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SPACES OF LOWER SEMICONTINUOUS SET-VALUED MAPS II

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ABSTRACT. This is a continuation of "Spaces of lower semicontinuous set-valued maps I". Together, these two parts contain two interrelated main theorems. In the previous part I, the Extension Theorem is proved, which says that for binormal spaces X and Y, every bimonotone homeomorphism between $C(X)$ and $C(Y)$ can be extended to an ordered homeomorphism between $L^-(X)$ and $L^-(Y)$. In this part II, the Factorization Theorem is proved, which says that for binormal spaces X and Y, every ordered homeomorphism between $L^-(X)$ and $L^-(Y)$ can be characterized by a unique factorization.

1. Introduction

The preceding "Spaces of lower semicontinuous set-valued maps I" introduces the lower semicontinuous analog, $L^-(X)$, of the well-studied space $L(X)$ of upper semicontinuous maps with values that are nonempty compact intervals in $\mathbb{R}$. Because the elements of $L^-(X)$ contain continuous selections, the space $C(X)$ of real-valued continuous functions on X can be used to establish properties of $L^-(X)$, such as the two interrelated main theorems. The first of these theorems, the Extension Theorem, is proved in part I. This Extension Theorem says that for binormal spaces X and Y, every bimonotone homeomorphism between $C(X)$ and $C(Y)$ can be extended to an ordered homeomorphism between $L^-(X)$ and $L^-(Y)$. We now prove in this part II the second main theorem, the Factorization Theorem, which says that for binormal spaces X and Y, every ordered homeomorphism between $L^-(X)$ and $L^-(Y)$ can be characterized by a unique

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factorization. So the two main theorems together give a unique factorization characterization of the bimonotone homeomorphisms between $C(X)$ and $C(Y)$. In this part, when we refer to a result in part I, we prefix its number with a I.

2. Regular open correspondences

Before we can understand exactly what the ordered homeomorphisms between $L^-(X)$ and $L^-(Y)$ are, we need to introduce a correspondence between X and Y that is more general than a homeomorphism. This will be needed for the Factorization Theorem in the next section.

Since X and Y are completely regular Hausdorff spaces, they are, in particular, regular spaces. Now let $\mathscr{T}_X$ and $\mathscr{T}_Y$ be the families of regular open subsets of X and Y, respectively (where U is regular open provided that $\operatorname{int}(\overline{U}) = U$). Then $\mathscr{T}_X$ and $\mathscr{T}_Y$ are bases for the topologies on X and Y. The next paragraph and the following lemma are stated in terms of the space X, but apply just as well to the space Y.

Let $\mathbb{T} = \mathbb{T}(X)$ be the collection of subfamilies of $\mathscr{T}_X$ that have the finite intersection property. If $\mathbb{T}$ is partially ordered by inclusion, then for every linearly ordered subcollection $\mathbb{L}$ of $\mathbb{T}$, $\bigcup \mathbb{L}$ is an upper bound of $\mathbb{L}$ in $\mathbb{T}$. So by Zorn's Lemma, every element of $\mathbb{T}$ is contained in a maximal element of $\mathbb{T}$. Let $\mathbb{T}^* = \mathbb{T}^*(X)$ be the collection of maximal elements of $\mathbb{T}$. For each $\mathscr{T} \in \mathbb{T}$, let $\overline{\mathscr{T}} = \{\overline{U} :\ U \in \mathscr{T}\}$.

Lemma 2.1. *For each $\mathscr{T} \in \mathbb{T}$, the following are true.*

(1) *If $\mathscr{T} \in \mathbb{T}^*$ and $U_1, \dots, U_n \in \mathscr{T}$, then $U_1 \cap \cdots \cap U_n \in \mathscr{T}$.*

(2) *The family $\mathscr{T} \in \mathbb{T}^*$ if and only if for every $U \in \mathscr{T}_X$ such that $U \cap U' \neq \emptyset$ for all $U' \in \mathscr{T}$, $U \in \mathscr{T}$.*

(3) *If $\mathscr{T} \in \mathbb{T}^*$, then either $\bigcap \overline{\mathscr{T}} = \emptyset$ or $\bigcap \overline{\mathscr{T}} = \{x\}$ for some $x \in X$.*

Proof.

(1) Let $\mathscr{T}' = \mathscr{T} \cup \{U_1 \cap \cdots \cap U_n\}$. Now let $U^1, \dots, U^m \in \mathscr{T}$. Then $U^1 \cap \cdots \cap U^m \cap U_1 \cap \cdots \cap U_n \neq \emptyset$ since $\mathscr{T}$ has the finite intersection property. Therefore, $\mathscr{T}'$ has the finite intersection property, so that $\mathscr{T}' \in \mathbb{T}$. Since $\mathscr{T}$ is maximal in $\mathbb{T}$, we have $\mathscr{T} = \mathscr{T}'$, and thus $U_1 \cap \cdots \cap U_n \in \mathscr{T}$.

(2) Let $\mathscr{T} \in \mathbb{T}^*$ and let $U \in \mathscr{T}_X$ be such that $U \cap U' \neq \emptyset$ for all $U' \in \mathscr{T}$. Let $\mathscr{T}' = \mathscr{T} \cup \{U\}$. If $U_1, \dots, U_n \in \mathscr{T}$, then by statement (1), $U_1 \cap \cdots \cap U_n \in \mathscr{T}$, so that $U_1 \cap \cdots \cap U_n \cap U \neq \emptyset$ by hypothesis. Therefore, $\mathscr{T}'$ has the finite intersection property, so that $\mathscr{T}' \in \mathbb{T}$. Since $\mathscr{T}$ is maximal in $\mathbb{T}$, we have $\mathscr{T} = \mathscr{T}'$, and hence $U \in \mathscr{T}$.

For the converse, suppose that for every $U \in \mathscr{T}_X$ such that $U \cap U' \neq \emptyset$ for all $U' \in \mathscr{T}$, we have $U \in \mathscr{T}$. Let $\mathscr{T}' \in \mathbb{T}$ with $\mathscr{T} \subseteq \mathscr{T}'$. Then for each $U \in \mathscr{T}'$, $U \cap U' \neq \emptyset$ for all $U' \in \mathscr{T}$ because $\mathscr{T}'$ has the finite intersection property. But then $U \in \mathscr{T}$, showing that $\mathscr{T}' = \mathscr{T}$, and that $\mathscr{T}$ is maximal in $\mathbb{T}$.

(3) Suppose, by way of contradiction, that there are x_1 and x_2 in $\bigcap \overline{\mathscr{T}}$ with $x_1 \neq x_2$. Let U_1 and U_2 be disjoint elements of $\mathscr{T}_X$ with $x_1 \in U_1$ and $x_2 \in U_2$. For each $U \in \mathscr{T}$, we have $U_1 \cap U \neq \emptyset$ and $U_2 \cap U \neq \emptyset$. So by statement (2), $U_1, U_2 \in \mathscr{T}$. Since $\mathscr{T}$ has the finite intersection property, $U_1 \cap U_2 \neq \emptyset$, which is a contradiction. □

Before we define a regular open correspondence from X to Y, we consider some preliminary properties of a bijection $\tau\colon \mathscr{T}_X \to \mathscr{T}_Y$.

Lemma 2.2. *If $\tau\colon \mathscr{T}_X \to \mathscr{T}_Y$ is a bijection, then the following are equivalent.*

(1) *For every $U_1, U_2 \in \mathscr{T}_X$, $U_1 \subseteq U_2$ if and only if $\tau(U_1) \subseteq \tau(U_2)$.*

(2) *For every $U_1, U_2 \in \mathscr{T}_X$, $\tau(U_1 \cap U_2) = \tau(U_2) \cap \tau(U_2)$.*

Proof.

(1) $\implies$ (2). Let $U_1, U_2 \in \mathscr{T}_X$. Since $U_1 \cap U_2 \subseteq U_1$ and $U_1 \cap U_2 \subseteq U_2$, we have $\tau(U_1 \cap U_2) \subseteq \tau(U_1) \cap \tau(U_2)$. For the reverse containment, since $\tau(U_1) \cap \tau(U_2) \subseteq \tau(U_1)$ and $\tau(U_1) \cap \tau(U_2) \subseteq \tau(U_2)$, it follows that $\tau^{-1}(\tau(U_1) \cap \tau(U_2)) \subseteq U_1 \cap U_2$. Therefore, $\tau(U_1) \cap \tau(U_2) \subseteq \tau(U_1 \cap U_2)$.

(2) $\implies$ (1). Let $U_1, U_2 \in \mathscr{T}_X$. Suppose first that $U_1 \subseteq U_2$. Then $U_1 \cap U_2 = U_1$, so that $\tau(U_1) = \tau(U_1 \cap U_2) = \tau(U_1) \cap \tau(U_2)$, which implies that $\tau(U_1) \subseteq \tau(U_2)$. Conversely, suppose that $\tau(U_1) \subseteq \tau(U_2)$. Then $\tau(U_1) \cap \tau(U_2) = \tau(U_1)$, so that $U_1 = \tau^{-1}(\tau(U_1) \cap \tau(U_2)) = \tau^{-1}(\tau(U_1) \cap U_2)) = U_1 \cap U_2$, which implies that $U_1 \subseteq U_2$. □

We will say that a bijection $\tau\colon \mathscr{T}_X \to \mathscr{T}_Y$ is an *ordered bijection* if it satisfies condition (1) of Lemma 2.2. Note that if τ is an ordered bijection, then $\tau(\emptyset) = \emptyset$. For each ordered bijection τ and each $\mathscr{T} \subseteq \mathscr{T}_X$, let $\tau(\mathscr{T}) = \{\tau(U) : U \in \mathscr{T}\}$.

Lemma 2.3. *Let $\tau\colon \mathscr{T}_X \to \mathscr{T}_Y$ be an ordered bijection, and let $\mathscr{T} \subseteq \mathscr{T}_X$. Then $\mathscr{T}$ has the finite intersection property if and only if $\tau(\mathscr{T})$ has the finite intersection property.*

Proof. Let $\mathscr{T}$ have the finite intersection property, and let $U_1, \dots, U_n \in \mathscr{T}$. If $U = U_1 \cap \dots \cap U_n$, then $U \neq \emptyset$, so that $\tau(U) \neq \emptyset$. But $\tau(U) \subseteq \tau(U_1) \cap \dots \cap \tau(U_n)$ since τ is ordered. This shows that $\tau(\mathscr{T})$ has the finite intersection property. The proof of the converse is similar. □

Note that Lemma 2.3 implies that for an ordered bijection $\tau\colon \mathscr{T}_X \to \mathscr{T}_Y$, if $\mathscr{T} \in \mathbb{T}(X)$, then $\tau(\mathscr{T}) \in \mathbb{T}(Y)$.

Lemma 2.4. *If $\tau\colon \mathscr{T}_X \to \mathscr{T}_Y$ is an ordered bijection and $\mathscr{T} \in \mathbb{T}^*(X)$, then $\tau(\mathscr{T}) \in \mathbb{T}^*(Y)$.*

Proof. Let $V \in \mathscr{T}_Y$ be such that $V \cap \tau(U) \neq \emptyset$ for all $U \in \mathscr{T}$. Then $\tau^{-1}(V) \cap U \neq \emptyset$ for all $U \in \mathscr{T}$, so that by Lemma 2.1, $\tau^{-1}(V) \in \mathscr{T}$, and hence $V \in \tau(\mathscr{T})$. But the proof of Lemma 2.1 also applies to $\tau(\mathscr{T})$ in $\mathbb{T}(Y)$, so that $\tau(\mathscr{T}) \in \mathbb{T}^*(Y)$. □

From this point on, we will use $\mathbb{T}^*$ for $\mathbb{T}^*(X)$. For each ordered bijection $\tau\colon \mathscr{T}_X \to \mathscr{T}_Y$ and $\mathscr{T} \in \mathbb{T}^*$, let $\overline{\tau(\mathscr{T})} = \{\overline{\tau(U)} : U \in \mathscr{T}\}$. Also let $\tau(\mathbb{T}^*) = \{\tau(\mathscr{T}) : \mathscr{T} \in \mathbb{T}^*\}$. Note that Lemmas 2.4 and 2.1 imply that if $\mathscr{T} \in \mathbb{T}^*$, then either $\bigcap \overline{\tau(\mathscr{T})} = \emptyset$ or $\bigcap \overline{\tau(\mathscr{T})} = \{y\}$ for some $y \in Y$.

Lemma 2.5. *If $\tau\colon \mathscr{T}_X \to \mathscr{T}_Y$ is an ordered bijection, then the following are equivalent.*

(1) *For every $U_1, U_2 \in \mathscr{T}_X$, $\overline{U_1} \cap \overline{U_2} \neq \emptyset$ if and only if $\overline{\tau(U_1)} \cap \overline{\tau(U_2)} \neq \emptyset$.*

(2) *For every $U_1, U_2 \in \mathscr{T}_X$, $\overline{U_1} \subseteq U_2$ if and only if $\overline{\tau(U_1)} \subseteq \tau(U_2)$.*

Proof.

(1) $\implies$ (2). Let $U_1, U_2 \in \mathscr{T}_X$. Suppose that $\overline{U_1} \subseteq U_2$. Define $U_0 = \tau^{-1}(Y \setminus \overline{\tau(U_2)})$. Then by Lemma 2.2, $\tau(U_0 \cap U_2) = \tau(U_0) \cap \tau(U_2) = (Y \setminus \overline{\tau(U_2)}) \cap \tau(U_2) = \emptyset$, and hence $U_0 \cap U_2 = \emptyset$. But then $\overline{U_0} \cap U_2 = \emptyset$. Since $\overline{U_1} \subseteq U_2$, we have $\overline{U_0} \cap \overline{U_1} = \emptyset$. Thus $\overline{Y \setminus \overline{\tau(U_2)}} \cap \overline{\tau(U_1)} = \overline{\tau(U_0)} \cap \overline{\tau(U_1)} = \emptyset$. But $\overline{\tau(U_1)} \subseteq \overline{\tau(U_2)}$, so that $\overline{\tau(U_1)} \cap \mathrm{bd}(\overline{\tau(U_2)}) = \emptyset$. Therefore, $\overline{\tau(U_1)} \subseteq \mathrm{int}(\overline{\tau(U_2)}) = \tau(U_2)$ because $\tau(U_2)$ is regular open. The proof of the converse is similar.

(2) $\implies$ (1). Let $U_1, U_2 \in \mathscr{T}_X$. Suppose that $\overline{U_1} \cap \overline{U_2} = \emptyset$. Define $U_0 = X \setminus \overline{U_2}$. Then $\overline{U_1} \subseteq X \setminus \overline{U_2} = U_0$. By statement (2), we have $\overline{\tau(U_1)} \subseteq \tau(U_0)$. Now $U_0 \cap U_2 = \emptyset$, so that by Lemma 2.2, $\tau(U_0) \cap \tau(U_2) = \tau(U_0 \cap U_2) = \emptyset$. Then $\tau(U_0) \cap \overline{\tau(U_2)} = \emptyset$, and hence $\overline{\tau(U_1)} \cap \overline{\tau(U_2)} = \emptyset$. The proof of the converse is similar. □

Now we define a *regular open correspondence from X to Y* to be an ordered bijection $\tau\colon \mathscr{T}_X \to \mathscr{T}_Y$ that satisfies condition (1) of Lemma 2.5. Let the family of such τ be denoted by $ROC(X, Y)$.

For $\tau \in ROC(X, Y)$, define

$$X^{\emptyset} = \left\{\mathscr{T} \in \mathbb{T}^* : \bigcap \overline{\mathscr{T}} = \emptyset \text{ and } \bigcap \overline{\tau(\mathscr{T})} \neq \emptyset\right\}$$

and

$$Y^{\emptyset} = \left\{\tau(\mathscr{T}) \in \tau(\mathbb{T}^*) : \bigcap \overline{\mathscr{T}} \neq \emptyset \text{ and } \bigcap \overline{\tau(\mathscr{T})} = \emptyset\right\}.$$

Also define equivalence relation $\sim$ on $X^{\emptyset}$ as follows. If $\mathscr{T}_1, \mathscr{T}_2 \in X^{\emptyset}$, then $\mathscr{T}_1 \sim \mathscr{T}_2$ provided that $\bigcap \overline{\tau(\mathscr{T}_1)} = \bigcap \overline{\tau(\mathscr{T}_2)}$. Then let X^* be the set of equivalence classes of $\sim$, and let us denote a typical element by $[\mathscr{T}]$ where $\mathscr{T}$ is an element of $X^{\emptyset}$ that is contained in this equivalence class. Similarly, define equivalence relation $\sim$ on $Y^{\emptyset}$ by $\tau(\mathscr{T}_1) \sim \tau(\mathscr{T}_2)$ provided that $\bigcap \overline{\mathscr{T}_1} = \bigcap \overline{\mathscr{T}_2}$, and let Y^* be its set of equivalence classes with typical element denoted by $[\tau(\mathscr{T})]$. Now define $\tau X = X \cup X^*$ and $\tau Y = Y \cup Y^*$. Finally, for each $U \in \mathscr{T}_X$, define

$$U^* = \bigl\{[\mathscr{T}] \in X^* : \ U \in \mathscr{T} \text{ and } \bigcap \overline{\tau(\mathscr{T})} \subseteq \tau(U)\bigr\},$$

and for each $V \in \mathscr{T}_Y$, define

$$V^* = \bigl\{[\tau(\mathscr{T})] \in Y^* : \ V \in \tau(\mathscr{T}) \text{ and } \bigcap \overline{\mathscr{T}} \subseteq \tau^{-1}(V)\bigr\}.$$

Lemma 2.6. *For $\tau \in ROC(X,Y)$, the family $\bigl\{U \cup U^* : \ U \in \mathscr{T}_X\bigr\}$ is a base for a topology on τX such that X is a dense subspace of τX. Also the family $\bigl\{V \cup V^* : \ V \in \mathscr{T}_Y\bigr\}$ is a base for a topology on τY such that Y is a dense subspace of τY.*

P r o o f. To show that $\bigl\{U \cup U^* : \ U \in \mathscr{T}_X\bigr\}$ is closed under finite intersections, it suffices to show that for each $U_1, U_2 \in \mathscr{T}_X$, $U_1^* \cap U_2^* = (U_1 \cap U_2)^*$. First let $[\mathscr{T}] \in U_1^* \cap U_2^*$. Then $U_1, U_2 \in \mathscr{T}$ and $\bigcap \overline{\tau(\mathscr{T})} \subseteq \tau(U_1) \cap \tau(U_2)$. But $U_1 \cap U_2 \in \mathscr{T}$ by Lemma 2.1, and $\tau(U_1) \cap \tau(U_2) = \tau(U_1 \cap U_2)$ by Lemma 2.2, so that $[\mathscr{T}] \in (U_1 \cap U_2)^*$. For the reverse containment, let $[\mathscr{T}] \in (U_1 \cap U_2)^*$. Then $U_1 \cap U_2 \in \mathscr{T}$ and $\bigcap \overline{\tau(\mathscr{T})} \subseteq \tau(U_1 \cap U_2) = \tau(U_1) \cap \tau(U_2)$. By the maximality of $\mathscr{T}$, $U_1, U_2 \in \mathscr{T}$, and hence $[\mathscr{T}] \in U_1^* \cap U_2^*$.

