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SPACES OF LOWER SEMICONTINUOUS SET-VALUED MAPS I

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ABSTRACT. We introduce a lower semicontinuous analog, $L^-(X)$, of the well-studied space of upper semicontinuous set-valued maps with nonempty compact interval images. Because the elements of $L^-(X)$ contain continuous selections, the space $C(X)$ of real-valued continuous functions on X can be used to establish properties of $L^-(X)$, such as the two interrelated main theorems. The first of these theorems, the Extension Theorem, is proved in this Part I. The Extension Theorem says that for binormal spaces X and Y, every bimonotone homeomorphism between $C(X)$ and $C(Y)$ can be extended to an ordered homeomorphism between $L^-(X)$ and $L^-(Y)$. The second main theorem, the Factorization Theorem, is proved in Part II. The Factorization Theorem says that for binormal spaces X and Y, every ordered homeomorphism between $L^-(X)$ and $L^-(Y)$ can be characterized by a unique factorization.

1. Introduction

Spaces of set-valued maps under a hyperspace topology are often spaces of upper semicontinuous maps. One reason for this is that the graphs of such maps are closed sets, which allows the topology on the space of such maps to be a Hausdorff topology. However, lower semicontinuous set-valued maps have a nice property that upper semicontinuous set-valued maps may not have. A lower semicontinuous set-valued map from a normal space to a separable Banach space has a continuous selection ([23]). That is, its graph contains the graph of a continuous function. This suggests that spaces of lower semicontinuous set-valued maps would be more naturally related to spaces of continuous functions than are the spaces of upper semicontinuous set-valued maps.

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In defining an appropriate space of lower semicontinuous set-valued maps, care must be taken to make sure that such a multifunction space is a completely regular Hausdorff space, despite the fact that the graphs of such maps will not necessarily be closed. There are several ways that one could define such a multifunction space. Our approach will use a fairly restrictive definition that gives a space that we denote by $L^-(X)$, which is a subset of the set of lower semicontinuous maps with values that are nonempty compact intervals in the space $\mathbb{R}$ of real numbers. This allows us to relate $L^-(X)$ to the well-studied space $L(X)$ of upper semicontinuous maps with values that are nonempty compact intervals in $\mathbb{R}$. In addition, we can then establish our two interrelated main theorems: the Extension Theorem found in this Part I, and the Factorization Theorem found in the following Part II. These theorems are similar to, and in some ways more general than, the corresponding theorems for $L(X)$ found in [22]. In this Part I, when we refer to a result in Part II, we prefix its number with a II.

The references contain a selection of papers that involve $L(X)$ and more general spaces of upper semicontinuous set-valued maps. We refer to Beer [1] for basic facts about set-valued maps and hyperspaces, and we refer to Engelking [7] for general topological facts. Finally, we will assume that all of our topological spaces are completely regular Hausdorff spaces.

2. Definition of $L^-(X)$

A set-valued map, or multifunction, from space X to space Y is a function that assigns to each element of X a subset of Y. If F is such a map from X to Y, then the graph of F is the set $\{\langle x, y\rangle \in X \times Y : \ y \in F(x)\}$. On the other hand, if F is any subset of $X \times Y$ and $x \in X$, we define $F(x) = \{y \in Y : \ \langle x, y\rangle \in F\}$. We see that F is the graph of a set-valued map whose value at each x is $F(x)$. In this way, we identify set-valued maps with their graphs.

If F is a set-valued map from X to Y (equivalently, F is a subset of $X \times Y$), then F is called *upper semicontinuous* (usc) provided that for each $x \in X$ and open subset V of Y containing $F(x)$, there exists a neighborhood U of x such that $F(x') \subseteq V$ for all $x' \in U$. On the other hand, F is called *lower semicontinuous* (lsc) provided that for each $x \in X$ and open subset V of Y intersecting $F(x)$, there exists a neighborhood U of x such that $F(x') \cap V \neq \emptyset$ for all $x' \in U$. A usc map F is called a usco map (see Christensen [3]) provided that $F(x)$ is a nonempty compact set for all $x \in X$. In addition, a usco map F is called a cusco map if $F(x)$ is connected for all $x \in X$. The obvious analogs, lsco maps and clsco maps, are defined by replacing upper semicontinuous with lower semicontinuous.

We will restrict our attention to $Y = \mathbb{R}$. In particular, we will be working with a lower semicontinuous analog to the space $L(X)$ of cusco maps from X

to $\mathbb{R}$, whose topological properties have been studied in [11], [18], [19], [20]. To obtain this analog, that we call $L^-(X)$, we want to restrict the set of clsco maps from X to $\mathbb{R}$ in such a way that $L^-(X)$ is a Hausdorff space under the Vietoris topology. To this end, let us first observe two properties that members of $L(X)$ have. We say that a subset F of $X \times \mathbb{R}$ is *locally bounded* provided that for each $x \in X$, there exist a neighborhood U of x and an $a \in \mathbb{R}$ such that $F(x') \subseteq [-a, a]$ for all $x' \in U$. Then every member of $L(X)$ is locally bounded and closed in $X \times \mathbb{R}$. This contrasts with the fact that a clsco map from X to $\mathbb{R}$ may be neither locally bounded nor closed in $X \times \mathbb{R}$. Also the closure of such a clsco map may not be lower semicontinuous, even though its closure is always upper semicontinuous. On the other hand, an open subset of $X \times \mathbb{R}$ is always lsc, but is in general not usc.

In order to both define $L^-(X)$ and to work with the Vietoris topology on this space, it will be useful to define the family $\mathscr{L}(X)$ of lsc subsets F of $X \times \mathbb{R}$ such that $F(x)$ is a nonempty interval for each $x \in X$. Then for every $F \in \mathscr{L}(X)$, let $F_{\max}$ be the union of all $G \in \mathscr{L}(X)$ such that $F \subseteq G \subseteq \overline{F}$ (the closure taken in $X \times \mathbb{R}$).

