $P\subseteq\{0,1\}^S$. In this case f' is the complement of f.*

Proof. $(P,\le,')$ is a distributive lattice with an antitone involution. Now assume $0\in P$. If $P\subseteq\{0,1\}^S$ then $(P,\le,')$ is a Boolean algebra according to Lemma 4.3. If $P\not\subseteq\{0,1\}^S$ then there exists an $f\in P$ and an $s\in S$ with $f(s)=1/2$. If $(P,\le)$ would be a Boolean algebra then there would exist a $g\in P$ which is a complement of f. Now $g(s)=1$ since $\max\big\{f(s),g(s)\big\}=1$ but then $0=\min\big\{f(s),g(s)\big\}=1/2$, a contradiction. Hence $(P,\le)$ is not a Boolean algebra in this case. $\square$

Summing up we obtain the following result:

Theorem 4.10. *Let $(L,\le,')$ be an algebra of S-probabilities in $\{0,1/2,1\}$ which is an ortholattice such that $1/2\cdot\lceil 2fg\rceil\in L$ for all $f,g\in L$. Then $(R(L),+_1,\cdot)$ is a classical GBQR of S-probabilities if and only if $L\subseteq\{0,1\}^S$.*

Proof. By Lemma 4.9, $(R(L),+_1,\cdot)$ is a Boolean ring if and only if $L\subseteq\{0,1\}^S$. $\square$

REFERENCES

[1] BELTRAMETTI, E. G.—MĄCZYŃSKI, M. J.: *On a characterization of classical and nonclassical probabilities*, J. Math. Phys. **32** (1991), 1280–1286.
[2] BELTRAMETTI, E. G.—MĄCZYŃSKI, M. J.: *On the characterization of probabilities: A generalization of Bell's inequalities*, J. Math. Phys. **34** (1993), 4919–4929.
[3] DORNINGER, D.—LÄNGER, H.—MĄCZYŃSKI, M. J.: *The logic induced by a system of homomorphisms and its various algebraic characterizations*, Demonstratio Math. **30** (1997), 215–232.
[4] DORNINGER, D.—LÄNGER, H.—MĄCZYŃSKI, M. J.: *On ring-like structures induced by Mackey's probability function*, Rep. Math. Phys. **43** (1999), 499–515.
[5] DORNINGER, D.—LÄNGER, H.—MĄCZYŃSKI, M. J.: *Lattice properties of ring-like quantum logics*, Internat. J. Theoret. Phys. **39** (2000), 1015–1026.
[6] DORNINGER, D.—LÄNGER, H.—MĄCZYŃSKI, M. J.: *Concepts of measures on ring-like quantum logics*, Rep. Math. Phys. **47** (2001), 167–176.
[7] MĄCZYŃSKI, M. J.—TRACZYK, T.: *A characterization of orthomodular partially ordered sets admitting a full set of states*, Bull. Polish Acad. Sci. Math. **21** (1973), 3–8.
[8] PTÁK, P.—PULMANNOVÁ, S.: *Orthomodular Structures as Quantum Logics*, Kluwer Acad. Publ., Dordrecht, 1991.

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