This shows that $\bigl\{U \cap U^* : \ U \in \mathscr{T}_X\bigr\}$ is closed under finite intersections, and since this family includes the empty set and covers τX, it is a base for a topology on τX. Also since $\mathscr{T}_X$ is a base for the topology on X, it is clear that X is a dense subspace of τX. We have a similar argument that $\bigl\{V \cap V^* : \ V \in \mathscr{T}_Y\bigr\}$ is a base for a topology on τY such that Y is a dense subspace of τY. $\square$

Lemma 2.7. *For $\tau \in ROC(X,Y)$, τX and τY are Hausdorff spaces.*

P r o o f. We only show that τX is Hausdorff since the proof that τY is Hausdorff is similar. First note that if $U_1, U_2 \in \mathscr{T}_X$ are such that $U_1 \cap U_2 = \emptyset$, then $(U_1 \cup U_1^*) \cap (U_2 \cup U_2^*) = \emptyset$. Since X is Huasdorff, we can therefore separate points of X in the space τX.

If $[\mathscr{T}_1], [\mathscr{T}_2] \in X^*$ with $[\mathscr{T}_1] \neq [\mathscr{T}_2]$, then $y_1 \neq y_2$ where $\bigcap \overline{\tau(\mathscr{T}_1)} = \{y_1\}$ and $\bigcap \overline{\tau(\mathscr{T}_2)} = \{y_2\}$. Let V_1 and V_2 be disjoint elements of $\mathscr{T}_Y$ with $y_1 \in V_1$ and $y_2 \in V_2$. So if $U_1 = \tau^{-1}(V_1)$ and $U_2 = \tau^{-1}(V_2)$, we have $U_1 \in \mathscr{T}_1$, $U_2 \in \mathscr{T}_2$, and $U_1 \cap U_2 = \emptyset$. Therefore, $U_1 \cup U_1^*$ and $U_2 \cup U_2^*$ are disjoint neighborhoods of $[\mathscr{T}_1]$ and $[\mathscr{T}_2]$ in τX.

Finally, if $x \in X$ and $[\mathscr{T}] \in X^*$, then there exists a $U_1 \in \mathscr{T}$ with $x \notin \overline{U_1}$. By the regularity of X, there exists a $U_2 \in \mathscr{T}_X$ with $x \in U_2$ and $\overline{U_1} \cap \overline{U_2} = \emptyset$. It follows that $\overline{\tau(U_1)} \cap \overline{\tau(U_2)} = \emptyset$. Let $U_3 = X \setminus \overline{U_2}$, which is in $\mathscr{T}_X$. Since $\overline{U_1} \subseteq U_3$, $\overline{\tau(U_1)} \subseteq \tau(U_3)$ by Lemma 2.5. If $\bigcap \overline{\tau(\mathscr{T})} = \{y\}$, then $y \in \overline{\tau(U_1)}$, so that $y \in \tau(U_3)$. By the maximality of $\tau(\mathscr{T})$, we have $\tau(U_3) \in \tau(\mathscr{T})$, and hence $U_3 \in \mathscr{T}$. Thus $[\mathscr{T}] \in U_3^*$. Note that $y \notin \overline{\tau(U_2)}$, so that $U_2 \notin \mathscr{T}$, and so $[\mathscr{T}] \notin U_2^*$. Then $U_2 \cup U_2^*$ and $U_3 \cup U_3^*$ are disjoint open subsets of τX containing x and $[\mathscr{T}]$, respectively. □

Now for $\tau \in ROC(X,Y)$, we have topological spaces τX and τY that are Hausdorff extensions of X and Y. Next we define $e_\tau \colon \tau X \to \tau Y$ as follows. First, let $x \in X$, and take $\mathscr{T}$ to be any element of $\mathbb{T}^*$ that contains $\{U \in \mathscr{T}_X : x \in U\}$ (and hence $\bigcap \overline{\mathscr{T}} = \{x\}$). If $\bigcap \overline{\tau(\mathscr{T})} = \emptyset$, then $\tau(\mathscr{T}) \in Y^\emptyset$, so we define $e_\tau(x) = [\tau(\mathscr{T})] \in Y^* \subseteq \tau Y$. If $\bigcap \overline{\tau(\mathscr{T})} \neq \emptyset$, then $\bigcap \overline{\tau(\mathscr{T})} = \{y\}$ for some $y \in Y$, so we define $e_\tau(x) = y \in Y \subseteq \tau Y$. Secondly, let $[\mathscr{T}] \in X^*$. Then $\mathscr{T} \in X^\emptyset$, so that $\bigcap \overline{\mathscr{T}} = \emptyset$. In this case also, $\bigcap \overline{\tau(\mathscr{T})} = \{y\}$ for some $y \in Y$, and we also define $e_\tau([\mathscr{T}]) = y$.

Lemma 2.8. *For $\tau \in ROC(X,Y)$, $e_\tau \colon \tau X \to \tau Y$ is a well-defined bijection.*

Proof. First let us show that e_τ is well-defined. Let $\mathscr{T}_1, \mathscr{T}_2$ be elements of $\mathbb{T}^*$ that both contain $\{U \in \mathscr{T}_X : x \in U\}$, and hence $\bigcap \overline{\mathscr{T}_1} = \{x\}$ and $\bigcap \overline{\mathscr{T}_2} = \{x\}$. Suppose, by way of contradiction, that $\bigcap \overline{\tau(\mathscr{T}_1)} = \emptyset$ and $\bigcap \overline{\tau(\mathscr{T}_2)} \neq \emptyset$. Then $\bigcap \overline{\tau(\mathscr{T}_2)} = \{y\}$ for some $y \in Y$. Now $y \notin \bigcap \overline{\tau(\mathscr{T}_1)}$, so there exits a $U_1 \in \mathscr{T}_1$ with $y \notin \overline{\tau(U_1)}$; let $V_1 = \tau(U_1)$. Then there exists a $V_2 \in \mathscr{T}_Y$ with $y \in V_2$ and $\overline{V_1} \cap \overline{V_2} = \emptyset$. Since $\tau(\mathscr{T}_2)$ is maximal in $\mathbb{T}(Y)$, by Lemma 2.4, we have $V_2 \in \tau(\mathscr{T}_2)$, so that $U_2 \in \mathscr{T}_2$ where $U_2 = \tau^{-1}(V_2)$. Now $x \in \overline{U_1} \cap \overline{U_2}$, which implies that $\overline{V_1} \cap \overline{V_2} = \overline{\tau(U_1)} \cap \overline{\tau(U_2)} \neq \emptyset$ by Lemma 2.5. But this is a contradiction, so that either both $\bigcap \overline{\tau(\mathscr{T}_1)} = \emptyset$ and $\bigcap \overline{\tau(\mathscr{T}_2)} = \emptyset$ or both $\bigcap \overline{\tau(\mathscr{T}_1)} \neq \emptyset$ and $\bigcap \overline{\tau(\mathscr{T}_2)} \neq \emptyset$.

Suppose first that $\bigcap \overline{\tau(\mathscr{T}_1)} = \{x\} = \bigcap \overline{\tau(\mathscr{T}_2)}$. Then since $\bigcap \overline{\mathscr{T}_1} = \{x\} = \bigcap \overline{\mathscr{T}_2}$, we have $\tau(\mathscr{T}_1), \tau(\mathscr{T}_2) \in Y^\emptyset$ and $[\tau(\mathscr{T}_1)] = [\tau(\mathscr{T}_2)]$, showing that $e_\tau(x)$ is well-defined. Now in the case that $\bigcap \overline{\tau(\mathscr{T}_1)} \neq \emptyset \neq \bigcap \overline{\tau(\mathscr{T}_2)}$, suppose $\bigcap \overline{\tau(\mathscr{T}_1)} = \{y_1\}$ and $\bigcap \overline{\tau(\mathscr{T}_2)} = \{y_2\}$. If $y_1 \neq y_2$, then there exists $V_1 \in \tau(\mathscr{T}_1)$ and $V_2 \in \tau(\mathscr{T}_2)$ with $y_1 \in V_1$, $y_2 \in V_2$, and $\overline{V_1} \cap \overline{V_2} = \emptyset$. But then if $U_1 = \tau^{-1}(V_1)$ and $U_2 = \tau^{-1}(V_2)$, we have $U_1 \in \mathscr{T}_1$ and $U_2 \in \mathscr{T}_2$, so that $x \in \overline{U_1} \cap \overline{U_2}$. Again, by Lemma 2.5, this is a contradiction, so that $y_1 = y_2$, showing that $e_\tau(x)$ is also well-defined in this case.

Finally, let $\mathscr{T}_1, \mathscr{T}_2 \in X^\emptyset$ be such that $[\mathscr{T}_1] = [\mathscr{T}_2]$. Then the definition of the equivalence relation $\sim$ gives $\bigcap \overline{\tau(\mathscr{T}_1)} = \bigcap \overline{\tau(\mathscr{T}_2)}$, so that in this final case, $e_\tau([\mathscr{T}_1]) = e_\tau([\mathscr{T}_2])$. Therefore, e_τ is a well-defined function from τX into τY.

Define the inverse $e_\tau^{-1} \colon \tau Y \to \tau X$ as follows. Let $y \in Y$. Take $\mathscr{T}$ to be any element of $\mathbb{T}^*$ that contains $\{\tau^{-1}(V) : y \in V \in \mathscr{T}_Y\}$; and hence $\bigcap \overline{\tau(\mathscr{T})} = \{y\}$. If $\bigcap \overline{\mathscr{T}} = \emptyset$, then $\mathscr{T} \in X^\emptyset$, so that we define $e_\tau^{-1}(y) = [\mathscr{T}] \in X^*$. If $\bigcap \overline{\mathscr{T}} \neq \emptyset$, then $\bigcap \overline{\mathscr{T}} = \{x\}$ for some $x \in X$, so we define $e_\tau^{-1}(y) = x$. Finally, let $[\tau(\mathscr{T})] \in Y^*$. Then $\tau(\mathscr{T}) \in Y^\emptyset$, so that $\bigcap \overline{\mathscr{T}} \neq \emptyset$. In this case also, $\bigcap \overline{\mathscr{T}} = \{x\}$ for some

$x \in X$, and we define $e_\tau^{-1}([\tau(\mathscr{T})]) = x$. The argument that e_τ^{-1} is well-defined is similar to the argument that e_τ is well-defined.

Now to show that $e_\tau^{-1}e_\tau$ is the identity map on τX, let $x \in X$. Let $\mathscr{T}$ be any element of $\mathbb{T}^*$ that contains $\{U \in \mathscr{T}_X : x \in U\}$; and hence $\bigcap \overline{\mathscr{T}} = \{x\}$. Suppose first that $\bigcap \overline{\tau(\mathscr{T})} = \emptyset$, so that $e_\tau(x) = [\tau(\mathscr{T})]$. Then $e_\tau^{-1}e_\tau(x) = x$ since $\bigcap \overline{\mathscr{T}} = \{x\}$. Now suppose that $\bigcap \overline{\tau(\mathscr{T})} = \{y\}$, so that $e_\tau(x) = y$. Then $e_\tau^{-1}e_\tau(x) = x$ since $\bigcap \overline{\mathscr{T}} = \{x\}$. Finally, let $[\mathscr{T}] \in X^*$. Then $e_\tau([\mathscr{T}]) = y$ where $\bigcap \overline{\tau(\mathscr{T})} = \{y\}$. But $\mathscr{T}$ contains $\{\tau^{-1}(V) : y \in V \in \mathscr{T}_Y\}$, so that $e_\tau^{-1}e_\tau([\mathscr{T}]) = [\mathscr{T}]$ since $\bigcap \overline{\mathscr{T}} = \emptyset$. Therefore, $e_\tau^{-1}e_\tau$ is the identity map on τX. A similar argument shows that $e_\tau e_\tau^{-1}$ is the identity map on τY, finishing the argument that e_τ is a well-defined bijection. $\square$

To show that $e_\tau \colon \tau X \to \tau Y$ is a homeomorphism, we need the next two technical lemmas. They have similar proofs, so we only give the proof of the first lemma.

Lemma 2.9. *Let $\tau \in ROC(X,Y)$, let $U \in \mathscr{T}_X$, and let $[\mathscr{T}] \in U^*$ with $\bigcap \overline{\tau(\mathscr{T})} = \{y\}$ for some $y \in Y$. Then there exists a $U_0 \in \mathscr{T}_X$ such that $y \in \tau(U_0)$, $\overline{U_0} \subseteq U$, and $[\mathscr{T}] \in U_0^* \subseteq U^*$.*

P r o o f. Since $[\mathscr{T}] \in U^*$, we have $U \in \mathscr{T}$ and $\{y\} = \bigcap \overline{\tau(\mathscr{T})} \subseteq \tau(U)$. Let $V = \tau(U)$, so that $y \in V \in \tau(\mathscr{T})$. Let $V_0 \in \mathscr{T}_Y$ with $y \in V_0$ and $\overline{V_0} \subseteq V$, and let $U_0 = \tau^{-1}(V_0)$. Then by Lemma 2.5, $\overline{U_0} \subseteq U$. Since $y \in V_0$, we have $V_0 \in \tau(\mathscr{T})$, so that $U_0 \in \mathscr{T}$. Also $\bigcap \overline{\tau(\mathscr{T})} = \{y\} \subseteq V_0 = \tau(U_0)$, and hence $[\mathscr{T}] \in U_0^*$. To show that $U_0^* \subseteq U^*$, let $[\mathscr{T}_0] \in U_0^*$. Then $U_0 \in \mathscr{T}_0$ and $\bigcap \overline{\tau(\mathscr{T}_0)} \subseteq \tau(U_0) = V_0$. But $U_0 \subseteq U$, so that $U \in \mathscr{T}_0$. Also $V_0 \subseteq V = \tau(U)$, so that $\bigcap \overline{\tau(\mathscr{T}_0)} \subseteq \tau(U)$. Thus $[\mathscr{T}_0] \in U^*$, showing that $U_0^* \subseteq U^*$. $\square$

Lemma 2.10. *Let $\tau \in ROC(X,Y)$, let $V \in \mathscr{T}_Y$, and let $[\tau(\mathscr{T})] \in V^*$ with $\bigcap \overline{\mathscr{T}} = \{x\}$ for some $x \in X$. Then there exists a $V_0 \in \mathscr{T}_Y$ such that $x \in \tau^{-1}(V_0)$, $\overline{V_0} \subseteq V$, and $[\tau(\mathscr{T})] \in V_0^* \subseteq V^*$.*

Proposition 2.11. *For $\tau \in ROC(X,Y)$, $e_\tau \colon \tau X \to \tau Y$ is a homeomorphism.*

P r o o f. We know that e_τ is a bijection by Lemma 2.8. To prove that e_τ is continuous, first let $x \in X$ and $V \in \mathscr{T}_Y$ with $e_\tau(x) \in V^* \subseteq V \cup V^*$. Let $\mathscr{T}$ be any element of $\mathbb{T}^*$ that contains $\{U \in \mathscr{T}_X : x \in U\}$; and hence $\bigcap \overline{\mathscr{T}} = \{x\}$. By Lemma 2.10, there exists a $V_0 \in \mathscr{T}_Y$ such that $x \in \tau^{-1}(V_0)$, $\overline{V_0} \subseteq V$, and $[\tau(\mathscr{T})] \in V_0^* \subseteq V^*$. Define $U_0 = \tau^{-1}(V_0)$. Then $x \in U_0 \in \mathscr{T}$, so that $U_0 \cup U_0^*$ is a neighborhood of x in X_τ.