Lemma 2.1. *For each $F \in \mathscr{L}(X)$, $F_{\max} \in \mathscr{L}(X)$ and $F_{\max}(x)$ is closed for all $x \in X$.*

Proof. To show that $F_{\max}$ is lsc, let $x \in X$ and let O be an open subset of $\mathbb{R}$ such that $F_{\max}(x) \cap O \neq \emptyset$. then $F_{\max}$ contains a $G \in \mathscr{L}(X)$ such that $F \subseteq G \subseteq \overline{F}$ and $G(x) \cap O \neq \emptyset$. Now G is lsc, so that x has a neighborhood U such that $G(x') \cap O \neq \emptyset$ for all $x' \in U$. But then $F_{\max}(x') \cap O \neq \emptyset$ for all $x' \in U$, showing that $F_{\max}$ is lsc.

If $x \in X$, $F(x) \neq \emptyset$, and hence $F_{\max}(x) \neq \emptyset$. To show that $F_{\max}(x)$ is an interval, let $a, b, c \in \mathbb{R}$ with $a < b < c$ and $a, c \in F_{\max}(x)$. Then $F_{\max}$ contains $G, H \in \mathscr{L}(X)$ such that $F \subseteq G \subseteq \overline{F}$, $F \subseteq H \subseteq \overline{F}$, $a \in G(x)$ and $c \in H(x)$. Now $F(x)$, $G(x)$ and $H(x)$ are connected with $F(x) \subseteq G(x)$ and $F(x) \subseteq H(x)$. Therefore, $G(x) \cup H(x)$ is connected, so that $b \in G(x) \cup H(x) \subseteq F_{\max}(x)$. This shows that $F_{\max}(x)$ is an interval.

Finally, to show that for each $x \in X$, $F_{\max}(x)$ is closed in $\mathbb{R}$, define

$$F'_{\max} = \big\{ \langle x, t \rangle : \ x \in X \text{ and } t \in \overline{F_{\max}(x)} \, \big\},$$

the closure taken in $\mathbb{R}$. Then each $F'_{\max}(x) = \overline{F_{\max}(x)}$ is an interval because $F_{\max}(x)$ is an interval. To show that $F'_{\max}$ is in $\mathscr{L}(X)$, it remains to show that it is lsc. To this end, let $x \in X$ and let O be an open subset of $\mathbb{R}$ such that $F'_{\max}(x) \cap O \neq \emptyset$. Then $\overline{F_{\max}(x)} \cap O \neq \emptyset$, so that $F_{\max}(x) \cap O \neq \emptyset$. Since $F_{\max}$ is lsc, x has a neighborhood U such that $F_{\max}(x') \cap O \neq \emptyset$ for all $x' \in U$. But then $F'_{\max}(x') \cap O \neq \emptyset$ for all $x' \in U$, showing that $F'_{\max}$ is lsc. Then $F'_{\max} \in \mathscr{L}(X)$, and clearly $F \subseteq F'_{\max} \subseteq \overline{F}$. Therefore, $F'_{\max} \subseteq F_{\max}$, so that $F_{\max} = F'_{\max}$. This shows that $F_{\max}(x)$ is closed in $\mathbb{R}$ for all $x \in X$. $\square$

Lemma 2.2. *Let $F \in \mathscr{L}(X)$, and let G be a lsc subset of $X \times \mathbb{R}$ with $F \subseteq G \subseteq \overline{F}$. Then there exists an $H \in \mathscr{L}(X)$ such that $G \subseteq H \subseteq \overline{F}$.*

Proof. Let us define

$$H = \{\langle x, t\rangle : \ x \in X \text{ and } t \in \mathrm{ch}(G(x))\}$$

where each $\mathrm{ch}(G(x))$ is the convex hull of $G(x)$ in $\mathbb{R}$ (i.e., the intersection of the intervals containing $G(x)$). Since each $H(x)$ is a nonempty interval, to show that $H \in \mathscr{L}(X)$, we must show that H is lsc. So let $x \in X$ and let O be an open subset of $\mathbb{R}$ such that there is some $s \in H(x) \cap O$; we may assume that $s \notin G(x)$. Then $G(x) \cap (-\infty, s) \neq \emptyset$ and $G(x) \cap (s, \infty) \neq \emptyset$. Now G is lsc, so that x has a neighborhood U such that for each $x' \in U$, there exist an $r \in G(x') \cap (-\infty, s)$ and a $t \in G(x') \cap (s, \infty)$. Then $s \in (r, t) \subseteq H(x')$, and hence $H(x') \cap O \neq \emptyset$ for all $x' \in U$. This shows that H is lsc, and thus $H \in \mathscr{L}(X)$.

To show that $H \subseteq \overline{F}$, let $\langle x, s\rangle \in H \setminus G$, and let $U \times O$ be a neighborhood of $\langle x, s\rangle$ in $X \times \mathbb{R}$. Then $G(x) \cap (-\infty, s) \neq \emptyset$ and $G(x) \cap (s, \infty) \neq \emptyset$. Now G is lsc, so that x has a neighborhood U' contained in U such that $G(x') \cap (-\infty, s) \neq \emptyset$ and $G(x') \cap (s, \infty) \neq \emptyset$ for all $x' \in U'$. Since $G \subseteq \overline{F}$, there exists a $\langle x', r'\rangle \in U' \times (-\infty, s) \cap F$. Using the lsc property of F, we obtain a neighborhood U'' of x' contained in U' such that $F(x'') \cap (-\infty, s) \neq \emptyset$ for all $x'' \in U''$. Now there exists a $\langle x'', t\rangle \in U'' \times (s, \infty) \cap F$. Let $r \in F(x'') \cap (-\infty, s)$. By the connectivity of $F(x'')$, we have $s \in (r, t) \subseteq F(x'')$. Then $\langle x'', s\rangle \in U \times O \cap F$, so that $\langle x, s\rangle \in \overline{F}$. This shows that $H \subseteq \overline{F}$. □

If we define a subset F of $X \times \mathbb{R}$ to be *maximally lower semicontinuous* (maximally lsc) provided that it is lsc and is not a proper subset of any lsc subset of its closure, then Lemma 2.1 implies that each $F \in \mathscr{L}(X)$ is densely contained in a maximally lsc member of $\mathscr{L}(X)$, namely $F_{\max}$. In fact, we have the following.