To show that $e_\tau(U_0 \cup U_0^*) \subseteq V \cup V^*$, first let $x_0 \in U_0$, and let $\mathscr{T}_0$ be any element of $\mathbb{T}^*$ that contains $\{U' \in \mathscr{T}_X : x_0 \in U'\}$; and hence $\bigcap \overline{\mathscr{T}_0} = \{x_0\}$. For the first case, suppose that $\bigcap \overline{\tau(\mathscr{T}_0)} = \emptyset$, so that $e_\tau(x_0) = [\tau(\mathscr{T}_0)]$. Now $x_0 \in U_0$, so that $U_0 \in \mathscr{T}_0$, and hence $V_0 \in \tau(\mathscr{T}_0)$. Also $\bigcap \overline{\mathscr{T}_0} = \{x_0\} \subseteq U_0 = \tau^{-1}(V_0)$, so that $e_\tau(x_0) = [\tau(\mathscr{T}_0)] \in V_0^* \subseteq V^* \subseteq V \cup V^*$. For the second case, suppose that $\bigcap \overline{\tau(\mathscr{T}_0)} = \{y\}$ for some $y \in Y$. Since $x_0 \in U_0$, we have $U_0 \in \mathscr{T}_0$, and thus $V_0 \in \tau(\mathscr{T}_0)$. Then $e_\tau(x_0) = y \in \overline{V_0} \subseteq V \subseteq V \cup V^*$. Finally, let $[\mathscr{T}_0] \in U_0^*$, so that $U_0 \in \mathscr{T}_0$, and hence $V_0 \in \tau(\mathscr{T}_0)$. Then $\bigcap \overline{\tau(\mathscr{T}_0)} = \{y\}$ for some $y \in Y$, so that $e_\tau([\mathscr{T}_0]) = y \in \overline{V_0} \subseteq V \subseteq V \cup V^*$. These cases finish the argument that $e_\tau(U_0 \cup U_0^*) \subseteq V \cup V^*$.

For our final case, let $[\mathscr{T}] \in X^*$ with $e_\tau([\mathscr{T}]) \in V \cup V^*$. Then $\bigcap \overline{\tau(\mathscr{T})} = \{y\}$ for some $y \in Y$, so that $e_\tau([\mathscr{T}]) = y$. Then $y \in V$, so that $V \in \tau(\mathscr{T})$. Let $U = \tau^{-1}(V)$, and thus $U \in \mathscr{T}$. Also $\bigcap \overline{\tau(\mathscr{T})} = \{y\} \subseteq V = \tau(U)$, so that $[\mathscr{T}] \in U^*$. By Lemma 2.9, there exists a $U_0 \in \mathscr{T}_X$ such that $y \in \tau(U_0)$, $\overline{U_0} \subseteq U$, and $[\mathscr{T}] \in U_0^* \subseteq U^*$. Then $U_0 \cup U_0^*$ is a neighborhood of $[\mathscr{T}]$ in τX. The argument that $e_\tau(x) \in V \subseteq V \cup V^*$ is similar to the argument in the previous paragraph, except we need to use Lemma 2.5 to know that $\overline{V_0} \subseteq V$ where $V_0 = \tau(U_0)$. This finishes the proof that e_τ is continuous. A similar argument shows that e_τ^{-1} is continuous. □

Finally, for $\tau \in ROC(X,Y)$, we will say that τ has the *lifting property* provided that every element of $C(X)$ has an extension in $C(\tau X)$ and every element of $C(Y)$ has an extension in $C(\tau Y)$. Let $LROC(X,Y)$ denote the set of elements of $ROC(X,Y)$ that have the lifting property. For $\tau \in LROC(X,Y)$ and for each $f \in C(X)$, let $e_X(f)$ denote the extension of f in $C(\tau X)$, and for each $g \in C(Y)$, let $e_Y(g)$ denote the extension of g in $C(\tau Y)$. This gives us bijections $e_X\colon C(X) \to C(\tau X)$ and $e_Y\colon C(Y) \to C(\tau Y)$. If $C(\tau X)$ and $C(\tau Y)$ have the fine topology, then the following is true.

Lemma 2.12. *If X and Y are binormal spaces and $\tau \in LROC(X,Y)$, then $e_X\colon C(X) \to C(\tau X)$ and $e_Y\colon C(Y) \to C(\tau Y)$ are strictly increasing homeomorphisms.*

Proof. Let $f_1, f_2 \in C(X)$ with $f_1 < f_2$. Since X is dense in τX, we have $e_X(f_1) \le e_X(f_2)$. Suppose, by way of contradiction, that $e_X(f_1)([\mathscr{T}]) = e_X(f_2)([\mathscr{T}])$ for some $[\mathscr{T}] \in X^*$. Then by the continuity of $e_X(f_1)$ and $e_X(f_2)$, for each $n \in \mathbb{N}$, there exists a $U_n \in \mathscr{T}_X$ with $[\mathscr{T}] \in U_n^*$ such that for each $x \in U_n$, $0 < f_2(x) - f_1(x) < 1/n$. Now define $f \in C(X)$ by $f(x) = 1/(f_2(x) - f_1(x))$. So for each $x \in U_n$, $f(x) > n$. But then the continuity of $e_X(f)$ implies that $e_X(f)([\mathscr{T}]) \ge n$ for all $n \in \mathbb{N}$, which is a contradiction. Therefore, $e_X(f_1) < e_X(f_2)$, showing that e_X is strictly increasing. Now since X and Y are binormal spaces, it follows that e_X is a homeomorphism. The proof that e_Y is a strictly increasing homeomorphism is similar. □

Now assume that X and Y are binormal spaces, and that $\tau \in LROC(X,Y)$. Define $\hat{e}_\tau\colon C(\tau X) \to C(\tau Y)$ by $\hat{e}_\tau(\hat{f}) = \hat{f} e_\tau^{-1}$ for all $\hat{f} \in C(\tau X)$. Then $\hat{e}_\tau$ is a strictly increasing homeomorphism with inverse given by $\hat{e}_\tau^{-1}(\hat{g}) = \hat{g}e_\tau$ for all $\hat{g} \in C(\tau Y)$. Define $\hat{\tau}\colon C(X) \to C(Y)$ by $\hat{\tau} = e_Y^{-1}\hat{e}_\tau e_X$. We see that $\hat{\tau}$ is a strictly increasing homeomorphism, so that the Extension Theorem I.5.1 ensures that $\hat{\tau}$ induces an ordered homeomorphism $\tau^*\colon L^-(X) \to L^-(Y)$ that is an extension of $\hat{\tau}$, where τ^* is defined by $\tau^*(F) = \bigcup\{\hat{\tau}(f) :\ f \in C(X) \text{ and } f \subseteq F\}$.

We need to point out that in the previous section, the Extension Theorem I.5.1 was only proved for $L^-(X)$ and $L^-(Y)$ with the upper Vietoris topology. However, the Extension Theorem I.5.1 is also true for $L^-(X)$ and $L^-(Y)$ with the Vietoris topology, but that follows from Proposition 3.6 in the next section, which in turn depends on τ^* being continuous with respect to the Vietoris topology. So we give an independent proof of the continuity of τ^* with respect to the lower Vietoris topology in the following proposition.

Proposition 2.13. *If X and Y are binormal spaces, then every $\tau \in LROC(X,Y)$ induces an ordered homeomorphism $\tau^*\colon L^-(X) \to L^-(Y)$.*

Proof. Since we know from the proof of the Extension Theorem I.5.1 that τ^* and $(\tau^*)^{-1}$ are continuous with respect to the upper Vietoris topology, we only need to show the continuity of τ^* with respect to the lower Vietoris topology. The argument for the continuity of $(\tau^*)^{-1}$ with respect to the lower Vietoris topology is similar.

Let $F \in L^-(X)$ and let $\tau^*(F) \in W^-$ where $W = V \times O$ for $V \in \mathscr{T}_Y$ and O open in $\mathbb{R}$. Then there exists an $f \in C(X)$ such that $f \subseteq F$ and $\hat{\tau}(f) \cap W$ contains some $\langle y, t\rangle$. Let $\hat{f} = e_X(f)$, let $\hat{x} = e_\tau^{-1}(y)$, and let $U \in \mathscr{T}_X$ be such that $\hat{x} \in U \cup U^* \subseteq e_\tau^{-1}(V \cup V^*)$. Since $\hat{f}(\hat{x}) = t \in O$ and $\hat{f}$ is continuous, we may choose U so that $\hat{f}(U \cup U^*) \subseteq O$. Define $W_0 = U \times O$. Then $f \cap W_0 \neq \emptyset$, so that W_0^- is a neighborhood of F in $L^-(X)$.

Now let $F_0 \in W_0^-$. Then there exists an $f_0 \in C(X)$ such that $f_0 \subseteq F_0$ and $f_0 \cap W_0$ contains some $\langle x, s\rangle$. Let $\hat{g} = \hat{e}_\tau e_X(f_0)$ and let $\hat{y} = e_\tau(x)$. Now $x \in U \subseteq U \cup U^*$, so that $\hat{y} \in V \cup V^*$. Since $\hat{g}(\hat{y}) = s \in O$ and $\hat{g}$ is continuous, there exists a $V_0 \in \mathscr{T}_Y$ such that $\hat{y} \in V_0 \cup V_0^* \subseteq V \cup V^*$ and $\hat{g}(V_0 \cup V_0^*) \subseteq O$. Then for $y_0 \in V_0$, $\hat{\tau}(f_0)(y_0) \in O$, so that $\emptyset \neq \hat{\tau}(f_0) \cap V_0 \times O \subseteq \hat{\tau}(f_0) \cap W$. Therefore, $\tau^*(F_0) \in W^-$, showing that $\tau^*(W_0^-) \subseteq W^-$, and hence τ^* is continuous with respect to the lower Vietoris topology. $\square$

Instead of relating X and Y by extensions via regular open correspondences, there is a second way of relating X and Y by using dense subspaces, as long as X and Y have some completeness. This will induce the same ordered homeomorphism between $L^-(X)$ and $L^-(Y)$ as was induced using extensions.

We define a *lifting dense homeomorphism from X to Y* to be a homeomorphism $\theta\colon X' \to Y'$ where X' is a dense subspace of X and Y' is a dense subspace of Y and θ is such that for every f in $C(X)$, $f\theta^{-1}$ extends to an element of $C(Y)$, and for every g in $C(Y)$, $g\theta$ extends to an element of $C(X)$. Denote the set of all such θ by $LDH(X,Y)$. Clearly, $H(X,Y) \subseteq LDH(X,Y)$, where $H(X,Y)$ is the set of homeomorphisms from X onto Y.

If $\theta \in LDH(X,Y)$, then for every $f \in C(X)$, let $\hat{\theta}(f)$ denote the unique extension of $f\theta^{-1}$ to $C(Y)$. This defines a function $\hat{\theta}\colon C(X) \to C(Y)$.

Lemma 2.14. *For each $\theta \in LDH(X,Y)$, the function $\hat{\theta}\colon C(X) \to C(Y)$ is an increasing homeomorphism.*

Proof. Define $\hat{\theta}^{-1}\colon C(Y) \to C(X)$ by letting $\hat{\theta}^{-1}(g)$ be the unique extension of $g\theta$ to $C(X)$. To show that $\hat{\theta}^{-1}$ is indeed the inverse of $\hat{\theta}$, let $f \in C(X)$. Then for each $x \in X'$, $\hat{\theta}^{-1}\hat{\theta}(f)(x) = \hat{\theta}(f)\theta(x) = f\theta^{-1}(\theta(x)) = f(x)$. Since X' is dense in X, $\hat{\theta}^{-1}\hat{\theta}(f)(x) = f(x)$ for all $x \in X$, and hence $\hat{\theta}^{-1}\hat{\theta}(f) = f$. This shows that $\hat{\theta}^{-1}\hat{\theta}$ is the identity on $C(X)$. A similar argument shows that $\hat{\theta}\hat{\theta}^{-1}$ is the identity on $C(Y)$, and hence $\hat{\theta}$ is a bijection with inverse $\hat{\theta}^{-1}$.

Because Y' is dense in Y, it is evident that if $f_1, f_2 \in C(X)$ with $f_1 \leq f_2$, then $\hat{\theta}(f_1) \leq \hat{\theta}(f_2)$. Similarly, if $g_1, g_2 \in C(Y)$ with $g_1 \leq g_2$, then $\hat{\theta}^{-1}(g_1) \leq \hat{\theta}^{-1}(g_2)$. So when we show that $\hat{\theta}$ is a homeomorphism, it will be an increasing homeomorphism.

To show that $\hat{\theta}$ is continuous, let $f \in C(X)$, and let W be an open subset of $Y \times \mathbb{R}$ with $\hat{\theta}(f) \in W^+$. Since Y is binormal, there exist $g_1, g_2 \in C(Y)$ with $g_1 < \hat{\theta}(f) < g_2$ and $G \subseteq W$ where $G = \big\{\langle y,t\rangle \in Y \times \mathbb{R} :\ g_1(y) \leq t \leq g_2(y)\big\}$. Let $f_1 = \hat{\theta}^{-1}(g_1)$ and $f_2 = \hat{\theta}^{-1}(g_2)$. Then let $W_0 = \big\{\langle x,t\rangle \in X \times \mathbb{R} :\ f_1(x) < t < f_2(x)\big\}$, which is an open subset of $X \times \mathbb{R}$.

We now argue that $f \in W_0^+$, by showing that $f_1 < f$; a similar argument shows that $f < f_2$. Define $g_0 \in C(Y)$ by $g_0(y) = 1/(\hat{\theta}(f)(y) - g_1(y))$ for all $y \in Y$, and let $f_0 = \hat{\theta}^{-1}(g_0)$. For each $x \in X'$, $f_0(x) = \hat{\theta}^{-1}(g_0)(x) = g_0\theta(x) = 1/(\hat{\theta}(f)\theta(x) - g_1\theta(x)) = 1/(f\theta^{-1}\theta(x) - \hat{\theta}^{-1}(g_1)(x)) = 1/(f(x) - f_1(x)) > 0$. Then for all $x \in X'$, $f_1(x) = f(x) - 1/f_0(x) < f(x)$. Since X' is dense in X, by the continuity of f, f_1 and f_0, $f_1(x) = f(x) - 1/f_0(x) < f(x)$ for all $x \in X$, and thus $f_1 < f$. Therefore, $f \in W_0^+$.

Now let $f' \in W_0^+$, so that $f_1 < f' < f_2$. Then $g_1 = \hat{\theta}(f_1) \leq \hat{\theta}(f') \leq \hat{\theta}(f_2) = g_2$. Hence $\hat{\theta}(f') \subseteq G \subseteq W$, so that $\hat{\theta}(f') \in W^+$. This shows that $\hat{\theta}$ is continuous, and a similar argument shows the continuity of $\hat{\theta}^{-1}$. □

Now Lemma 2.14 and the Extension Theorem I.5.1 imply that $\hat{\theta}$ induces an ordered homeomorphism $\theta^*\colon L^-(X) \to L^-(Y)$.

Proposition 2.15. *Let X and Y be Čech-complete binormal spaces and let $\tau \in LROC(X,Y)$. Then there exists a $\theta \in LDH(X,Y)$ such that e_τ is an extension of θ, and such that $\theta^* = \tau^*$.*

Proof. Let $\{\mathscr{U}_n : n \in \mathbb{N}\}$ and $\{\mathscr{V}_n : n \in \mathbb{N}\}$ be Čech-complete sieves for X and Y, respectively (see [7, Theorem 3.9.2]). Let $\mathbb{T}^*_C$ be the collection of $\mathscr{T}$ in $\mathbb{T}^*$ that satisfy:

(1) for each $n \in \mathbb{N}$, there exist $U \in \mathscr{T}$ and $U_n \in \mathscr{U}_n$ such that $\overline{U} \subseteq U_n$; and

(2) for each $n \in \mathbb{N}$, there exist $V \in \tau(\mathscr{T})$ and $V_n \in \mathscr{V}_n$ such that $\overline{V} \subseteq V_n$.

Because of the Čech-completeness of X and Y, for each $\mathscr{T} \in \mathbb{T}^*_C$, $\bigcap \overline{\mathscr{T}} = \{x_{\mathscr{T}}\}$ for some $x_{\mathscr{T}} \in X$ and $\bigcap \overline{\tau(\mathscr{T})} = \{y_{\mathscr{T}}\}$ for some $y_{\mathscr{T}} \in Y$. Let $X' = \{x_{\mathscr{T}} : \mathscr{T} \in \mathbb{T}^*_C\}$ and $Y' = \{y_{\mathscr{T}} : \mathscr{T} \in \mathbb{T}^*_C\}$. Define $\theta \colon X' \to Y'$ by letting $\theta(x_{\mathscr{T}}) = y_{\mathscr{T}}$ for each $\mathscr{T} \in \mathbb{T}^*_C$.

To show that $\theta \colon X' \to Y'$ is a well-defined function, let $\mathscr{T}_1, \mathscr{T}_2 \in \mathbb{T}^*_C$ with $x_{\mathscr{T}_1} = x_{\mathscr{T}_2}$. Suppose, by way of contradiction, that $y_{\mathscr{T}_1} \neq y_{\mathscr{T}_2}$. Then let $V_1, V_2 \in \mathscr{T}_Y$ be such that $\overline{V_1} \cap \overline{V_2} = \emptyset$, $y_{\mathscr{T}_1} \in V_1$, and $y_{\mathscr{T}_2} \in V_2$. Let $U_1 = \tau^{-1}(V_1)$ and $U_2 = \tau^{-1}(V_2)$. Because of the maximality of $\tau(\mathscr{T}_1)$ and $\tau(\mathscr{T}_2)$, we have $V_1 \in \tau(\mathscr{T}_1)$ and $V_2 \in \tau(\mathscr{T}_2)$. That means $U_1 \in \mathscr{T}_1$ and $U_2 \in \mathscr{T}_2$, so that $x_{\mathscr{T}_1} \in \overline{U_1} \cap \overline{U_2}$. But then by definition of τ, $\overline{V_1} \cap \overline{V_2} \neq \emptyset$, which is a contradiction. Therefore, $y_{\mathscr{T}_1} = y_{\mathscr{T}_2}$, showing that θ is a well-defined function. A similar argument shows that θ^{-1} is a well-defined function, and thus θ is a bijection. It follows from the definition of e_τ that e_τ is an extension of θ. Therefore, θ is a homeomorphism from X' onto Y'.

To show that $\theta \in LDH(X, Y)$, let $f \in C(X)$. Define $\hat{\theta}(f) \in C(Y)$ by $\hat{\theta}(f) = e_X(f)e_\tau^{-1}|_Y$. To see that $\hat{\theta}(f)$ is an extension of $f\theta^{-1}$, let $y \in Y'$. Then $e_\tau^{-1}(y) = \theta^{-1}(y) \in X' \subseteq X$, so that $e_X(f)(e_\tau^{-1}(y)) = f(\theta^{-1}(y))$. We can make a similar argument by starting with a $g \in C(Y)$ and getting extension $\hat{\theta}^{-1}(g)$ of $g\theta$ where $\hat{\theta}^{-1}(g) = e_Y(g)e_\tau|_X$. Therefore, $\theta \in LDH(X, Y)$. Note that for each $f \in C(X)$, $\hat{\tau}(f) = e_Y^{-1}\hat{e}_\tau e_X(f) = e_Y^{-1}e_X(f)e_\tau^{-1} = e_X(f)e_\tau^{-1}|_Y = \hat{\theta}(f)$. This shows that $\hat{\theta} = \hat{\tau}$, and hence $\theta^* = \tau^*$. □

Now the set $H(X, Y)$ of homeomorphisms from X onto Y can also be related to $LROC(X, Y)$ if we make some additional hypotheses about X and Y. In particular, the next proposition follows from Proposition 2.15 by using basically the same techniques as in the proofs of [22, Lemmas 3.4, 3.5]. The term E_0-space means that singleton sets are G_δ-sets.

Proposition 2.16. *Let X and Y be Čech-complete binormal spaces that are either both realcompact or both E_0-spaces, and let $\tau \in LROC(X, Y)$. Then there exists an $h \in H(X, Y)$ such that e_τ is an extension of h and $h^* = \tau^*$.*

This is a good place to observe that if $h \in H(X, Y)$, then there is an associated $\tau \in LROC(X, Y)$ defined by $\tau(U) = h(U)$ for all $U \in \mathscr{T}_X$. In this case, $\tau X = X$, $\tau Y = Y$, and $e_\tau = h$. If $\hat{h} \colon C(X) \to C(Y)$ is defined by $\hat{h}(f) = fh^{-1}$ for all $f \in C(X)$, then $\hat{h} = \hat{\tau}$, and $h^* = \tau^*$ is an ordered homeomorphism from $L^-(X)$ onto $L^-(Y)$.

Example 2.17. Let X be the space of countable ordinals, and let Y be its (one point) compactification. Then X and Y are Čech-complete binormal spaces. Thinking of X as a subspace of Y, define $\tau\colon \mathscr{T}_X \to \mathscr{T}_Y$ as follows. For each $U \in \mathscr{T}_X$, let $\tau(U)$ be the interior of the closure of U in Y. Then X^* is a singleton set that can be identified with ω_1 in Y, and so $\tau X = Y = \tau Y$. Thus $e_\tau\colon \tau X \to \tau Y$ is the identity. It is well-known that every f in $C(X)$ extends to an element of $C(Y)$, so this gives homeomorphism $\hat{\tau}\colon C(X) \to C(Y)$. Therefore, $\tau \in LROC(X,Y)$, and τ induces the ordered homeomorphism $\tau^*\colon L^-(X) \to L^-(Y)$. Note that if θ is the identity map from X onto its image in Y, then $\theta \in LDH(X,Y)$ and e_τ is an extension of θ with $\theta^* = \tau^*$. Of course $H(X,Y) = \emptyset$, and as we see, X is not realcompact and Y is not an E_0-space.

Although we did not need to know that τX and τY are binormal spaces, it happens that they are. We now show this with a general proposition. In this proposition, when we say that a subspace X' of a space X is *liftable*, we mean that every element of $C(X')$ extends to an element of $C(X)$.

Proposition 2.18. *If X is a T_1-space that has a dense liftable binormal subspace, then X is binormal.*

Proof. We use Theorem I.3.3 to show that X is binormal. So let $f \in USC(X)$ and $g \in LSC(X)$ with $f < g$. For every $x \in X$, let $a(x) = (g(x) - 3f(x))/4$ and $b(x) = (3g(x) + f(x))/4$, and let $U(x)$ be a neighborhood of x such that $f(U(x)) \subseteq (-\infty, a(x))$ and $g(U(x)) \subseteq (b(x), \infty)$. Also let A be the closure in $X \times \mathbb{R}$ of $\bigcup\{U(x) \times (-\infty, f(x)]:\ x \in X\}$, and let B be the closure in $X \times \mathbb{R}$ of $\bigcup\{U(x) \times [g(x), \infty):\ x \in X\}$.

Observe that for each $x \in X$, if $x' \in U(x)$, then $f(x') < a(x)$. Therefore, for each $x \in X$, $\sup A(x) \leq a(x) < g(x)$. By the same argument, for each $x \in X$, $\inf B(x) \geq b(x) > f(x)$. Now define $f_0\colon X \to \mathbb{R}$ by $f_0(x) = \sup A(x)$ for all $x \in X$, and define $g_0\colon X \to \mathbb{R}$ by $g_0(x) = \inf B(x)$ for all $x \in X$. Then f_0 and g_0 are finite-valued functions in $USC(X)$ and $LSC(X)$, respectively. It is clear from the definitions of A and B that $f \leq f_0$ and $g_0 \leq g$. As we saw before, for each $x \in X$, $f_0(x) \leq a(x) < b(x) \leq G_0(x)$. Therefore, $f_0 < g_0$.

Let X' be a dense liftable binormal subspace of X, and let $f_0' = f_0|_{X'}$ and $g_0' = g_0|_{X'}$. Then $f_0' \in USC(X)$, $g_0' \in LSC(X)$, and $f_0' < g_0'$. From the binormality of X', there exist $h', h_1', h_2' \in C(X')$ such that $f_0' < h_1' < h' < h_2' < g_0'$. Let h, h_1 and h_2 be the extensions of h', h_1' and h_2' in $C(X)$. Since X' is dense in X, the continuity of h, h_1 and h_2 ensures that $h_1 \leq h \leq h_2$.

To show that $h_1 < h$, suppose not. Then there exists an $x \in X \setminus X'$ such that $h_1(x) = h(x)$. Define $k' \in C(X')$ by $k'(x') = 1/(h'(x') - h_1'(x'))$ for each $x' \in X'$. We know k' has an extension k in $C(X)$. Let $n \in \mathbb{N}$. By the continuity of h and h_1, there exists a neighborhood U of x in X such that $h(x') - h_1(x') < 1/n$ for all $x' \in U$. Then $k(x') > n$ for all $x' \in U \cap X'$. Since X' is dense in X and

k is continuous, we have $k(x) \geq n$. But this is true for all $n \in \mathbb{N}$, which is a contradiction. Therefore, $h_1 < h$. A similar argument shows that $h < h_2$.

Finally, to show that $f \leq h_1$, let $x \in X$. For each $x' \in U(x) \cap X'$, we have $f(x) \leq f_0'(x') < h_1'(x')$. Since X' is dense in X and h_1 is continuous, it follows that $f(x) \leq h_1(x)$. By a similar argument, $h_2 \leq g$. So we have $f < h < g$, showing that X is binormal. $\square$

Corollary 2.19. *If X and Y are binormal spces and $\tau \in LROC(X, Y)$, then τX and τY are binormal.*

Corollary 2.19 gives us another way of expressing an ordered homeomorphism $\tau^*\colon L^-(X) \to L^-(Y)$ induced from a $\tau \in LROC(X, Y)$. If X' is a binormal extension of X, we will call X' a *lifting* binormal extension provided that every element of $C(X)$ extends to an element of $C(X')$. We now define a *lifting binormal extension homeomorphism from X to Y* to be a homeomorphism from a lifting binormal extension of X onto a lifting binormal extension of Y. Then if $\tau \in LROC(X, Y)$, we know that e_τ is a lifting binormal extension homeomorphism from X to Y.

If $\eta\colon X' \to Y'$ is a lifting binormal extension homeomorphism from X to Y, define $\hat{\eta}\colon C(X) \to C(Y)$ by $\hat{\eta}(f) = \hat{f}\eta^{-1}|_Y$ for all $f \in C(X)$ where $\hat{f}$ is the extension of f in $C(X')$. Then $\hat{\eta}$ is a bimonotone homeomorphism, and by the Extension Theorem I.5.1, $\hat{\eta}$ induces an ordered homeomorphism $\eta^*\colon L^-(X) \to L^-(Y)$. If $\eta = e_\tau$ for some $\tau \in LROC(X, Y)$, then $\eta^* = \tau^*$.

3. Factorization Theorem

In the previous section we discovered one kind of ordered homeomorphism from $L^-(X)$ onto $L^-(Y)$, obtained as the induced map τ^* from a $\tau \in LROC(X, Y)$. As a special case, if h is a homeomorphism from X onto Y, we can associate h with a $\tau \in LROC(X, Y)$ defined by $\tau(U) = h(U)$ for all $U \in \mathscr{T}_X$, and h^* is an ordered homeomorphism from $L^-(X)$ onto $L^-(Y)$. In this case, the restriction of h^* to $C(X)$ is a bimonotone homeomorphism $\hat{h}\colon C(X) \to C(Y)$ where $\hat{h}(f) = fh^{-1}$ for all $f \in C(X)$.

We now look at a second kind of ordered homeomorphism, this one mapping $L^-(X)$ onto itself (or mapping $L^-(Y)$ onto itself). We define a *fiber homeomorphism on X* to be a homeomorphism $\phi\colon X \times \mathbb{R} \to X \times \mathbb{R}$ such that ϕ maps $\{x\} \times \mathbb{R}$ onto itself for each $x \in X$. Let $FH(X)$ denote the set of fiber homeomorphisms on X. If $\phi \in FH(X)$, then for every $f \in C(X)$, let $\hat{\phi}(f)$ be equal to $\phi(f)$ where f is identified with its graph in $X \times \mathbb{R}$. This defines a function $\hat{\phi}\colon C(X) \to C(X)$.

Lemma 3.1. *For each $\phi \in FH(X)$, the function $\hat{\phi}\colon C(X) \to C(X)$ is a bimonotone homeomorphism.*

Proof. Since ϕ is a bijection of $X \times \mathbb{R}$ onto itself, it is clear that $\hat{\phi}$ is a bijection with inverse $\hat{\phi}^{-1}$ defined by $\hat{\phi}^{-1}(f) = \phi^{-1}(f)$ for each $f \in C(X)$. Let us check that $\hat{\phi}$ is bimonotone. The argument that $\hat{\phi}^{-1}$ is bimonotone is similar. Let $f_1, f_2 \in C(X)$ with $f_1 \leq f_2$, and let $f \in C(X)$. Suppose first that $f_1 \leq f \leq f_2$, and let $x \in X$. Since ϕ restricted to $\{x\} \times \mathbb{R}$ is a homeomorphism from $\{x\} \times \mathbb{R}$ onto itself, $\hat{\phi}(f)(x)$ lies between $\hat{\phi}(f_1)(x)$ and $\hat{\phi}(f_2)(x)$. Therefore, $\min\{\hat{\phi}(f_1)(x), \hat{\phi}(f_2)(x)\} \leq \hat{\phi}(f)(x) \leq \max\{\hat{\phi}(f_1)(x), \hat{\phi}(f_2)(x)\}$. This is true for all $x \in X$, so that $\min\{\hat{\phi}(f_1), \hat{\phi}(f_2)\} \leq \hat{\phi}(f) \leq \max\{\hat{\phi}(f_1), \hat{\phi}(f_2)\}$. Now suppose that $\min\{\hat{\phi}(f_1), \hat{\phi}(f_2)\} \leq \hat{\phi}(f) \leq \max\{\hat{\phi}(f_1), \hat{\phi}(f_2)\}$, and let $x \in X$. We assume that $\hat{\phi}(f_1)(x) \leq \hat{\phi}(f_2)(x)$ since the other case is similar. Then $\hat{\phi}(f_1)(x) \leq \hat{\phi}(f)(x) \leq \hat{\phi}(f_2)(x)$. So $\hat{\phi}^{-1}(\hat{\phi}(f))(x)$ lies between $\hat{\phi}^{-1}(\hat{\phi}(f_1))(x)$ and $\hat{\phi}^{-1}(\hat{\phi}(f_2))(x)$, and hence $f(x)$ lies between $f_1(x)$ and $f_2(x)$. Since $f_1(x) \leq f_2(x)$, we have $f_1(x) \leq f(x) \leq f_2(x)$. This is true for all $x \in X$, so that $f_1 \leq f \leq f_2$. Therefore, $\hat{\phi}$ is bimonotone.

Finally, we show that $\hat{\phi}$ is continuous. The argument that $\hat{\phi}^{-1}$ is continuous is similar. Let $f \in C(X)$, and let W be an open subset of $X \times \mathbb{R}$ with $\hat{\phi}(f) \in W^+$. Since X is binormal, there exist $f_1, f_2 \in C(X)$ with $f_1 < \hat{\phi}(f) < f_2$ and $f_1, f_2 \subseteq W$. Since $\hat{\phi}^{-1}$ is bimonotone, it follows that $\min\{\hat{\phi}^{-1}(f_1), \hat{\phi}^{-1}(f_2)\} < f < \max\{\hat{\phi}^{-1}(f_1), \hat{\phi}^{-1}(f_2)\}$. The strict inequality is because ϕ is a bijection. Let $W_0 = \{\langle x, t\rangle \in X \times \mathbb{R} : \min\{\hat{\phi}^{-1}(f_1)(x), \hat{\phi}^{-1}(f_2)(x)\} < t < \max\{\hat{\phi}^{-1}(f_1)(x), \hat{\phi}^{-1}(f_2)(x)\}\}$. Then W_0^+ is a neighborhood of f in $C(X)$. If $f' \in C(X)$ with $f' \in W_0^+$, then $\min\{\hat{\phi}^{-1}(f_1), \hat{\phi}^{-1}(f_2)\} < f' < \max\{\hat{\phi}^{-1}(f_1), \hat{\phi}^{-1}(f_2)\}$. Again, since $\hat{\phi}^{-1}$ is bimonotone, we have $f_1 < \hat{\phi}(f') < f_2$, with the inequality strict because ϕ is a bijection. Therefore, $f \in W_0^+$ and $\hat{\phi}(W_0^+) \subseteq W^+$, showing that $\hat{\phi}$ is continuous. □

Now the Extension Theorem I.5.1 says that $\hat{\phi}$ induces an ordered homeomorphism $\phi^*\colon L^-(X) \to L^-(X)$ that is an extension of $\hat{\phi}$, where ϕ^* is defined by $\phi^*(F) = \bigcup\{\hat{\phi}(f) : f \in C(X) \text{ and } f \subseteq F\}$ for all $F \in L^-(X)$. Since at this stage, we only have the Extension Theorem I.5.1 proved for $L^-(X)$ and $L^-(Y)$ with the upper Vietoris topology, we need to give an independent proof of the continuity of τ^* with respect to the lower Vietoris topology.

Proposition 3.2. *For a binormal space X and for each $\phi \in FH(X)$, the induced map $\phi^*\colon L^-(X) \to L^-(X)$ is an ordered homeomorphism.*

Proof. The argument for the continuity of $(\phi^*)^{-1} = (\phi^{-1})^*$ is the same as that for the continuity of ϕ^*. We already know that ϕ^* is an ordered homeomorphism with respect to the upper Vietoris topology, so it suffices to consider a subbasic open set W^- of $L^-(X)$ in the lower Vietoris topology. Let $F \in L^-(X)$ with $\phi^*(F) \in W^-$. We may assume that $W = U \times (a,b)$ where U is an open set in X and (a,b) is an open interval. Then there exists a $\langle x,t\rangle \in \phi^*(F) \cap U \times (a,b)$. So there is an $f \in C(X)$ with $f \subseteq F$ and $\hat{\phi}(f)(x) = \phi^*(f)(x) = t$. Let $p,q \in \mathbb{R}$ with $a < p < t < q < b$. Let $f_1, f_2 \in C(X)$ be defined by $f_1(x') = f(x') - t + p$ for all $x' \in X$ and $f_2(x') = f(x') - t + q$ for all $x' \in X$. Let U_0 be a neighborhood of x contained in U such that $f_1(U_0) \subseteq (a,\infty)$ and $f_2(U_0) \subseteq (-\infty, b)$.

Define W_0 to be the set of $\langle x',s\rangle$ in $U_0 \times \mathbb{R}$ such that $\min\{\hat{\phi}^{-1}(f_1)(x'), \hat{\phi}^{-1}(f_2)(x')\} < s < \max\{\hat{\phi}^{-1}(f_1)(x'), \hat{\phi}^{-1}(f_2)(x')\}$. Then W_0 is an open subset of $X \times \mathbb{R}$ with $f \in W_0^-$, and hence $F \in W_0^-$. Let $F' \in W_0^-$, so there exists a $\langle x',s\rangle \in F \cap W_0$. Then there is an $f' \in C(X)$ with $f' \subseteq F'$ such that $f'(x') = s$. Then $a < f_1(x') < \hat{\phi}(f')(x') < f_2(x') < b$, so that $\phi^*(f')(x') = \hat{\phi}(f')(x') \in (a,b)$. That means $\langle x', \phi^*(f')(x')\rangle \in \phi^*(F') \cap U \times (a,b)$, so that $\phi^*(F') \in W^-$. Thus $F \in W_0^-$ and $\phi^*(W_0^-) \subseteq W^-$, showing that ϕ^* is continuous with respect to the lower Vietoris topology. □

Our two kinds of ordered homeomorphisms are τ^* and ϕ^* for $\tau \in LROC(X,Y)$ and $\phi \in FH(X)$. In this section we prove the Factorization Theorem that says every ordered homeomorphism from $L^-(X)$ onto $L^-(Y)$ can be uniquely factored as the composition of these two kinds of ordered homeomorphisms.