Lemma 2.3. *For each $F \in \mathscr{L}(X)$, F is maximally lsc if and only if $F = F_{\max}$.*

Proof. Let $F \in \mathscr{L}(X)$ be maximally lsc. Since $F \subseteq F_{\max}$ and $F_{\max}$ is an lsc subset of $\overline{F}$, we have $F = F_{\max}$. Conversely, let $F \in \mathscr{L}(X)$ be such that $F = F_{\max}$. To show that F is maximally lsc, let G be an lsc set with $F \subseteq G \subseteq \overline{F}$. We need to show that $F = G$. Lemma 2.2 gives us an $H \in \mathscr{L}(X)$ such that $F \subseteq G \subseteq H \subseteq \overline{F}$. It now follows that $H \subseteq F_{\max}$. Therefore, $G \subseteq H \subseteq F_{\max} = F \subseteq G$, so that $F = G$. This completes the argument that F is maximally lsc. □

The next lemma gives a convenient characterization of F being maximally lsc. If $F \in \mathscr{L}(X)$ and $\langle x, t\rangle \in X \times \mathbb{R}$, we define $\langle x, t\rangle$ to be an *almost lsc point of F* provided that for every neighborhood O of t, there exists a neighborhood U of x such that every nonempty open subset of U contains a point x' with $F(x') \cap O \neq \emptyset$.

Lemma 2.4. *Let $F \in \mathscr{L}(X)$. Then $F_{\max}$ is equal to the set of almost lsc points of F. It follows that F is maximally lsc if and only if it is equal to its set of almost lsc points.*

Proof. Let F' be the set of almost lsc points of F. To show that $F' \subseteq F_{\max}$, let $\langle x_0, t_0 \rangle \in F'$. Suppose, by way of contradiction, that $\langle x_0, t_0 \rangle \notin F_{\max}$; say $t_0 > d$ where $d = \sup F_{\max}(x_0)$. Let $s \in (d, t_0)$. Now x_0 has a neighborhood U_0 such that for every $x \in U_0$, $F_{\max}(x) \cap (-\infty, s) \neq \emptyset$, and such that every nonempty open subset of U_0 contains a point x with $F_{\max}(x) \cap (s, \infty) \neq \emptyset$. Let $a = \inf F_{\max}$, and define

$$F_0 = F_{\max} \cup \big\{ \langle x, t \rangle \in U_0 \times \mathbb{R} : \ a(x) \leq t \leq s \big\}.$$

Obviously $F_{\max}$ is a proper subset of F_0.

To obtain a contradiction, we need to show that $F_0 \in \mathscr{L}(X)$ and $F_0 \subseteq \overline{F}$. Clearly each $F_0(x)$ is a nonempty interval. Since $F_{\max}$ is lsc, to show that F_0 is lsc, we need only consider an $x \in U_0$. Let O be an open interval with $[a(x), s] \cap O \neq \emptyset$. For each $x' \in U_0$, if $F_{\max}(x') \cap O = \emptyset$, then $a(x') < t$ for all $t \in O$. It follows that $F_0(x') \cap O \neq \emptyset$, showing that F_0 is lsc.

To show that $F_0 \subseteq \overline{F}$, let $\langle x, t \rangle \in F_0$. Since $F_{\max} \subseteq \overline{F}$, we may assume that $x \in U_0$ and $t \in [a(x), s]$. It suffices to show that $\langle x, t \rangle \in \overline{F_{\max}}$. For each neighborhood U of x contained in U_0, there exists an $x_U \in U$ such that $F_{\max}(x_U) \cap (-\infty, s) \neq \emptyset$, and hence $s \in F_{\max}(x_U)$. Now the net $\langle x_U \rangle$ converges to x, so that $\langle x, s \rangle \in \overline{F_{\max}}$. Since also $\langle x, a(x) \rangle \in \overline{F_{\max}}$, it follows that $\langle x, t \rangle \in \overline{F_{max}}$. This completes the argument that $F_0 \subseteq \overline{F}$, which in turn completes the contradiction argument and shows that $F' \subseteq F_{\max}$.

Conversely, to show that $F_{\max} \subseteq F'$, let $\langle x, t \rangle \in F_{\max}$. Suppose, by way of contradiction, that $\langle x, t \rangle \notin F'$. Then t has a neighborhood O such that for every neighborhood U of x, there exists a nonempty open subset U' of U with $F(x') \cap O = \emptyset$ for all $x' \in U'$. Since $F_{\max}$ is lsc, x has a neighborhood U_0 such that for every $x' \in U_0$, $F_{\max}(x') \cap O \neq \emptyset$. So there exists a nonempty open subset U' of U_0 such that for every $x' \in U'$, $F(x') \cap O = \emptyset$. Then $U' \times O \cap F = \emptyset$. But $U' \times O \cap F_{\max} \neq \emptyset$, so that $F_{\max} \not\subseteq \overline{F}$; which is a contradiction. Therefore, $F_{\max} \subseteq F'$, which finishes the argument that $F_{\max} = F'$. □

We now define $L^-(X)$ to be the set of clsco maps from X to $\mathbb{R}$ that are maximally lsc and locally bounded. Specifically, $F \in L^-(X)$ if and only if

(1) $F \subseteq X \times \mathbb{R}$;

(2) $F(x)$ is a nonempty compact interval for all $x \in X$;

(3) F is maximally lsc; and

(4) F is locally bounded.

Note that $L^-(X)$ is a subset of $\mathscr{L}(X)$.