Factorization Theorem 3.3. *Let X and Y be binormal spaces, and let $M\colon L^-(X) \to L^-(Y)$ be an ordered homeomorphism. Then there exist $\tau \in LROC(X,Y)$, $\phi \in FH(X)$, and $\psi \in FH(Y)$ such that M can be uniquely factored as $M = \tau^*\phi^*$ and can be uniquely factored as $M = \psi^*\tau^*$. If, in addition, X and Y are both Čech-complete, then $\tau^* = \theta^*$ for some $\theta \in LDH(X,Y)$. Furthermore, if X and Y are either both realcompact or both E_0-spaces, then $\tau^* = h^*$ for some $h \in H(X,Y)$; in particular, X and Y are homeomorphic.*

By combining the Extension Theorem I.5.1 and the Factorization Theorem 3.3, we can now characterize the bimonotone homeomorphisms between $C(X)$ and $C(Y)$.

Corollary 3.4. *Let X and Y be binormal spaces, and let $\mu\colon C(X) \to C(Y)$ be a bimonotone homeomorphism. Then there exist $\tau \in LROC(X,Y)$, $\phi \in FH(X)$, and $\psi \in FH(Y)$ such that μ can be uniquely factored as $\mu = \tau^*\phi^*$ and can be uniquely factored as $\mu = \psi^*\tau^*$. If, in addition, X and Y are both Čech-complete, then $\tau^* = \theta^*$ for some $\theta \in LDH(X,Y)$. Furthermore, if X and Y are either both realcompact or both E_0-spaces, then $\tau^* = h^*$ for some $h \in H(X,Y)$; in particular, X and Y are homeomorphic.*

Note that the τ^*, ϕ^*, ψ^*, θ^*, and h^* in Corollary 3.4 are the restrictions of these maps to $C(X)$ or $C(Y)$, and as such may have simpler ways of expressing their definitions in terms of elements of $C(X)$ or $C(Y)$.

The next corollary is also a consequence of both the Extension Theorem I.5.1 and the Factorization Theorem 3.3.

Corollary 3.5. *If X and Y are binormal spaces, then the following are equivalent.*

(1) *There exists an increasing homeomorphism from $C(X)$ onto $C(Y)$.*

(2) *There exists a bimonotone homeomorphism from $C(X)$ onto $C(Y)$.*

(3) *There exists an ordered homeomorphism from $L^-(X)$ onto $L^-(Y)$.*

(4) *There exists a lifting regular open correspondence from X to Y.*

(5) *There exists a lifting binormal extension homeomorphism from X to Y.*

As we go through the proof of the Factorization Theorem 3.3 showing that an ordered homeomorphism $M\colon L^-(X) \to L^-(Y)$ can be uniquely factored, we will see that the only places that the continuity of M and M^{-1} are used is when we need the restriction $M_*\colon L^-(X) \to L^-(Y)$ of M to be continuous or have continuous inverse. So only the upper Vietoris topology will be needed. As a result, the Factorization Theorem 3.3 is true when the topology on $L^-(X)$ and $L^-(Y)$ is only the upper Vietoris topology. Of course it is also true for the Vietoris topology as well. Now because of this, Propositions 2.13 and 3.2, that were proved independently using the lower Vietoris topology, imply the following fact.

Proposition 3.6. *If X and Y are binormal spaces and if $M\colon L^-(X) \to L^-(Y)$ is an ordered homeomorphism under the upper Vietoris topology on $L^-(X)$ and $L^-(Y)$, then M is also an ordered homeomorphism under the Vietoris topology on $L^-(X)$ and $L^-(Y)$.*

Proposition 3.6 now completes the proof of the Extension Theorem I.5.1 for the full Vietoris topology.

For the rest of this section we prove the Factorization Theorem 3.3 by breaking the argument into a number of lemmas. So for the following lemmas, let X and Y be binormal spaces and let $M\colon L^-(X) \to L^-(Y)$ be a given ordered homeomorphism.

Lemma 3.7. *Let $F_1, F_2 \in L^-(X)$ be such that $F_1(x) \cap F_2(x) \neq \emptyset$ for all $x \in X$. Then the following are true.*

(1) *$F_1 \cap F_2$ and $(F_1 \cup F_2)_{\max}$ are in $L^-(X)$, and $M(F_1) \cap M(F_2)$ and $(M(F_1) \cup M(F_2))_{\max}$ are in $L^-(Y)$.*

(2) *$M(F_1 \cap F_2) = M(F_1) \cap M(F_2)$.*

(3) *$M((F_1 \cup F_2)_{\max}) = (M(F_1) \cup M(F_2))_{\max}$.*

P r o o f. The parts involving intersection were proved in Lemma I.5.2. It is immediate that $F_1 \cup F_2$ is a locally bounded member of $\mathscr{L}(X)$, so that $(F_1 \cup F_2)_{\max} \in L^-(X)$. It is also immediate that $M(F_1) \cup M(F_2)$ is a locally bounded member of $\mathscr{L}(Y)$, and hence $(M(F_1) \cup M(F_2))_{\max} \in L^-(Y)$. Now $M(F_1) \subset M((F_1 \cup F_2)_{\max})$ and $M(F_2) \subseteq M((F_1 \cup F_2)_{\max})$, so that $(M(F_1) \cup M(F_2))_{\max} \subseteq M((F_1 \cup F_2)_{\max})$. Using this argument, but with M^{-1}, we have $(F_1 \cup F_2)_{\max} = (M^{-1}(M(F_1)) \cup M^{-1}(M(F_2)))_{\max} \subseteq M^{-1}((M(F_1) \cup M(F_2))_{\max})$. Therefore, $M((F_1 \cup F_2)_{\max}) \subseteq (M(F_1) \cup M(F_2))_{\max}$, which gives the equality. □

For $f_0, f_1 \in C(X)$, we use the notation $f_0 \wedge f_1$ for $\min\{f_0, f_1\}$ and the notation $f_0 \vee f_1$ for $\max\{f_0, f_1\}$. Now for each $f_0, f_1 \in C(X)$ with $f_0 \le f_1$, define

$$F(f_0, f_1) = \{\langle x, t\rangle \in X \times \mathbb{R} : \ f_0(x) \le t \le f_1(x)\},$$

which is an element of $L^-(X)$. Also for each $g_0, g_1 \in C(Y)$ with $g_0 \le g_1$, define

$$G(g_0, g_1) = \{\langle y, t\rangle \in Y \times \mathbb{R} : \ g_0(y) \le t \le g_1(y)\},$$

which is an element of $L^-(Y)$. Lemmas I.5.3 and I.5.5 imply the following lemma.

Lemma 3.8. *Let $f_0, f_1 \in C(X)$, and let $g_0 = M(f_0)$ and $g_1 = M(f_1)$. Then the following are true.*

(1) *If $f_0 \le f_1$, then $M(F(f_0, f_1)) = G(g_0 \wedge g_1, g_0 \vee g_1)$.*
(2) *If $g_0 \le g_1$, then $M(F(f_0 \wedge f_1, f_0 \vee f_1)) = G(g_0, g_1)$.*
(3) *If $f_0 < f_1$, then $g_0 \wedge g_1 < g_0 \vee g_1$.*
(4) *If $g_0 < g_1$, then $f_0 \wedge f_1 < f_0 \vee f_1$.*

For each $f_0, f_1 \in C(X)$ with $f_0 \le f_1$, for each open subset U of X, and for each $i \in \{0, 1\}$ let $F_i(f_0, f_1, U) = F(f_0, f_1) \cap U \times \mathbb{R} \cup f_i$. Also for each $g_0, g_2 \in C(Y)$, for each open subset V of Y, and for each $i \in \{0, 1\}$, let $G_i(g_0, g_1, V) = G(g_0, g_1) \cap V \times \mathbb{R} \cup g_i$. The following lemma shows when $F_i(f_0, f_1, U)$ and $G_i(g_0, g_1, V)$ are in $L^-(X)$ and $L^-(Y)$.

Lemma 3.9. *Let $f_0, f_1 \in C(X)$ with $f_0 \le f_1$, let U be an open subset of X, and let $i \in \{0, 1\}$. If $U \in \mathscr{T}_X$, then $F_i(f_0, f_1, U) \in L^-(X)$. Conversely, if $f_0 < f_1$ and $F_i(f_0, f_1, U) \in L^-(X)$, then $U \in \mathscr{T}_X$.*

P r o o f. We only prove this for $i = 0$ since the other case is similar. Let $\langle x, t\rangle$ be an almost lsc point of $F_0(f_0, f_1, U)$. If $\langle x, t\rangle \in f_0$, then $\langle x, t\rangle \in F_0(f_0, f_1, U)$; so we assume that $\langle x, t\rangle \notin f_0$, and hence that $f_0(x) < t \le f_1(x)$. Since U is regular open, it suffices to show that $x \in \operatorname{int}(\overline{U})$. Let $m = (t + f_0(x))/2$. By the continuity of f_0, there exists a neighborhood U_0 of x such that $f_0(U_0) \subseteq (-\infty, m)$. By the almost lsc property of $\langle x, t\rangle$, there exists a neighborhood U_1 of x contained in U_0 such that every nonempty open subset of U_1 contains a point x'

with $F_0(f_0, f_1, U)(x') \cap (m, \infty) \neq \emptyset$. To see that $U_1 \subseteq \overline{U}$, let $x_1 \in U_1$, and let U_2 be a neighborhood of x_1 contained in U_1. Then there exists an $x' \in U_2$ such that $F_0(f_0, f_1, U)(x') \cap (m, \infty)$ contains a point; say s. Then $f_0(x') < m < s$, and thus $x' \in U$. This shows that $x \in U_1 \subseteq \overline{U}$, and hence that $x \in \operatorname{int}(\overline{U}) = U$. Therefore, $\langle x, t\rangle \in F_0(f_0, f_1, U)$, which shows that $F_0(f_0, f_1, U) = F_0(f_0, f_1, U)_{\max}$, and it follows that $F_0(f_0, f_1, U) \in L^-(X)$.

Conversely, let $f_0 < f_1$ and $F_0(f_0, f_1, U) \in L^-(X)$. To show that U is regular open, let $x \in \operatorname{int}(\overline{U})$. We will show that $\langle x, f_1(x)\rangle$ is an almost lsc point of $F_0(f_0, f_1, U)$. Let O be a neighborhood of $f_1(x)$. By the continuity of f_1, there exists a neighborhood U_0 of x contained in $\operatorname{int}(\overline{U})$ such that $f_1(U_0) \subseteq O$. Let U_1 be any nonempty open subset of U_0. Since $U_1 \subseteq \overline{U}$, there exists an $x_1 \in U_1 \cap U$. But then $f_1(x_1) \in F_0(f_0, f_1, U)(x_1) \cap O$. This shows that $\langle x, f_1(x)\rangle$ is an almost lsc point of $F_0(f_0, f_1, U)$. Now $F_0(f_0, f_1, U) = F_0(f_0, f_1, U)_{\max}$, so that $\langle x, f_1(x)\rangle \in F_0(f_0, f_1, U)$. Since $f_0(x) < f_1(x)$, we have $x \in U$, showing that U is regular open. $\square$

Lemma 3.9 has its obvious analog in the space $L^-(Y)$ involving the $G_i(g_0, g_1, V)$. Now for each $F \in L^-(X)$ and $G \in L^-(Y)$, define $X_F = \{x \in X : |F(x)| > 1\}$ and $Y_G = \{y \in Y : |G(y)| > 1\}$. For example, if $F = F_i(f_0, f_1, U)$ given above where $U \in \mathscr{T}_X$ and $f_0 < f_1$, then $X_F = U$.

Lemma 3.10. *For each $F \in L^-(X)$ and $G \in L^-(Y)$, X_F is open in X and Y_G is open in Y.*

Proof. Let $x \in X_F$. Then $F(x) = [a, b]$ where $a < b$. Let $m = (a + b)/2$. Since F is lsc, x has a neighborhood U in X such that for every $x' \in U$, $F(x') \cap (-\infty, m) \neq \emptyset$ and $F(x') \cap (m, \infty) \neq \emptyset$. Then for every $x' \in U$, $|F(x')| > 1$, so that $x \in U \subseteq X_F$. Therefore, X_F is open in X. A similar proof shows that Y_G is open in Y. $\square$

Lemma 3.11. *Let $f_0, f_1 \in C(X)$ with $f_0 < f_1$, let $g_0 = M(f_0)$, let $g_1 = M(f_1)$, and let $i \in \{0, 1\}$. Then the following are true.*

1. *If $U \in \mathscr{T}_X$ and $F_1, F_2 \in L^-(X)$ are such that $F_1 \cap F_2 = f_i$, $(F_1 \cup F_2)_{\max} = F_i(f_0, f_1, U)$ and $F_1 \neq f_i$, then there exists a $U_1 \in \mathscr{T}_X$ such that $F_1 = F_i(f_0, f_1, U_1)$.*
2. *If $V \in \mathscr{T}_Y$ and $G_1, G_2 \in L^-(Y)$ are such that $G_1 \cap G_2 = g_i$, $(G_1 \cup G_2)_{\max} = G_i(g_0 \wedge g_1, g_0 \vee g_1, V)$ and $G_1 \neq g_i$, then there exists a $V_1 \in \mathscr{T}_Y$ such that $G_1 = G_i(g_0 \wedge g_1, g_0 \vee g_1, V_1)$.*

Proof. We only prove (2) since the proof of (1) is similar. Define $Y_1 = \{y \in Y : g_0(y) < g_1(y)\}$ and $Y_2 = \{y \in Y : g_0(y) > g_1(y)\}$. Clearly, Y_1 and Y_2 are disjoint sets that are open in Y because of the continuity of g_0 and g_1; but also $Y_1 \cup Y_2 = Y$ by Lemma 3.8.

Let $G = G_i(g_0 \wedge g_1, g_0 \vee g_1, V)$. We show that $V_1 = Y_{G_1} \in \mathscr{T}_Y$. Suppose, by way of contradiction, that there exists a $y_0 \in Y_{G_1}$ such that $\langle y_0, g_{1-i}(y_0)\rangle \notin G_1$. Since $G_1 \subseteq G$, we know that $y_0 \in V$. We may assume that $y_0 \in Y_1$ since the other case that $y_0 \in Y_2$ is similar. Because $Y_1 \cap Y_{G_1}$ is a neighborhood of y_0 by Lemma 3.10, the continuity of g_i ensures that there exists a neighborhood V_0 of y_0 contained in $Y_1 \cap Y_{G_1}$ and a neighborhood O_0 of $g_{1-i}(y_0)$ such that $g_i \cap V_0 \times O_0 = \emptyset$.

Now $G_1 = (G_1)_{\max}$, so that $\langle y_0, g_{1-i}(y_0)\rangle \notin (G_1)_{\max}$. Then $\langle y_0, g_{1-i}(y_0)\rangle$ is not an almost lsc point of G_1, so there exists a neighborhood O of $g_{1-i}(y_0)$ contained in O_0 such that for every neighborhood V' of y_0 there exists a nonempty open subset V'' of V' with $G_1(y) \cap O = \emptyset$ for all $y \in V''$.

Since $y_0 \in V$, we have $\langle y_0, g_{1-i}(y_0)\rangle \in G = (G_1 \cup G_2)_{\max}$, so that there exists a neighborhood $V'$of y_0 contained in V_0 such that for every nonempty open subset V'' of V', $(G_1 \cap G_2)(y) \cup O \neq \emptyset$ for some $y \in V''$. From the previous paragraph, there exists a nonempty open subset V'' of V' such that $G_1(y) \cap O = \emptyset$ for all $y \in V''$. Let $y \in V''$ with $(G_1 \cap G_2)(y) \cap O \neq \emptyset$.

Now $G_1 \cap G_2 = g_i$ and $g_i \cap V'' \times O = \emptyset$, so that there exists a $t \in G_2(y) \cap O$. Then $t \neq g_i(y)$, and since $y \in Y_{G_1}$, there exists an $s \in G_1(y)$ such that s is between $g_i(y)$ and t. Since $G_1 \cap G_2 = g_i$, we have $s \notin G_2(y)$. But $g_i(y), t \in G_2(y)$, which contradicts the connectedness of $G_2(y)$.

Therefore, for every $y \in Y_{G_1}$, $\langle y, g_{1-i}(y)\rangle \in G_1$. Now define $V_1 = Y_{G_1}$. Since $g_0 \wedge g_1 < g_0 \vee g_1$, we have $G_1 = G_i(g_0 \wedge g_1, g_0 \vee g_1, V_1)$. Then by Lemma 3.9, it follows that $V_1 \in \mathscr{T}_Y$. □

Lemma 3.12. *Let $f_0, f_1 \in C(X)$, let $g_0 = M(f_0)$, let $g_1 = M(f_1)$, and let $i \in \{0, 1\}$. Then the following are true.*

(1) *If $f_0 < f_1$, then for each $U \in \mathscr{T}_X$, there exists a $V_i \in \mathscr{T}_Y$ such that $M(F_i(f_0, f_1, U)) = G_i(g_0 \wedge g_1, g_0 \vee g_1, V_i)$.*
(2) *If $f_0 < f_1$, then for each $V \in \mathscr{T}_Y$, there exists a $U_i \in \mathscr{T}_X$ such that $M(F_i(f_0, f_1, U_i)) = G_i(g_0 \wedge g_1, g_0 \vee g_1, V)$.*
(3) *If $g_0 < g_1$, then for each $U \in \mathscr{T}_X$, there exists a $V_i \in \mathscr{T}_Y$ such that $M(F_i(f_0 \wedge f_1, f_0 \vee f_1, U)) = G_i(g_0, g_1, V_i)$.*
(4) *If $g_0 < g_1$, then for each $V \in \mathscr{T}_Y$, there exists a $U_i \in \mathscr{T}_X$ such that $M(F_i(f_0 \wedge f_1, f_0 \vee f_1, U_i)) = G_i(g_0, g_1, V)$.*

Proof. To prove statement (1), let $U \in \mathscr{T}_X$, let $F_1 = F_i(f_0, f_1, U)$, and let $G_1 = M(F_1)$. Since $F_1 \subseteq F(f_0, f_1)$, we have $G_1 \subseteq G(g_0 \wedge g_1, g_0 \vee g_1)$ by Lemma 3.8. Define $V = Y_{G_1}$ and $G' = G_i(g_0 \wedge g_1, g_0 \vee g_1, V)$, and let $F' = M^{-1}(G')$. Then $G_1 \subseteq G'$, so that $F_1 \subseteq F'$. Define $F_2 = F' \cap (X \setminus \overline{U}) \times \mathbb{R} \cup f_i$, and let $G_2 = M(F_2)$. Observe that $F_1 \cap F_2 = f_i$, so that by Lemma 3.7, $G_1 \cap G_2 = M(F_1) \cap M(F_2) = M(F_1 \cap F_2) = M(f_i) = g_i$.

To show that $(F_1 \cup F_2)_{\max} = F'$, first note that $F_1 \cup F_2 \subseteq F'$, which implies that $(F_1 \cup F_2)_{\max} \subseteq F'$. Let $\langle x, t\rangle \in F'$. To show that $\langle x, t\rangle \in (F_1 \cup F_2)_{\max}$, we need to show that $\langle x, t\rangle$ is an almost lsc point of $F_1 \cup F_2$. Let O be a neighborhood of t. Since F' is lsc at $\langle x, t\rangle$, there exists a neighborhood U of x such that for all $x' \in U$, $F'(x') \cap O \neq \emptyset$. Let U' be a nonempty open subset of U. Now U' is not contained in the boundary of X_{F_1}, so either $U' \cap X_{F_1} \neq \emptyset$ or $U' \cap X_{F_2} \neq \emptyset$. If $x' \in U' \cap X_{F_1}$ or if $x' \in U' \cap X_{F_2}$, then in either case, $(F_1 \cup F_2)(x') \cap O = F'(x') \cap O \neq \emptyset$. This finishes the argument that $\langle x, t\rangle$ is an almost lsc point of $F_1 \cup F_2$. Therefore, $F' \subseteq (F_1 \cup F_2)_{\max}$, so that $(F_1 \cup F_2)_{\max} = F'$.

Now by Lemma 3.7, $(G_1 \cup G_2)_{\max} = (M(F_1) \cup M(F_2))_{\max} = M((F_1 \cup F_2)_{\max}) = M(F') = G'$. Then since $G' \neq g_i$, statement (2) of Lemma 3.11 says that $G_1 = G_i(g_0 \wedge g_1, g_0 \vee g_1, V_i)$ for some $V_i \in \mathscr{T}_Y$. This finishes the proof of statement (1) of this lemma. Statement (2) of this lemma can be proved in a similar way, except statement (1) of Lemma 3.11 must be used instead of statement (2). Statements (3) and (4) also have similar proofs. □

For each $t \in \mathbb{R}$, let $f^t = X \times \{t\}$ be the constant t function on X, and let $g^t = M(f^t)$. Lemma 3.12 allows us to define for each $U \in \mathscr{T}_X$ a $\tau_0(U) \in \mathscr{T}_Y$ such that $M(F_0(f^0, f^1, U)) = G_0(g^0 \wedge g^1, g^0 \vee g^1, \tau_0(U))$ and for each $V \in \mathscr{T}_Y$ a $\tau_0^{-1}(V) \in \mathscr{T}_X$ such that $M(F_0(f^0, f^1, \tau_0^{-1}(V))) = G_0(g^0 \wedge g^1, g^0 \vee g^1, V)$. Since M is a bijection, we see that $\tau_0^{-1}(\tau_0(U)) = U$ and $\tau_0(\tau_0^{-1}(V)) = V$. Therefore, this defines a bijection $\tau_0 \colon \mathscr{T}_X \to \mathscr{T}_Y$. Observe that the ordered property of M ensures that τ_0 is ordered in the sense that if $U_1, U_2 \in \mathscr{T}_X$ with $U_1 \subseteq U_2$ then $\tau_0(U_1) \subseteq \tau_0(U_2)$, and if $V_1, V_2 \in \mathscr{T}_Y$ with $V_1 \subseteq V_2$ then $\tau_0^{-1}(V_1) \subseteq \tau_0^{-1}(V_2)$. Also if $\mathscr{T}$ is a subfamily of $\mathscr{T}_X$ that has the finite intersection property, then $\{\tau_0(U) : U \in \mathscr{T}\}$ has the finite intersection property.

In a similar way Lemma 3.12 allows us to define an ordered bijection $\tau_1 \colon \mathscr{T}_X \to \mathscr{T}_Y$ such that for each $U \in \mathscr{T}_X$, $M(F_1(f^{-1}, f^0, U)) = G_1(g^{-1} \wedge g^0, g^{-1} \vee g^0, \tau_1(U))$ and for each $V \in \mathscr{T}_Y$, $M(F_1(f^{-1}, f^0, \tau_1^{-1}(V))) = G_1(g^{-1} \wedge g^0, g^{-1} \vee g^0, V)$. Now τ_1 has the same properties as those given above for τ_0.

Lemma 3.13. *Let $f_0, f_1 \in C(X)$, let $g_0 = M(f_0)$, let $g_1 = M(f_1)$, let $U \in \mathscr{T}_X$, and let $i \in \{0, 1\}$. Then the following are true.*

1. *If $f_0 \leq f_1$, then $M(F_i(f_0, f_1, U)) = G_i(g_0 \wedge g_1, g_0 \vee g_1, \tau_i(U))$.*
2. *If $g_0 \leq g_1$, then $M(F_i(f_0 \wedge f_1, f_0 \vee f_1, U)) = G_i(g_0, g_1, \tau_i(U))$.*

Proof. The proofs of these two statements are similar, so we only give the proof of (1). Also, for convenience, we only show this for $i = 0$ since the proof for $i = 1$ is similar. Let $F = F_0(f_0, f_1, U)$ and $G = M(F)$. We first prove this for the case that $f_0 < f_1$. Let $f_0' = f_0 \wedge f^0$ and $f_1' = f_1 \vee f^1$, and let $g_0' = M(f_0')$ and $g_1' = M(f_1')$. Note that $f_0' < f_1'$.

First let us show that $M(F_0(f_0', f_1', U)) = G_0(g_0' \wedge g_1', g_0' \vee g_1', \tau_0(U))$. By Lemma 3.12, there exists a $V \in \mathscr{T}_Y$ such that $M(F_0(f_0', f_1', U)) = G_0(g_0' \wedge g_1', g_0' \vee g_1', V)$. Since $F_0(f^0, f^1, U) \subseteq F_0(f_0', f_1', U)$, we have $G_0(g^0 \wedge g^1, g^0 \vee g^1, \tau_0(U)) \subseteq G_0(g_0' \wedge g_1', g_0' \vee g_1', V)$, so that $\tau_0(U) \subseteq V$. Also by Lemma 3.12, there exists a $U' \in \mathscr{T}_X$ such that $M(F_0(f_0', f_1', U')) = G_0(g_0' \wedge g_1', g_0' \vee g_1', \tau_0(U))$. Since $G_0(g_0' \wedge g_1', g_0' \vee g_1', \tau_0(U)) \subseteq G_0(g_0' \wedge g_1', g_0' \vee g_1', V)$, we have $F_0(f_0', f_1', U') \subseteq F_0(f_0', f_1', U)$, so that $U' \subseteq U$. But also $G_0(g^0 \wedge g^1, g^0 \vee g^1, \tau_0(U)) \subseteq G_0(g_0' \wedge g_1', g_0' \vee g_1', \tau_0(U))$, so that $F_0(f^0, f^1, U) \subseteq F_0(f_0', f_1', U')$, which implies that $U \subseteq U'$. Then $U' = U$, and hence $G_0(g_0' \wedge g_1', g_0' \vee g_1', V) = M(F_0(f_0', f_1', U)) = M(F_0(f_0', f_1', U')) = G_0(g_0' \wedge g_1', g_0' \vee g_1', \tau_0(U))$, showing that $V = \tau_0(U)$.

Now we can use this argument in reverse to show that $M(F_0(f_0, f_1, U)) = G_0(g_0 \wedge g_1, g_0 \vee g_1, \tau_0(U))$. By Lemma 3.12, there exists a $V \in \mathscr{T}_Y$ such that $G = G_0(g_0 \wedge g_1, g_0 \vee g_1, V)$. Since $F \subseteq F_0(f_0', f_1', U)$, we have $G \subseteq G_0(g_0' \wedge g_1', g_0' \vee g_1', \tau_0(U))$, so that $V \subseteq \tau_0(U)$. Also by Lemma 3.12, there exists a $U' \in \mathscr{T}_X$ such that $M(F_0(f_0, f_1, U')) = G_0(g_0 \wedge g_1, g_0 \vee g_1, \tau_0(U))$. Since $G \subseteq G_0(g_0 \wedge g_1, g_0 \vee g_1, \tau_0(U))$, it follows that $F \subseteq F_0(f_0, f_1, U')$, so that $U \subseteq U'$. But also $G_0(g_0 \wedge g_1, g_0 \vee g_1, \tau_0(U)) \subseteq G_0(g_0' \wedge g_1', g_0' \vee g_1', \tau_0(U))$, so that $F_0(f_0, f_1, U') \subseteq F_0(f_0', f_1', U)$, and hence $U' \subseteq U$. Then $U' = U$, and thus $G = M(F_0(f_0, f_1, U)) = M(F_0(f_0, f_1, U')) = G_0(g_0 \wedge g_1, g_0 \vee g_1, \tau_0(U))$, showing that $V = \tau_0(U)$. This finishes the argument that $G = G_0(g_0 \wedge g_1, g_0 \vee g_1, \tau_0(U))$ when $f_0 < f_1$.

We now give an argument for the general case that $f_0 \leq f_1$. Let $\mathscr{W}$ be the family of open subset of $X \times \mathbb{R}$ containing f_0, where $\mathscr{W}$ is directed downward by inclusion. For each $W \in \mathscr{W}$, let $f_0^W \in C(X)$ with $f_0^W \subseteq W$ and $f_0^W < f_0$, and let $g_0^W = M(f_0^W)$. Also for each $W \in \mathscr{W}$, define $F_W = F_0(f_0^W, f_1, U)$, and let $G_W = M(F_W)$. Since $f_0^W < f_1$, by our argument above, each $G_W = G_0(g_0^W \wedge g_1, g_0^W \vee g_1, \tau_0(U))$. Now it is evident that the net $\langle F_W \rangle_{\mathscr{W}}$ converges to F in $L^-(X)$. So by the continuity of M, the net $\langle G_W \rangle_{\mathscr{W}}$ converges to G in $L^-(Y)$. For each W, $F \subseteq F_W$, so that $G \subseteq G_W$. Also, since $\langle f_0^W \rangle_{\mathscr{W}}$ converges to f_0 in $L^-(X)$, we see that $\langle g_0^W \rangle_{\mathscr{W}}$ converges to g_0 in $L^-(Y)$. From this it follows that $G = G_0(g_0 \wedge g_1, g_0 \vee g_1, \tau_0(U))$. □

Lemma 3.14. *If $\mathscr{U}$ is a subfamily of $\mathscr{T}_X$ having the finite intersection property, then $\{\tau_0(U) :\ U \in \mathscr{U}\} \cup \{\tau_1(U) :\ U \in \mathscr{U}\}$ has the finite intersection property. Also if $\mathscr{V}$ is a subfamily of $\mathscr{T}_Y$ having the finite intersection property, then $\{\tau_0^{-1}(V) :\ V \in \mathscr{V}\} \cup \{\tau_1^{-1}(V) :\ V \in \mathscr{V}\}$ has the finite intersection property.*

Proof. First let us show that for each $U \in \mathscr{T}_X$, $\tau_0(U) \cap \tau_1(U) \neq \emptyset$. From this it easily follows that $\{\tau_0(U) :\ U \in \mathscr{U}\} \cup \{\tau_1(U) :\ U \in \mathscr{U}\}$ has the finite intersection property. Suppose, by way of contradiction, that there exists a $U \in \mathscr{T}_X$ with $\tau_0(U) \cap \tau_1(U) = \emptyset$. Let $V_0 \in \mathscr{T}_Y$ be such that $\overline{V_0} \subseteq \tau_0(U)$, and let $U_0 = \tau_0^{-1}(V_0)$. Then $\tau_1(U_0) \subseteq \tau_1(U)$, so that $\overline{\tau_0(U_0)} \cap \overline{\tau_1(U_0)} = \emptyset$.

Now there exists a $g \in C(Y)$ with $g \subseteq G(g^{-1} \wedge g^1, g^{-1} \vee g^1)$ such that $g(y) = (g^{-1} \vee g^1)(y)$ for all $y \in \tau_0(U_0)$ and $g(y) = (g^{-1} \wedge g^1)(y)$ for all $y \in \tau_1(U_0)$. Let $f = M^{-1}(g)$, so that $f \subseteq F(f^{-1}, f^1)$. From Lemma 3.13, we see that $M(F_0(f^{-1}, f^1, U_0)) = G_0(g^{-1} \wedge g^1, g^{-1} \vee g^1, \tau_0(U_0))$ and $M(F_1(f^{-1}, f^1, U_0)) = G_1(g^{-1} \wedge g^1, g^{-1} \vee g^1, \tau_1(U_0))$. Let $f_0 = M^{-1}(g^{-1} \wedge g^1)$ and $f_1 = M^{-1}(g^{-1} \vee g^1)$.

We have $M(F(f_0 \wedge f_1, f_0 \vee f_1)) = G(g^{-1} \wedge g^1, g^{-1} \vee g^1) = M(F(f^{-1}, f^1))$. Therefore, $F(f_0 \wedge f_1, f_0 \vee f_1) = F(f^{-1}, f^1)$, and it follows that $f_0 \wedge f_1 = f^{-1}$ and $f_0 \vee f_1 = f^1$. From this we see that for any $x \in X$, either $f_0(x) = -1$ and $f_1(x) = 1$, or $f_0(x) = 1$ and $f_1(x) = -1$. Now $G_0(g^{-1} \wedge g^1, g^{-1} \vee g^1, \tau_0(U_0)) \subseteq G(g^{-1} \wedge g^1, g)$, so that $F_0(f^{-1}, f^1, U_0) \subseteq F(f_0 \wedge f, f_0 \vee f)$ by Lemma 3.8. Then for any $x \in U_0$, either $f_0(x) = -1$ and $f(x) = 1$, or $f_0(x) = 1$ and $f(x) = -1$. Also $G_1(g^{-1} \wedge g^1, g^{-1} \vee g^1, \tau_1(U_0)) \subseteq G(g, g^{-1} \vee g)$, so that $F_1(f^{-1}, f^1, U_0) \subseteq F(f \wedge f_1, f \vee f_1)$ by Lemma 3.8. Then for any $x \in U_0$, either $f_1(x) = -1$ and $f(x) = 1$, or $f_1(x) = 1$ and $f(x) = -1$. But for $x \in U_0$, these three either/or values for $f_0(x)$, $f_1(x)$ and $f(x)$ cannot all hold. Since U_0 is nonempty, this gives us our contradiction.

Let us show that for each $V \in \mathscr{T}_Y$, $\tau_0^{-1}(V) \cap \tau_1^{-1}(V) \neq \emptyset$. Although this argument is similar to that of the first part, there are enough differences that we give the details. Suppose, by way of contradiction, that there exists a $V \in \mathscr{T}_Y$ with $\tau_0^{-1}(V) \cap \tau_1^{-1}(V) = \emptyset$. Let $U_0 \in \mathscr{T}_X$ be such that $\overline{U_0} \subseteq \tau_0^{-1}(V)$, and let $V_0 = \tau_0(U_0)$. Then $\tau_1^{-1}(V_0) \subseteq \tau_1^{-1}(V)$, so that $\overline{\tau_0^{-1}(V_0)} \cap \overline{\tau_1^{-1}(V_0)} = \emptyset$. Let $g_0 = g^{-1} \wedge g^1$, let $g_1 = g^{-1} \vee g^1$, let $f_0 = M^{-1}(g_0)$, and let $f_1 = M^{-1}(g_1)$. Since $f^{-1} < f^1$, we have $g_0 < g_1$ by Lemma 3.8. As we saw earlier, $M(F(f_0 \wedge f_1, f_0 \vee f_1)) = G(g_0, g_1) = M(F(f^{-1}, f^1))$.

Now there exists an $f \in C(X)$ with $f \subseteq F(f_0 \wedge f_1, f_0 \vee f_1)$ such that $f(x) = (f_0 \vee f_1)(x)$ for all $x \in \tau_0^{-1}(V_0)$ and $f(x) = (f_0 \wedge f_1)(x)$ for all $x \in \tau_1^{-1}(V_0)$. Let $g = M(f)$, so that $g \subseteq G(g_0, g_1)$. Then by Lemma 3.13, $M(F_0(f_0 \wedge f_1, f_0 \vee f_1, \tau_0^{-1}(V_0)) = G_0(g_0, g_1, V_0)$ and $M(F_1(f_0 \wedge f_1, f_0 \vee f_1, \tau_1^{-1}(V_0)) = G_1(g_0, g_1, V_0)$. Define $g_0' = M(f_0 \wedge f_1)$ and $g_1' = M(f_0 \vee f_1)$.