Unless otherwise indicated, the topology on $L^-(X)$ will be the Vietoris topology, where the basic open subsets of $L^-(X)$ are the subsets of the form $W^+ \cap W_1^- \cap \cdots \cap W_n^-$ where $W, W_1, \dots, W_n$ are open subsets of $X \times \mathbb{R}$, $W^+ = \{F \in L^-(X) : \ F \subseteq W\}$, and each $W_i^- = \{F \in L^-(X) : \ F \cap W_i \neq \emptyset\}$. Also the topology generated by the sets of the form W^+ is called the upper Vietoris topology, and the topology generated by the sets of the form W^- is called the lower Vietoris topology.

We can use $\mathscr{L}(X)$ to give another useful base for the Vietoris topology on $L^-(X)$.

Lemma 2.5. *A base for $L^-(X)$ consists of sets of the form $W^+ \cap W_1^- \cap \cdots \cap W_n^-$ where W is an open element of $\mathscr{L}(X)$ with $W(x)$ bounded for all $x \in X$, and where each W_i is an open subset of W such that $W_i = U_i \times O_i$ with O_i an interval.*

Proof. Let $F \in L^-(X)$ and let $W^+ \cap W_1^- \cap \cdots \cap W_n^-$ contain F where $W, W_1, \dots, W_n$ are open subsets of $X \times \mathbb{R}$. Define

$$W' = \{\langle x, s\rangle \in X \times \mathbb{R} : \ \inf F(x) - 1 < s < \sup F(x) + 1\},$$

which is a subset of $X \times \mathbb{R}$ containing F. The fact that W' is open in $X \times \mathbb{R}$ follows from Lemmas 3.1 and 3.2 in the next section. Now for each $x \in X$, $F(x)$ is compact, so there exist neighborhood U_x of x and open interval O_x such that $\{x\} \times F(x) \subseteq U_x \times O_x \subseteq W \cap W'$. Define

$$W_0' = \bigcup \{U_x \times O_x : \ x \in X\},$$

which is open in $X \times \mathbb{R}$ and $F \subseteq W_0' \subseteq W \cap W'$. Note that each $W_0'(x)$ is bounded since $W'(x)$ is bounded. However, $W_0'(x)$ may not be connected for some $x \in X$. So for each $x \in X$, let $W_0(x)$ be the component of $W_0'(x)$ that contains the nonempty set $F(x)$. Then define

$$W_0 = \{\langle x, t\rangle : \ x \in X \text{ and } t \in W_0(x)\}.$$

Now $W_0(x)$ is nonempty and connected for each $x \in X$, and $F \subseteq W_0 \subseteq W \cap W'$.

To show that $W_0 \in \mathscr{L}(X)$, it remains to show that W_0 is open in $X \times \mathbb{R}$. So let $\langle x, t\rangle \in W_0$, and let $s \in F(x)$. Since $\{x\} \times W_0(x)$ is connected and covered by $\{U_{x'} \times O_{x'} : \ x' \in X\}$, there exist $x_1, \dots, x_k \in X$ such that $s \in O_{x_1}$, $t \in O_{x_k}$, $O_{x_i} \cap O_{x_{i+1}} \neq \emptyset$ for $i = 1, \dots, k-1$, and $x \in U_{x_i}$ for $i = 1, \dots, k$. Let $U = U_{x_1} \cap \cdots \cap U_{x_k}$ and let $O = O_{x_i} \cup \cdots \cup O_{x_k}$. Note that O is a connected subset of $W_0'(x)$ that intersects $F(x)$, so that $O \subseteq W_0(x)$. Now since F is lsc, x has a neighborhood U' contained in U such that $F(x') \cap O_{x_1} \neq \emptyset$ for all $x' \in U'$. Then for every $x' \in U'$, O is a connected subset of $W_0'(x')$ that intersects $F(x')$, so that $O \subseteq W_0(x')$. Therefore, $U' \times O$ is a neighborhood of $\langle x, t\rangle$ that is contained in W_0, showing that W_0 is open in $X \times \mathbb{R}$, and hence W_0 is an open element of $\mathscr{L}(X)$ with $W_0(x)$ bounded for all $x \in X$.

Finally, let $\langle x_i, t_i\rangle \in F \cap W_i$ for $i = 1, \dots, n$. Then for each i, there exist neighborhood U_{i0} of x_i and open interval O_{i0} containing t_i such that $U_{i0} \times O_{i0} \subseteq W_i \cap W_0$. For each $i = 1, \dots, n$, let $W_{i0} = U_{i0} \times O_{i0}$. Then

$$F \in W_0^+ \cap W_{10}^- \cap \cdots \cap W_{n0}^- \subseteq W^+ \cap W_1^- \cap \cdots \cap W_n^-,$$

which shows that sets like $W_0^+ \cap W_{10}^- \cap \cdots \cap W_{n0}^-$ form a base for $L^-(X)$. □

We end this section with a lemma that will help us show that $L^-(X)$ is a Hausdorff space.

Lemma 2.6. *If $F, G \in L^-(X)$ are such that $\overline{F} = \overline{G}$, then $F = G$.*

Proof. Since the union of lsc sets is lsc, $F \cup G$ is a lsc set. Now $F \subseteq F \cup G \subseteq \overline{F}$, so that by Lemma 2.2, there exists an $H \in \mathscr{L}(X)$ such that $F \cup G \subseteq H \subseteq \overline{F}$. Then $F_{\max} = F \subseteq H \subseteq \overline{F}$, which implies that $H = F_{\max} = F$. Similarly, $H = G_{\max} = G$, so that $F = G$. □

3. Relation to C(X)

The set $C(X)$ of continuous functions from X to $\mathbb{R}$ is a subset of both $L(X)$ and $L^-(X)$. One can show that, as a subspace of both $L(X)$ and $L^-(X)$ with the Vietoris topology, the subspace topology induced on $C(X)$ can be generated by the basic open sets of the form W^+ where W is an open subset of $X \times \mathbb{R}$ and $W^+ = \{f \in C(X) : f \subseteq W\}$. In particular, the Vietoris topology on $C(X)$ is the same as the upper Vietoris topology on $C(X)$. This topology on $C(X)$ is also called the graph topology ([24]), and will be the topology that $C(X)$ will have throughout this study. From Lemma 2.4, we see that $C(X)$ has as a base for its topology the family of sets of the form W^+ where W is an open element of $\mathscr{L}(X)$.