We have $G(g_0' \wedge g_1', g_0' \vee g_1') = M(F(f_0 \wedge f_1, f_0 \vee f_1)) = G(g_0, g_1)$. It follows that $g_0' \wedge g_1' = g_0$ and $g_0' \vee g_1' = g_1$, so that for any $y \in Y$, either $g_0'(y) = g_0(y)$ and $g_1'(y) = g_1(y)$, or $g_0'(y) = g_1(y)$ and $g_1'(y) = g_0(y)$. Now $F_0(f_0 \wedge f_1, f_0 \vee f_1, \tau_0^{-1}(V_0)) \subseteq F(f_0 \wedge f_1, f)$, so that $G_0(g_0, g_1, V_0) \subseteq G(g_0' \wedge g, g_0' \vee g)$ by Lemma 3.8. Then for any $y \in V_0$, either $g_0'(y) = g_0(y)$ and $g(y) = g_1(y)$, or $g_0'(y) = g_1(y)$ and $g(y) = g_0(y)$. Also $F_1(f_0 \wedge f_1, f_0 \vee f_1, \tau_1^{-1}(V_0)) \subseteq F(f, f_0 \vee f_1)$, so that $G_1(g_0, g_1, V_0) \subseteq G(g \wedge g_1', g \vee g_1')$ by Lemma 3.8. Then for any $y \in V_0$, either $g_1'(y) = g_0(y)$ and $g(y) = g_1(y)$, or $g_1'(y) = g_1(y)$ and $g(y) = g_0(y)$. Since $g_0(y) < g_1(y)$, these three either/or values for $g_0'(y)$, $g_1'(y)$ and $g(y)$, when $y \in V_0$, cannot all hold. Since V_0 is nonempty, this gives the desired contradiction. $\square$

Lemma 3.15. *The ordered bijections $\tau_0 \colon \mathscr{T}_X \to \mathscr{T}_Y$ and $\tau_1 \colon \mathscr{T}_X \to \mathscr{T}_Y$ are equal.*

Proof. Suppose there exists a $U \in \mathscr{T}_X$ such that $\tau_0(U) \neq \overline{\tau_1(U)}$, say $\tau_0(U) \setminus \tau_1(U) \neq \emptyset$. Since $\tau_1(U)$ is regular open, we have $\tau_0(U) \setminus \overline{\tau_1(U)} \neq \emptyset$. Let $V \in \mathscr{T}_Y$ be nonempty with $V \subseteq \tau_0(U) \setminus \overline{\tau_1(U)}$. Now $V \subseteq \tau_0(U)$, so that $\tau_0^{-1}(V) \subseteq U$. Also since $V \cap \tau_1(U) = \emptyset$, we have $\tau_1^{-1}(V) \cap U = \emptyset$. But then $\tau_0^{-1}(V) \cap \tau_1^{-1}(V) = \emptyset$, which contradicts Lemma 3.14. $\square$

We now define $\tau\colon \mathscr{T}_X \to \mathscr{T}_Y$ to be $\tau_0\colon \mathscr{T}_X \to \mathscr{T}_Y$ (and also $\tau_1\colon \mathscr{T}_X \to \mathscr{T}_Y$). To show that $\tau \in ROC(X,Y)$, we need the following lemma.

Lemma 3.16. *For every $U_1, U_2 \in \mathscr{T}_X$, $\overline{U_1} \cap \overline{U_2} \neq \emptyset$ if and only if $\overline{\tau(U_1)} \cap \overline{\tau(U_2)} \neq \emptyset$.*

Proof. Suppose that $\overline{U_1} \cap \overline{U_2} \neq \emptyset$ while $\overline{\tau(U_1)} \cap \overline{\tau(U_2)} = \emptyset$. Let V_1 and V_2 be disjoint elements of $\mathscr{T}_Y$ with $\overline{\tau(U_1)} \subseteq V_1$ and $\overline{\tau(U_2)} \subseteq V_2$. Let $g_1, g_2 \in C(Y)$ be contained in $G(g^0 \wedge g^1, g^0 \vee g^1)$ such that $g_1(y) = g^1(y)$ for $y \in \overline{\tau(U_1)}$, $g_1(y) = g^0(y)$ for $y \in Y \setminus V_1$, $g_2(y) = g^1(y)$ for $y \in \overline{\tau(U_2)}$, and $g_2(y) = g^0(y)$ for $y \in Y \setminus V_2$. Let $f_1 = M^{-1}(g_1)$ and $f_2 = M^{-1}(g_2)$. Note that $f^0 \leq f_1$ and $f^0 \leq f_2$. Let $G_1 = G(g^0 \wedge g_1, g^0 \vee g_1)$, $G_2 = G(g^0 \wedge g_2, g^0 \vee g_2)$, $F_1 = M^{-1}(G_1)$, and $F_2 = M^{-1}(G_2)$. Then by Lemma 3.8, $F_1 = F(f^0, f_1)$ and $F_2 = F(f^0, f_2)$. Now $G_1 \subseteq G_0(g^0 \wedge g^1, g^0 \vee g^1, V_1)$ and $G_2 \subseteq G_0(g^0 \wedge g^1, g^0 \vee g^1, V_2)$, so that $F_1 \subseteq F_0(f^0, f^1, \tau^{-1}(V_1))$ and $F_2 \subseteq F_0(f^0, f^1, \tau^{-1}(V_2))$. Since $V_1 \cap V_2 = \emptyset$, we have $\tau^{-1}(V_1) \cap \tau^{-1}(V_2) = \emptyset$. But then $f_1 \wedge f_2 = f^0$. Also we have $G_0(g^0 \wedge g^1, g^0 \vee g^1, \tau(U_1)) \subseteq G_1$ and $G_0(g^0 \wedge g^1, g^0 \vee g^1, \tau(U_2)) \subseteq G_2$, so that $F_0(f^0, f^1, U_1) \subseteq F_1$ and $F_0(f^0, f^1, U_2) \subseteq F_2$. Then for every $x \in U_1$, $f_1(x) = 1$, and for every $x \in U_2$, $f_2(x) = 1$. But there exists an $x \in \overline{U_1} \cap \overline{U_2}$. For this x, $f_1(x) = 1 = f_2(x)$, so that $f_1 \wedge f_2 \neq f^0$. This contradiction shows that if $\overline{U_1} \cap \overline{U_2} \neq \emptyset$, then $\overline{\tau(U_1)} \cap \overline{\tau(U_2)} \neq \emptyset$. The converse is proved in a similar way. $\square$

This shows that $\tau \in ROC(X,Y)$, so that we have extensions τX and τY of X and Y defined, and we have the homeomorphism $e_\tau\colon \tau X \to \tau Y$. To show that $\tau \in LROC(X,Y)$, we need the next lemma.

Lemma 3.17. *Every element of $C(X)$ can be extended to an element of $C(\tau X)$, and every element of $C(Y)$ can be extended to an element of $C(\tau Y)$.*

Proof. Let $f \in C(X)$, and let $[\mathscr{T}] \in X^*$. In order to extend f to $X \cup \{[\mathscr{T}]\}$, we want to show that for each $n \in \mathbb{N}$, there exists a $U \in \mathscr{T}_X$ such that $[\mathscr{T}] \in U^*$ and such that for every $x_1, x_2 \in U$, $|f(x_1) - f(x_2)| < 1/n$. Suppose, by way of contradiction, that there exists an $n \in \mathbb{N}$ such that for every $U \in \mathscr{T}_X$ with $[\mathscr{T}] \in U^*$, there are $x_1, x_2 \in U$ so that $|f(x_1) - f(x_2)| \geq 1/n$. Define $\mathscr{U} = \{U \in \mathscr{T}_X : [\mathscr{T}] \in U^*\}$. Now $\bigcap \overline{\tau(\mathscr{T})} = \{y\}$ for some $y \in Y$, and for this y, we have $y \in \tau(U)$ for all $U \in \mathscr{U}$.

Observe that $\inf\{\sup\{f(x) : x \in U\} : U \in \mathscr{U}\} - \sup\{\inf\{f(x) : x \in U\} : U \in \mathscr{U}\} \geq 1/n$. Then let $a, b \in \mathbb{R}$ with $\sup\{\inf\{f(x) : x \in U\} : U \in \mathscr{U}\} <$

$a < b < \inf\{\sup\{f(x) : x \in U\} : U \in \mathscr{U}\}$. For each $U \in \mathscr{U}$, by the continuity of f, there exist $U_a, U_b \in \mathscr{T}_X$ that are contained in U, and are such that $f(U_a) \subseteq (-\infty, a]$ and $f(U_b) \subseteq [b, \infty)$. Let U_* be the interior of the closure of $\bigcup\{U_a : U \in \mathscr{U}\}$ in X, so that $U_* \in \mathscr{T}_X$. Note that $U_a \subseteq U_*$ for all $U \in \mathscr{U}$. It is evident that for each $U \in \mathscr{U}$, $U_b \cap U_* = \emptyset$.

Define $F = F_0(f \wedge f^a, f \vee f^a, U_*)$, which is an element of $L^-(X)$. Also let $g_a = M(f \wedge f^a)$, $g_b = M(f \vee f^b)$, and $G = M(F) = G_0(g_a \wedge g_b, g_a \vee g_b, \tau(U_*))$. Now for every $U \in \mathscr{U}$, $\tau(U_a) \subseteq \tau(U_*)$ and $\tau(U_b) \cap \tau(U_*) = \emptyset$. Also for each $U \in \mathscr{U}$, $F_0(f^a, f^b, U_a) \subseteq F$, so that $G_0(g^a \wedge g^b, g^a \vee g^b, \tau(U_a)) \subseteq G$. Since every neighborhood of y contains $\tau(U_a)$ for some $U \in \mathscr{U}$, it follows that $\inf G(y) \leq g^a \wedge g^b(y) < g^a \vee g^b(y) \leq \sup G(y)$. Now for each $U \in \mathscr{U}$, $F_0(f^a, f^b, U_b) \cap F = f^a$, so that $G_0(g^a \wedge g^b, g^a \vee g^b, \tau(U_b)) \cap G = g^a$.

Let $t \in \mathbb{R}$ be such that $g^a \wedge g^b(y) < t < g^a \vee g^b(y)$ and $t \neq g^a(y)$. Note that $\langle y, t\rangle \in G$. Let O_1 and O_2 be disjoint open intervals contained in the interval $(g^a \wedge g^b(y), g^a \vee g^b(y))$ such that $t \in O_1$ and $g^a(y) \in O_2$. Let V_0 be a neighborhood of y with $g^a(V_0) \subseteq O_2$. Now for each neighborhood V of y contained in V_0, there is a $U \in \mathscr{U}$ such that $\tau(U) \subseteq V$, and hence $U_b \times O_1 \cap G = \emptyset$. But then G is not lsc at $\langle y, t\rangle$, which is a contradiction.

This shows that for each $n \in \mathbb{N}$, there exists a $U(n) \in \mathscr{T}_X$ such that $[\mathscr{T}] \in U(n)^*$ and such that for every $x_1, x_2 \in U(n)$, $|f(x_1) - f(x_2)| < 1/n$. Since $y \in \tau(U(n))$ for each n, $\{\tau(U(n)) : n \in \mathbb{N}\}$ has the finite intersection property, and hence $\{U(n) : n \in \mathbb{N}\}$ has the finite intersection property. So we may assume that each $U(n+1) \subseteq U(n)$. Now for each n, choose an $x_n \in U(n)$. Then if $m, n \in \mathbb{N}$ with $m \leq n$, we have $|f(x_m) - f(x_n)| < 1/m$. Therefore, the sequence $\langle f(x_n)\rangle$ is a Cauchy sequence and thus has a limit, that we denote by $\hat{f}([\mathscr{T}])$. Now by letting $\hat{f}(x) = f(x)$ for all $x \in X$, we have defined $\hat{f}\colon X \cup \{[\mathscr{T}]\} \to \mathbb{R}$ that is an extension of f.

To show that $\hat{f}$ is continuous at $[\mathscr{T}]$, let $n \in \mathbb{N}$. Since $\langle f(x_m)\rangle$ converges to $\hat{f}([\mathscr{T}])$, there exists an $m \geq 2n$ such that $|f(x_m) - \hat{f}([\mathscr{T}])| < 1/(2n)$. Let $x \in U(2n)$. Then $|\hat{f}(x) - \hat{f}([\mathscr{T}])| = |f(x) - \hat{f}([\mathscr{T}])| \leq |f(x) - f(x_m)| + |f(x_m) - \hat{f}([\mathscr{T}])| < 1/(2n) + 1/(2n) = 1/n$. So $\hat{f}(U(2n) \cup \{[\mathscr{T}]\}) \subseteq (\hat{f}([\mathscr{T}]) - 1/n, \hat{f}([\mathscr{T}]) + 1/n)$, showing that $\hat{f}$ is continuous at $[\mathscr{T}]$.

We have shown that we can extend f continuously to X^* one point at a time. Then by the regularity of $\mathbb{R}$, f can be extended continuously to all of X^*, giving us an element of $C(\tau X)$ that extends f. A similar proof shows that each element of $C(Y)$ can be extended to an element of $C(\tau Y)$. $\square$

Therefore, $\tau \in LROC(X, Y)$, so that τ induces an ordered homeomorphism $\tau^*\colon L^-(X) \to L^-(Y)$.

Our final goal is to define a $\phi \in FH(X)$ such that $M = \tau^*\phi^*$. Consider the ordered homeomorphism $(\tau^*)^{-1}M\colon L^-(X) \to L^-(X)$. To simplify notation, we denote $(\tau^*)^{-1}M$ by N. Also for each $t \in \mathbb{R}$, denote $N(f^t)$ by h^t. Observe that Lemmas 3.7 through 3.12 are true for any ordered homeomorphism, so that they apply to N. Now define $\phi\colon X \times \mathbb{R} \to X \times \mathbb{R}$ by $\phi(\langle x, t\rangle) = \langle x, h^t(x)\rangle$ for each $\langle x, t\rangle \in X \times \mathbb{R}$.

Lemma 3.18. *Let $f_0, f_1 \in C(X)$, let $\hat{g}_0 = e_Y M(f_0)$, let $\hat{g}_1 = e_Y M(f_1)$, let $x \in X$, and let $\hat{y} = e_\tau(x)$. Then $f_0(x) = f_1(x)$ if and only if $\hat{g}_0(\hat{y}) = \hat{g}_1(\hat{y})$.*

Proof. Both directions have similar proofs, so we only argue the implication in one direction. Suppose, by way of contradiction, that $f_0(x) = f_1(x)$ but $\hat{g}_0(\hat{y}) \neq \hat{g}_1(\hat{y})$. Let $g_0 = M(f_0)$, let $g_1 = M(f_1)$, let $f_2 = f_0 \wedge f_1$, let $f_3 = f_0 \vee f_1$, let $g_2 = M(f_2)$, let $g_3 = M(f_3)$, let $\hat{g}_2 = e_Y(g_2)$, and let $\hat{g}_3 = e_Y(g_3)$. Finally, define $F(f_2, f_3)$ and $G = M(F) = G(g_2 \wedge g_3, g_2 \vee g_3)$.

Let $\mathscr{T} \in \mathbb{T}^*$ contain $\{U \in \mathscr{T}_X :\ x \in U\}$, so that $\bigcap \overline{\mathscr{T}} = \{x\}$. For each $U \in \mathscr{T}$. Let $F_U = F_0(f_2, f_3, U)$ and $G_U = M(F_U) = G_0(g_2 \wedge g_3, g_2 \vee g_3, \tau(U))$. We can think of $\mathscr{T}$ as a directed set, directed downward by inclusion. So $\langle F_U\rangle_{U\in\mathscr{T}}$ and $\langle G_U\rangle_{U\in\mathscr{T}}$ are nets in $L^-(X)$ and $L^-(Y)$. We will obtain our contradiction by showing that $\langle F_U\rangle_{U\in\mathscr{T}}$ converges to f_2 in $L^-(X)$ while $\langle G_U\rangle_{U\in\mathscr{T}}$ does not converge to any element of $C(Y)$ in $L^-(Y)$; then because M maps $C(X)$ onto $C(Y)$, this would contradict the continuity of M.

To show that $\langle F_U\rangle_{U\in\mathscr{T}}$ converges to f_2 in $L^-(X)$, let W be an open subset of $X \times \mathbb{R}$ with $f_2 \subseteq W$. Then there exist a $U_0 \in \mathscr{T}$ and an open interval O containing $f_2(x)$ such that $U_0 \times O \subseteq W$. Since f_2 and f_3 are continuous and $f_2(x) = f_3(x)$, there is a $U_1 \in \mathscr{T}$ with $U_1 \subseteq U_0$ such that $f_2(U_1) \subseteq O$ and $f_3(U_1) \subseteq O$. Then $F_{U_1} \subseteq U_1 \times O \cup f_2 \subseteq W$. So for all $U \in \mathscr{T}$ with $U \subseteq U_1$, we have $F_U \subseteq F_{U_1} \subseteq W$; showing that $\langle F_U\rangle_{U\in\mathscr{T}}$ converges to f_2 in $L^-(X)$.

To show that $\langle G_U\rangle_{U\in\mathscr{T}}$ does not converge to any element of $C(Y)$ in $L^-(Y)$, let us start by observing that $f_0, f_1 \subseteq F$, so that $g_0, g_1 \subseteq G$. That means $g_2 \wedge g_3 \leq g_0 \leq g_2 \vee g_3$ and $g_2 \wedge g_3 \leq g_1 \leq g_2 \vee g_3$. Since Y is dense in τY, we have $\hat{g}_2 \wedge \hat{g}_3 \leq \hat{g}_0 \leq \hat{g}_2 \vee \hat{g}_3$ and $\hat{g}_2 \wedge \hat{g}_3 \leq \hat{g}_1 \leq \hat{g}_2 \vee \hat{g}_3$. Now $\hat{g}_0(\hat{y}) \neq \hat{g}_1(\hat{y})$, so that $\hat{g}_2(\hat{y}) \neq \hat{g}_3(\hat{y})$; say $\hat{g}_2(\hat{y}) < \hat{g}_3(\hat{y})$. Let $a, b \in \mathbb{R}$ with $\hat{g}_2(\hat{y}) < a < b < \hat{g}_3(\hat{y})$, and let I be the closed interval $[a, b]$. By the continuity of $\hat{g}_2$ and $\hat{g}_3$, there exists a $V_0 \in \mathscr{T}_Y$ with $\hat{y} \in V_0 \cup V_0^*$ such that $\hat{g}_2(V_0 \cup V_0^*) \subseteq (-\infty, a)$ and $\hat{g}_3(V_0 \cup V_0^*) \subseteq (b, \infty)$. Now $V_0 \in \tau(\mathscr{T})$, so that $U_0 \in \mathscr{T}$ where $U_0 = \tau^{-1}(V_0)$. Note that if $U \in \mathscr{T}$ with $U \subseteq U_0$, then for each $y \in \tau(U)$, $I \subseteq G_U(y)$. Let g be any element of $C(Y)$, let $\varepsilon = (b-a)/2$, and let $W = \{\langle y, t\rangle \in Y \times \mathbb{R} :\ g(y) - \varepsilon < t < g(y) + \varepsilon\}$, which is open in $Y \times \mathbb{R}$. Then for every $y \in Y$, $I \not\subseteq W(y)$. That means for each $U \in \mathscr{T}$ with $U \subseteq U_0$, $G_U \not\subseteq W$. Since $g \subseteq W$, we see that $\langle G_U\rangle_{U\in\mathscr{T}}$ does not converge to g in $L^-(Y)$. $\square$

Lemma 3.19. *Let $s, t \in \mathbb{R}$ with $s < t$, and let $i \in \{0, 1\}$. Then for each $U \in \mathscr{T}_X$, $N(F_i(f^s, f^t, U)) = F_i(h^s \wedge h^t, h^s \vee h^t, U)$.*

P r o o f. We only prove this for $i = 0$ since the proof of the other case is similar. By Lemma 3.12, there exists a $U_0 \in \mathscr{T}_X$ such that $N(F_0(f^s, f^t, U)) = F_0(h^s \wedge h^t, h^s \vee h^t, U_0)$. Since U and U_0 are regular open sets, we will know that $U_0 = U$ if we show that $\overline{U_0} = \overline{U}$. Suppose, by way of contradiction, that $\overline{U_0} \neq \overline{U}$.

First suppose that $\overline{U} \setminus \overline{U_0} \neq \emptyset$. Then there exists an $x \in U \setminus \overline{U_0}$. Let $f \in C(X)$ with $f \subseteq F_0(f^s, f^t, U)$ such that $f(x) = t$ and $f(x') = s$ for all $x' \in X \setminus U$. Then by Lemma 3.18, $N(f)(x) = e_Y M(f)(e_\tau(x)) \neq e_Y M(f^s)(e_\tau(x)) = N(f^s)(x)$. But $N(f), N(f^s) \subseteq F_0(h^s \wedge h^t, h^s \vee h^t, U_0)$ and $F_0(h^s \wedge h^t, h^s \vee h^t, U_0)(x) = \{h^s \wedge h^t(x)\}$, a singleton set; which is a contradiction.

Now suppose that $\overline{U_0} \setminus \overline{U} \neq \emptyset$. Then there exists an $x \in U_0 \setminus \overline{U}$. Let $h \in C(X)$ with $h \subseteq F_0(h^s \wedge h^t, h^s \vee h^t, U_0)$ such that $h(x) = h^s \vee h^t(x)$ and $h(x') = h^s \wedge h^t(x')$ for all $x' \in X \setminus U_0$. Let $f = N^{-1}(h)$ and $f' = N^{-1}(h^s \wedge h^t)$. Then $e_Y M(f)(e_\tau(x)) = N(f)(x) = h(x) \neq h^s \wedge h^t(x) = N(f')(x) = e_Y M(f')(e_\tau(x))$. Now Lemma 3.18 implies that $f(x) \neq f'(x)$. But $f, f' \subseteq F_0(f^s, f^t, U)$ and $F_0(f^s, f^t, U)(x) = \{s\}$, a singleton; which is a contradiction. $\square$

Lemma 3.20. *For each $x \in X$, either $h^t(x)$ increases as t increases in $\mathbb{R}$, or $h^t(x)$ decreases as t increases in $\mathbb{R}$.*

P r o o f. Let us fix $x \in X$. This argument makes repeated use of Lemma 3.8 as applied to N, rather than M. Since $f^0 < f^1$, it follows that either $h^0(x) < h^1(x)$ or $h^0(x) > h^1(x)$; say the former. Let $s, t \in \mathbb{R}$ with $s < t$. We assume that $1 \leq s$, since the other cases have similar arguments. Since $F(f^0, f^1) \subseteq F(f^0, f^s) \subseteq F(f^0, f^t)$, we have $F(h^0 \wedge h^1, h^0 \vee h^1) \subseteq F(h^0 \wedge h^s, h^0 \vee h^s) \subseteq F(h^0 \wedge h^t, h^0 \vee h^t)$. From the first containment, we see that $h^1(x)$ is between $h^0(x)$ and $h^s(x)$. Now $h^0(x) < h^1(x)$, so that $h^1(x) \leq h^s(x)$. The second containment gives us $h^s(x)$ between $h^0(x)$ and $h^t(x)$, and hence $h^s(x) \leq h^t(x)$. But $f^s < f^t$, so that $h^s(x) < h^t(x)$. Thus $h^t(x)$ is increasing for this fixed x as t increases in $\mathbb{R}$. $\square$

Lemma 3.21. *Let $x \in X$, let $r, t \in \mathbb{R}$ with $r < t$, and let $q \in F(h^r \wedge h^t, h^r \vee h^t)(x)$. Then there exists an $s \in [r, t]$ such that $h^s(x) = q$.*

P r o o f. Suppose, by way of contradiction, that for every $s \in [r, t]$, $h^s(x) \neq q$. By Lemma 3.20, we may assume that $h^s(x)$ is increasing as s increases in $[r, t]$, since the other case that $h^s(x)$ is decreasing has a similar argument. Let $A = \{s \in [r, t] : \ h^s(x) < q\}$ and let $B = \{s \in [r, t] : \ h^s(x) > q\}$. Clearly $A \cap B = \emptyset$ and $A \cup B = [r, t]$. Since $h^s(x)$ is increasing, $h^r(x) < h^t(x)$, so that $r \in A$ and $t \in B$. Also since $h^s(x)$ is increasing, every element of A is less than each element of B. Therefore, A and B are intervals, and there exists an $s \in (r, t)$ such that either $A = [r, s]$ and $B = (s, t]$, or $A = [r, s)$ and $B = [s, t]$. Let us suppose that $A = [r, s]$ and $B = (s, t]$. In this case we need to use $i = 0$, whereas in the other case we would need to use $i = 1$.

Now $h^s(x) < q < h^t(x)$, so by the continuity of h^s and h^t, there exists an open neighborhood U_0 of x in $\mathscr{T}_X$ such that $h^s(x') < q < h^t(x')$ for all $x' \in U_0$. Let $\mathscr{U}$ be the set of $U \in \mathscr{T}_X$ with $x \in U \subseteq U_0$; so $\mathscr{U}$ is a base for x. For each $U \in \mathscr{U}$, define $F_U = F(h^s \wedge h^t, h^s \vee h^t) \cap U \times (-\infty, q] \cup h^s$, and let $F'_U = N^{-1}(F_U)$. This defines nets $\langle F_U \rangle_{\mathscr{U}}$ and $\langle F'_U \rangle_{\mathscr{U}}$ in $L^-(X)$ over the set $\mathscr{U}$ directed downward by inclusion. Since $\{x\} \times [h^s(x), q] \subseteq F_U$ for every $U \in \mathscr{U}$, we see that $\langle F_U \rangle_{\mathscr{U}}$ does not converge to h^s in $L^-(X)$.

If we can show that $\langle F'_U \rangle_{\mathscr{U}}$ converges to f^s in $L^-(X)$, then we have a contradiction to the fact that N is continuous. Since $f^s \in C(X)$, it does not matter whether $L^-(X)$ has the Vietoris topology or the upper Vietoris topology. Let W be an open subset of $X \times \mathbb{R}$ with $f^s \in W^+$. Then there exist a $U_1 \in \mathscr{U}$ and an open interval (a, b) that contains s and is contained in $[r, t]$ such that $U_1 \times [a, b] \subseteq W$. Now $q < h^b(x)$, so by the continuity of h^b, x has a neighborhood $U_2 \in \mathscr{T}_X$ contained in U_1 such that $q < h^b(x')$ for all $x' \in U_2$. Then $N(F_0(f^s, f^b, U_2)) = F_0(h^s \wedge h^b, h^s \vee h^b, U_2)$ by Lemma 3.19. Now $F_{U_2} \subseteq F_0(h^s \wedge h^b, h^s \vee h^b, U_2)$, so that $F'_{U_2} \subseteq F_0(f^s, f^b, U_2) \subseteq U_2 \times [s, b] \cup f^s \subseteq W$. Then for every $U \in \mathscr{U}$ that is contained in U_2, we have $F'_U \subseteq F'_{U_2} \subseteq W$, showing that $\langle F'_U \rangle_{\mathscr{U}}$ converges to f^s in $L^-(X)$. □

Lemma 3.22. *The function $\phi\colon X \times \mathbb{R} \to X \times \mathbb{R}$ is a bijection.*

Proof. Lemma 3.8 ensures that for each $s, t \in \mathbb{R}$ with $s \neq t$, $h^s(x) \neq h^t(x)$ for all $x \in X$. Therefore, ϕ is one-to-one.

For each $x \in X$, define $R(x) = \bigcup_{n=1}^{\infty} F(h^{-n} \wedge h^n, h^{-n} \vee h^n)(x)$, which is a connected subset of $\mathbb{R}$. Now Lemma 3.21 implies that $\phi(\{x\} \times \mathbb{R}) = \{x\} \times R(x)$. So it remains to show that each $R(x) = \mathbb{R}$.

Suppose, by way of contradiction, that for some $x \in X$, there exists a $b \in \mathbb{R}$ such that $t < b$ for all $t \in R(x)$. The argument for the case that all $t > b$ is similar, but uses $i = 1$ instead of $i = 0$. From Lemma 3.20, we may assume that $h^t(x)$ is increasing as t increases in $\mathbb{R}$, because if $h^t(x)$ is decreasing, the proof is similar. Then $h^0(x) < b$; define $c = b - h^0(x)$. Let $h \in C(X)$ be defined by $h(x') = h^0(x') + c$ for all $x' \in X$. Since for each $n \in \mathbb{N}$, $h^n(x) < b = h(x)$, by the continuity of h^n and h, x has a neighborhood $U_n \in \mathscr{T}_X$ such that $h^n(x') < h(x')$ for all $x' \in U_n$. We may choose these U_n such that each $U_{n+1} \subseteq U_n$. Observe that for each n, $F_0(h^0 \wedge h^n, h^0 \vee h^n, U_n) \subseteq F_0(h^0 \wedge h, h^0 \vee h, U_n) \in L^-(X)$. Define $F = \bigcup_{n=1}^{\infty} F_0(h^0 \wedge h^n, h^0 \vee h^n, U_n)$, which is thus an element of $L^-(X)$. Note that for each n, $N(F_0(f^0, f^n, U_n)) = F_0(h^0 \wedge h^n, h^0 \vee h^n, U_n)$ by Lemma 3.19. Then if $F' = N^{-1}(F)$, we have for each n, $F_0(f^0, f^n, U_n) \subseteq F'$. But each $F_0(f^0, f^n, U_n)(x) = [0, n]$, which implies that $[0, \infty) \subseteq F'(x)$, contradicting the local boundedness of F'. □

Lemma 3.23. *The bijection $\phi\colon X \times \mathbb{R} \to X \times \mathbb{R}$ is a fiber homeomorphism.*

Proof. To show that ϕ is continuous, let $\langle x, s\rangle \in X\times\mathbb{R}$, let U be a neighborhood of x, and let (a, b) be an open interval containing $h^s(x)$. By Lemma 3.20, we may assume that $h^t(x)$ increases as t increases in $\mathbb{R}$, because if $h^t(x)$ decreases, the proof is similar. Let $p, q \in \mathbb{R}$ be such that $a < p < h^s(x) < q < b$. By Lemma 3.21, there exist $r, t \in \mathbb{R}$ with $r < s < t$ such that $h^r(x) = p$ and $h^t(x) = q$. Since $a < h^r(x)$ and $h^t(x) < b$, by the continuity of h^r and h^t, x has a neighborhood U_0 contained in U such that $h^r(U_0) \subseteq (a, \infty)$ and $h^t(U_0) \subseteq (-\infty, b)$. Now let $\langle x', s'\rangle \in U_0 \times (r, t)$. Then $a < h^r(x') < h^{s'}(x') < h^t(x') < b$, so that $h^{s'}(x') \in (a, b)$. Therefore, $U_0 \times (r, t)$ is a neighborhood of $\langle x, s\rangle$ and $\phi(U_0 \times (r, t)) \subseteq U \times (a, b)$. This shows the continuity of ϕ.

To show that ϕ^{-1} is continuous, let $\langle x, s\rangle \in X \times \mathbb{R}$, let U be a neighborhood of x, and let (a, b) be an open interval containing s. Again, we may assume that $h^t(x)$ increases as t increases in $\mathbb{R}$. Let $p, q \in \mathbb{R}$ be such that $h^a(x) < p < h^s(x) < q < h^b(x)$. By the continuity of h^a and h^b, x has a neighborhood U_0 contained in U such that $h^a(U_0) \subseteq (-\infty, p)$ and $h^b(U_0) \subseteq (q, \infty)$. Now let $\langle x', s'\rangle \in U_0 \times (p, q)$. By Lemma 3.21, there exits a $t \in \mathbb{R}$ such that $h^t(x') = s'$. Then $h^a(x') < p < h^t(x') < q < h^b(x')$, which implies that $a < t < b$, and hence $\langle x', t\rangle \in U \times (a, b)$. Therefore, $U_0 \times (p, q)$ is a neighborhood of $\langle x, h^s(x)\rangle$ and $\phi^{-1}(U_0 \times (p, q)) \subseteq U \times (a, b)$. This shows the continuity of ϕ^{-1}. Now it is evident that ϕ is a fiber homeomorphism. □

Now since $\phi \in FH(X)$, we know from Proposition 3.2 that ϕ induces an ordered homeomorphism $\phi^*\colon L^-(X) \to L^-(X)$.

Lemma 3.24. *The ordered homeomorphism ϕ^* is equal to the ordered homeomorphism N.*

Proof. The Extension Theorem I.5.1 implies that two ordered homeomorphisms from $L^-(X)$ onto $L^-(X)$ are equal if they agree on $C(X)$. So let $f \in C(X)$ and let $x \in X$. Then $\phi^*(f) = \phi(f)$, where the second f is equal to its graph in $X\times\mathbb{R}$. Now $\phi(f)(x) = h^{f(x)}(x) = N(f^{f(x)})(x) = (\tau^*)^{-1}M(f^{f(x)})(x) = e_Y M(f^{f(x)})(e_\tau(x))$. But since $f^{f(x)}(x) = f(x)$, we have Lemma 3.18 implying that $e_Y M(f^{f(x)})(e_\tau(x)) = e_Y M(f)(e_\tau(x)) = (\tau^*)^{-1}M(f)(x) = N(f)(x)$, showing that $\phi^* = N$. □

So Lemma 3.24 gives us our factorization $M = \tau^*\phi^*$. For the uniqueness of this factorization, we have the following.

Lemma 3.25. *If $\tau_1, \tau_2 \in LROC(X, Y)$ and $\phi_1, \phi_2 \in FH(X)$ are such that $\tau_1^*\phi_1^* = \tau_2^*\phi_2^*$, then $\tau_1^* = \tau_2^*$ and $\phi_1^* = \phi_2^*$.*

Proof. Now $\tau_1^*\phi_1^* = \tau_2^*\phi_2^*$ implies that $(\tau_2^*)^{-1}\tau_1^* = \phi_2^*(\phi_1^*)^{-1} = \phi_2^*(\phi_1^{-1})^* = (\phi_2\phi_1^{-1})^*$. Let t be any element of $\mathbb{R}$. Then $\tau_1^*(f^t) = \hat{\tau}_1(f^t) = e_Y^{-1}\hat{e}_\tau e_X(f^t)$.

Since f^t is the constant t function on X, $e_X(f^t)$ is the constant t function on τX. Also $\hat{e}_\tau e_X(f^t)$ is the constant t function on τY, and hence $\tau_1^*(f^t)$ is the constant t function on Y; call this last function g_t. Now $(\tau_2^*)^{-1}(g_t) = f^t$ by the same reasoning. Therefore, for each $t \in \mathbb{R}$, $(\phi_2\phi_1^{-1})^*(f^t) = f^t$, which says that $\phi_2\phi_1^{-1}$ is the identity map on $X \times \mathbb{R}$. Thus $\phi_2^*(\phi_1^*)^{-1} = (\phi_2\phi_1^{-1})^*$ is the identity map on $L^-(X)$. Therefore, $\phi_1^* = \phi_2^*$, and it follows that $\tau_1^* = \tau_2^*$. $\square$

This shows that M can be uniquely factored as $M = \tau^*\phi^*$ where $\tau \in LROC(X,Y)$ and $\phi \in FH(X)$. Similar arguments also show that M can be uniquely factored as $M = \psi^*\tau^*$ where $\tau \in LROC(X,Y)$ and $\psi \in FH(Y)$. These lemmas now complete the proof of the Factorization Theorem 3.3.

We end by considering some questions that we frame in terms of three open ended problems.

Problem 1. To what extent can these results for $L^-(X)$ be generalized to spaces of lower semicontinuous set-valued maps with nonempty compact convex images in a separable Banach space?

Problem 2. Can one characterize various topological properties of $L^-(X)$ in terms of properties of X? For example, it is known that $L(X)$ is metrizable if and only if X is compact and metrizable ([11]). Is the same true for $L^-(X)$?

Problem 3. We have a characterization of the bimonotone homeomorphisms between $C(X)$ and $C(Y)$. Can one understand the general homeomorphisms between $C(X)$ and $C(Y)$, and is this somehow related to homeomorphisms between $L^-(X)$ and $L^-(Y)$?

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