The terms upper semicontinuous and lower semicontinuous, when applied to real-valued functions as opposed to set-valued maps, has a well-known different definition. In particular, a function f from X to $\mathbb{R}$ is called *upper semicontinuous* (respectively, *lower semicontinuous*) provided that for every $x \in X$ and $\varepsilon > 0$, there exists a neighborhood U of x such that $f(U) \subseteq (-\infty, f(x)+\varepsilon)$ (respectively, $f(U) \subseteq (f(x)-\varepsilon, \infty)$). Let $USC(X)$ and $LSC(X)$ denote such upper semicontinuous real-valued functions and lower semicontinuous real-valued functions, respectively. Although a member of $USC(X)$ (or $LSC(X)$) can be considered as a set-valued map, it will not necessarily be usc (or lsc) in the sense that was previously defined for set-valued maps. However, $USC(X)$ and $LSC(X)$ are related to $L(X)$ and $L^-(X)$ as follows.

For each F in either $L(X)$ or $L^-(X)$, we have $F(x)$ nonempty and bounded for all $x \in X$. Then there are two real-valued functions $\sup F$ and $\inf F$ defined on X by $\sup F(x) = \sup\{t : t \in F(x)\}$ and $\inf F = \inf\{t : t \in F(x)\}$ for all

$x \in X$. Because each $F(x)$ is compact, $\sup F \subseteq F$ and $\inf F \subseteq F$. We can think of $\sup F$ as the upper boundary of F and $\inf F$ as the lower boundary of F.

Lemma 3.1. *Let $F \subseteq X \times \mathbb{R}$ such that $F(x)$ is nonempty and bounded for all $x \in X$. If F is usc, then $\sup F \in USC(X)$ and $\inf F \in LSC(X)$. If F is lsc, then $\sup F \in LSC(X)$ and $\inf F \in USC(X)$.*

Proof. Since this follows directly from definitions, we only illustrate with a proof of one case. Let F be lsc, let $x \in X$ and let $\varepsilon > 0$. Then there exists a $t \in F(x)$ with $t > \sup F(x) - \varepsilon$. By the lsc property of F, x has a neighborhood U such that for every $x' \in U$, $F(x') \cap (\sup F(x) - \varepsilon, \infty) \neq \emptyset$. Then $\sup F(U) \subseteq (\sup F(x) - \varepsilon, \infty)$, so that $\sup F \in LSC(X)$. $\square$

Since $L^-(X) \subseteq \mathscr{L}(X)$, we see that for each $F \in L^-(X)$, the upper boundary of F is a lower semicontinuous real-valued function, and the lower boundary of F is an upper semicontinuous real-valued function.

Lemma 3.2. *If $a \in USC(X)$ and $b \in LSC(X)$ with $a < b$, then $\big\{\langle x, s\rangle \in X \times \mathbb{R} : a(x) < s < b(x)\big\}$ is open in $X \times \mathbb{R}$.*

Proof. Let $W = \big\{\langle x, s\rangle \in X \times \mathbb{R} : \ a(x) < s < b(x)\big\}$, let $\langle x, s\rangle \in W$, and let $r, t \in \mathbb{R}$ with $a(x) < r < s < t < b(x)$. Then x has a neighborhood U such that $a(U) \subseteq (-\infty, r)$ and $b(U) \subseteq (t, \infty)$. So $\langle x, s\rangle \in U \times (r, t) \subseteq W$, showing that W is open in $X \times \mathbb{R}$. $\square$

The following theorem (see [6], [26], or [7, Problems 2.7.4, 5.5.20]) allows us to use $C(X)$ to study $L^-(X)$. Here the term *binormal space* means a normal countably paracompact space.

Theorem 3.3. *The following are true for a T_1-space X.*

(1) *The space X is normal if and only if for every $f \in USC(X)$ and $g \in LSC(X)$ with $f \leq g$, there exists an $h \in C(X)$ such that $f \leq h \leq g$.*
(2) *The space X is binormal if and only if for every $f \in USC(X)$ and $g \in LSC(X)$ with $f < g$, there exists an $h \in C(X)$ such that $f < h < g$.*

From Theorem 3.3, we see that if X is a normal space and $F \in L^-(X)$, then there exists an $f \in C(X)$ such that $\inf F \leq f \leq \sup F$, and therefore $f \subseteq F$. So every member of $L^-(X)$ contains a continuous selection. On the other hand, the members of $L(X)$ do not in general contain continuous selections.

Theorem 3.3 allows us to work with the topology on $C(X)$. In particular, let X be a binormal space, let $h \in C(X)$, and let W^+ be a basic neighborhood of h where W is an open element of $\mathscr{L}(X)$ with $W(x)$ bounded for all $x \in X$. Then $\inf W < h < \sup W$, so there exist $f, g \in C(X)$ with $\inf W < f < h < g < \sup W$. Then $\big\{h' \in C(X) : \ f < h' < g\big\}$ is a neighborhood of h in $C(X)$ contained in W^+. This shows that for binormal X, the Vietoris (graph) topology on $C(X)$ is equal to the fine topology on $C(X)$ (see [4, Section 2]).

Theorem 3.3 also allows us to work with the topology on $L(X)$, and in so doing we can relate $L^-(X)$ to $L(X)$.

Proposition 3.4. *If X is a binormal space, then the function $\iota\colon L^-(X) \to L(X)$ defined by $\iota(F) = \overline{F}$ for all $F \in L^-(X)$ is a well-defined continuous injection.*

Proof. To show that ι is well-defined, first note that $\overline{F}$ is locally bounded, and so $\overline{F}(x)$ is a nonempty closed bounded set for all $x \in X$. To finish showing that $\overline{F} \in L(X)$, as indicated by [11, Lemma 3.1], we need to show that each $\overline{F}(x)$ is connected. To this end, let $x \in X$, let $[a,b] = F(x)$, and let $t \in \overline{F}(x)$. We only consider the case that $t > b$ since the case that $t < a$ is similar. It suffices to show that $[b,t] \subseteq \overline{F}(x)$, so let $s \in (b,t)$. Since F is lsc, x has a neighborhood U_0 such that for every $x_0 \in U_0$, $F(x_0) \cap (-\infty, s) \neq \emptyset$. Also since $\langle x, t\rangle \in \overline{F}$, for every neighborhood U of x contained in U_0, there exists a $\langle x_U, t_U\rangle \in F \cap U \times (s,\infty)$. Then for this x_U, let $r_U \in F(x_U) \cap (-\infty, s)$. By the connectedness of $F(x_U)$, $s \in (r_U, t_U) \subseteq F(x_U)$. So we have a net $\langle x_U, s\rangle_U$ in F that converges to $\langle x, s\rangle$, showing that $\langle x, s\rangle \in \overline{F}$, and hence $[b,t] \subseteq \overline{F}(x)$. This completes the argument that $\overline{F}(x)$ is connected, and we see that ι is a well-defined function.

The fact that ι is one-to-one follows from Lemma 2.6. Finally, to show that ι is continuous, let $F \in L^-(X)$ and $\overline{F} \in W^+ \cap W_1^- \cap \cdots \cap W_n^-$ in $L(X)$ with $W(x)$ bounded for all $x \in X$. Since $\inf W \in USC(X)$, $\inf \overline{F} \in LSC(X)$, and $\inf W < \inf \overline{F}$, the binormality of X insures, by Theorem 3.3, that there exists an $f \in C(X)$ such that $\inf W < f < \inf \overline{F}$. Similarly, there exists a $g \in C(X)$ such that $\sup \overline{F} < g < \sup W$. Let

$$W_0 = \big\{\langle x, t\rangle \in X \times \mathbb{R} :\ f(x) < t < g(x)\big\},$$

which is an open subset of $X \times \mathbb{R}$. Also observe that $\overline{F} \subseteq W_0$ and $\overline{W_0} \subseteq W$. Now W_0^+ is a neighborhood of F in $L^-(X)$, and if $G \in L^-(X)$ with $G \in W_0^+$, we have $\overline{G} \subseteq \overline{W_0} \subseteq W$, so that $\iota(G) \in W^+$. Finally, note that since $\overline{F} \cap W_i \neq \emptyset$ for each $i = 1, \dots, n$, then $F \cap W_i \neq \emptyset$ for each i. Certainly if $G \in L^-(X)$ with $G \cap W_i \neq \emptyset$ for each i, then $\overline{G} \cap W_i \neq \emptyset$ for each i. Therefore, $W_0^+ \cap W_1^- \cap \cdots \cap W_n^-$ is a neighborhood of F in $L^-(X)$ that maps into $W^+ \cap W_1^- \cap \cdots \cap W_n^-$ under ι, showing the continuity of ι. $\square$

We might note that, in general, the continuous injection $\iota\colon L^-(X) \to L(X)$ in Proposition 3.4 is not a bijection and is not an embedding.

Our last two lemmas in this section give us additional tools for using $C(X)$ to work with $L^-(X)$.

Lemma 3.5. *Let X be a normal space, and let F be an lsc subset of $X \times \mathbb{R}$ such that $F(x)$ is a nonempty compact interval for all $x \in X$. Then $F = \bigcup\{f \in C(X) :\ f \subseteq F\}$.*

P r o o f. We certainly have $\bigcup\{f \in C(X):\ f \subseteq F\} \subseteq F$. To show containment in the other direction, let $\langle x,t\rangle \in F$. Define $a\colon X \to \mathbb{R}$ and $b\colon X \to \mathbb{R}$ as follows. For every $x' \in X$ with $x' \neq x$, let $a(x') = \inf F(x')$ and $b(x') = \sup F(x')$, and let $a(x) = b(x) = t$. Since $\inf F(x) \leq t \leq \sup F(x)$, and $\inf F \in USC(X)$ and $\sup F \in LSC(X)$ by Lemma 3.1, we see that $a \in USC(X)$ and $b \in LSC(X)$. Also since X is normal and $a \leq b$, Theorem 3.3 tells us that there exists an $f \in C(X)$ such that $a \leq f \leq b$. Then $\langle x,t\rangle \in f \subseteq F$, showing that $F = \bigcup\{f \in C(X):\ f \subseteq F\}$. □

Lemma 3.6. *Let X be a normal space, and let $F \in \mathscr{L}(X)$ be locally bounded. Then $F_{\max} = \bigcup\{f \in C(X):\ f \subseteq \overline{F}\}$.*

P r o o f. Since F is locally bounded, it follows that $\overline{F}(x)$ is bounded for all $x \in X$. Therefore, $F_{\max}$ is a lsc subset of $X \times \mathbb{R}$ such that $F_{\max}(x)$ is nonempty and bounded for all $x \in X$, so that by Lemma 3.5, we have $F_{\max} = \bigcup\{f \in C(X):\ f \subseteq F_{\max}\} \subseteq \bigcup\{f \in C(X):\ f \subseteq \overline{F}\}$.

To show containment in the other direction, let $f \in C(X)$ with $f \subseteq \overline{F}$. Now $F_{\max}$ is lsc, and clearly f is lsc. Since it is evident that the union of two lsc subsets of $X \times \mathbb{R}$ is lsc, we see that $F_{\max} \cup f$ is lsc. But $F_{\max}$ is maximally lsc and $F_{\max} \cup f \subseteq \overline{F}$. It follows that $F_{\max} \cup f = F_{\max}$, and $f \subseteq F_{\max}$. □

Now from these two lemmas we see that when X is normal, each F in $L^-(X)$ has the property that $F = \bigcup\{f \in C(X):\ f \subseteq F\} = \bigcup\{f \in C(X):\ f \subseteq \overline{F}\}$.

4. Separation properties

Recall that we are assuming that our spaces are completely regular Hausdorff spaces. In particular, the regularity of X now ensures that $L^-(X)$ is Hausdorff, as shown in our first proposition.

Proposition 4.1. *The space $L^-(X)$ is a Hausdorff space.*

P r o o f. Let $F, G \in L^-(X)$ with $F \neq G$. Then by Lemma 2.6, $\overline{F} \neq \overline{G}$; say there is an $\langle x,t\rangle \in \overline{G} \setminus \overline{F}$. Since $X \times \mathbb{R}$ is regular, there exist disjoint open subsets W and W_1 of $X \times \mathbb{R}$ such that $\overline{F} \subseteq W$ and $\langle x,t\rangle \in W_1$. Note that $G \cap W_1 \neq \emptyset$. Then W^+ and W_1^- are disjoint open subsets of $L^-(X)$ containing F and G, respectively. □

In order to show that $L^-(X)$ is completely regular, we need the following two lemmas. In the first, for an $F \in \mathscr{L}(X)$, the notation F^+ is used for the set of all $G \in L^-(X)$ such that $G \subseteq F$.

Lemma 4.2. *For each $W \in \mathscr{L}(X)$ with $W(x)$ bounded for all $x \in X$, the following are true.*

(1) *The closure of* W^+ *in* $L^-(X)$ *is contained in* $W^+_{\max}$.

(2) *The set* $W^+_{\max}$ *is closed in* $L^-(X)$.

Proof. Statement (2) follows from statement (1) because $(W_{\max})_{\max} = W_{\max}$. For statement (1), let $F \in L^-(X)$ be such that $F \notin W^+_{\max}$. Then there is some $\langle x, t\rangle \in F \setminus W_{\max}$. Since $W_{\max}(x)$ is closed, either $t > \sup W_{\max}(x)$ or $t < \inf W_{\max}(x)$; say the former. Let $s \in \mathbb{R}$ with $\sup W_{\max}(x) < s < t$, and let O_1 and O_2 be disjoint open intervals with $t \in O_1$ and $s \in O_2$. Since F is lsc, x has a neighborhood U_1 such that for all $x' \in U_1$, $F(x') \cap O_1 \neq \emptyset$. By Lemma 2.4, $\langle x, s\rangle$ is not an almost lsc point of W, so that there exist neighborhood O_3 of s contained in O_2 and nonempty open subset U_2 of U_1 such that $U_2 \times O_3 \cap W = \emptyset$. But then $U_2 \times O_1 \cap W = \emptyset$. So define $W_0 = U_2 \times O_1$. Then $F \in W_0^-$ and $W^+ \cap W_0^- = \emptyset$, showing that F is not in the closure of W^+ in $L^-(X)$. $\square$

Lemma 4.3. *If* $F \in L^-(X)$ *and* W *is an open element of* $\mathscr{L}(X)$ *with* $F \subseteq W$ *and* $W(x)$ *bounded for all* $x \in X$, *then there exists an open element* W' *of* $\mathscr{L}(X)$ *such that* $F \subseteq W'$ *and* $W'_{\max} \subseteq W$.

Proof. Let $a = \inf F$, $b = \sup F$, $c = \inf W$, and $d = \sup W$. Define $c_1 = (a+c)/2$ and $d_1 = (b+d)/2$, which, by Lemma 3.1, are in $USC(X)$ and $LSC(X)$, respectively. Then $c < c_1 < a \le b < d_1 < d$. Define $W_1 = \{\langle x, t\rangle \in X \times \mathbb{R} : c_1(x) < t < d_1(x)\}$, which, by Lemma 3.2, is an open subset of $X \times \mathbb{R}$ such that $F \subseteq W_1$. Also if $\hat{W}_1 = \{\langle x, t\rangle \in X \times \mathbb{R} : c_1(x) \le t \le d_1(x)\}$, then $\hat{W}_1 \subseteq W$.

Let $x \in X$ be momentarily fixed. Define $r(x) = (c_1(x) + a(x))/2$ and $s(x) = (b(x) + d_1(x))/2$. Let U_x be a neighborhood of x such that $U_x \times [r(x), s(x)] \subseteq W_1$, $c_1(U_x) \subseteq (-\infty, r(x))$, and $d_1(U_x) \subseteq (s(x), \infty)$. Since, by Proposition 2.4, $\langle x, c_1(x)\rangle$ and $\langle x, d_1(x)\rangle$ are not almost lsc points of F, there exist neighborhoods V_1 and V_2 of $c_1(x)$ and $d_1(x)$, respectively, such that for every neighborhood U of x contained in U_x, there exist nonempty open subsets O^1_U and O^2_U of U such that for all $x' \in O^1_U$, $F(x') \cap V_1 = \emptyset$ and for all $x' \in O^2_U$, $F(x') \cap V_2 = \emptyset$. For each neighborhood U of x contained in U_x, let $K^1_{x,U} = O^1_U \times (-\infty, c_1(x)]$ and $K^2_{x,U} = O^2_U \times [d_1(x), \infty)$. Note that $K^1_{x,U}$ and $K^2_{x,U}$ are disjoint from F. Define $K^1_x = \bigcup\{K^1_{x,U} :\ U$ is a neighborhood of x contained in $U_x\}$, $K^2_x = \bigcup\{K^2_{x,U} :$ U is a neighborhood of x contained in $U_x\}$, and $K_x = K^1_x \cup K^2_x$.

Now let x vary over X, and define $W' = W_1 \setminus \overline{K}$ where $K = \bigcup\{K_x :\ x \in X\}$. We need to show that $W' \in \mathscr{L}(X)$, $F \subseteq W'$, and $W'_{\max} \subseteq W$.

Since $F \subseteq W_1$, to show that $F \subseteq W'$, we need to show that $F \cap \overline{K} = \emptyset$. Suppose, by way of contradiction, that there exits a $\langle x, t\rangle \in F \cap \overline{K}$. Then there exists a net $\langle x_\lambda, t_\lambda\rangle_\lambda$ in K that converges to $\langle x, t\rangle$; we may assume that each $\langle x_\lambda, t_\lambda\rangle \in U_x \times (r(x), s(x))$. For each λ, there exists an $x'_\lambda \in X$ with $\langle x_\lambda, t_\lambda\rangle \in K_{x'_\lambda}$. Since we can use a subnet, if necessary, we may assume that each $\langle x_\lambda, t_\lambda\rangle \in K^2_{x'_\lambda}$; so then $d_1(x'_\lambda) \le t_\lambda$. Since each $x'_\lambda \in U_x$, $s(x) < d_1(x'_\lambda)$.

But then $t_\lambda < s(x) < d_1(x'_\lambda) \le t_\lambda$, which is a contradiction. This shows that $F \subseteq W'$, and that each $W'(x) \ne \emptyset$.

Now to show that $W' \in \mathscr{L}(X)$, it suffices to show that each $W'(x)$ is connected. Suppose, by way of contradiction, that for some $x \in X$, $W'(x)$ is not connected; say $r < s < t$ and $r, t \in W'(x)$ while $s \notin W'(x)$. Then $\langle x, s\rangle \in \overline{K}$, so that there exists a net $\langle x_\lambda, s_\lambda\rangle$ in K that converges to $\langle x, s\rangle$. Then for each λ, there exists an $x'_\lambda \in X$ with $\langle x_\lambda, s_\lambda\rangle \in K_{x'_\lambda}$. As before, we may assume that each $\langle x_\lambda, s_\lambda\rangle \in K^2_{x'_\lambda}$. Let $p = (s+t)/2$. Since W' is open, x has a neighborhood U such that for all $x' \in U$, $W'(x') \cap (p, \infty) \ne \emptyset$. Now there exists a λ such that $\langle x_\lambda, s_\lambda\rangle \in U \times (-\infty, p)$. But there exists some $t_\lambda \in W'(x_\lambda) \cap (p, \infty)$, so that since $s_\lambda < t_\lambda$, we have $\langle x_\lambda, t_\lambda\rangle \in K^2_{x'_\lambda}$. This contradicts the fact that $K^2_{x'_\lambda} \cap W' = \emptyset$. Therefore, each $W'(x)$ is connected, so that $W' \in \mathscr{L}(X)$.

It remains to show that $W'_{\max} \subseteq W$. Now $\hat{W}_1 \subseteq W$, so it suffices to show that $W'_{\max} \subseteq \hat{W}_1$. Let $\langle x, t\rangle \in X \times \mathbb{R} \setminus \hat{W}_1$; say $d_1(x) < t$. Let $q = (d_1(x)+t)/2$, and let $V = (q, \infty)$. For each neighborhood U of x contained in U_x, O^2_U is a nonempty open subset of U such that for all $x' \in O^2_U$, $\langle x', d_1(x)\rangle \in K^2_{x,U} \subseteq K_x \subseteq X \times \mathbb{R} \setminus W'$. Then for each $x' \in O^2_U$, $W'(x') \cap V = \emptyset$. Therefore, $\langle x, t\rangle$ is not an almost lsc point of W', so that by Proposition 2.4, $\langle x, t\rangle \notin W'_{\max}$. This shows that $W'_{\max} \subseteq \hat{W}_1 \subseteq W$, which finishes the proof. ☐

Proposition 4.4. *The space $L^-(X)$ is a completely regular space.*

Proof. Let $F \in L^-(X)$ and let $W_0^+ \cap W_1^- \cap \cdots \cap W_n^-$ be a basic open subset of $L^-(X)$ containing F such that W_0 is an open element of $\mathscr{L}(X)$ with each $W_0(x)$ bounded, as given by Lemma 2.5. By Lemma 4.3, there exists an open $W \in \mathscr{L}(X)$ such that $F \subseteq W$ and $W_{\max} \subseteq W_0$. For each $x \in X$, define $a(x) = \inf F(x)$, $b(x) = \sup F(x)$, $c(x) = \inf W_{\max}(x)$, and $d(x) = \sup W_{\max}(x)$. This defines $a, c \in USC(X)$ and $b, d \in LSC(X)$ such that $c < a \le b < d$. Now for each $t \in [0,1]$ and $x \in X$, define $p_t(x) = tc(x) + (1-t)a(x)$ and $q_t(x) = td(x) + (1-t)b(x)$. Then for each $t \in [0,1]$, define

$$F_t = \big\{\langle x, y\rangle \in X \times \mathbb{R} :\ p_t(x) \le y \le q_t(x)\big\}.$$

We now show that for each $t \in [0,1]$, $F_t = (F_t)_{\max}$. Suppose, by way of contradiction, that there exits a $\langle x_0, y_0\rangle \in (F_t)_{\max} \setminus F_t$; say $y_0 > q_t(x_0)$. Let $\varepsilon = y_0 - q_t(x_0)$. Since $(F_t)_{\max}$ and $W_{\max}$ are lsc, x_0 has a neighborhood U_0 such that for all $x \in U_0$, $(F_t)_{\max}(x) \cap (y_0 - \varepsilon/4, y_0 + \varepsilon/4) \ne \emptyset$ and $W_{\max}(x) \cap (d(x_0) - \varepsilon/4, d(x_0) + \varepsilon/4) \ne \emptyset$. Let $y_1 = b(x_0) + \varepsilon/4$, so that $\langle x_0, y_1\rangle \notin F$. Now $F = F_{\max}$, so that by Lemma 2.4, there exists an open subset $U \times (r, s)$ contained in $U_0 \times (y_1 - \varepsilon/4, y_1 + \varepsilon/4)$ with $U \times (r, s) \cap F = \emptyset$; in particular, $U \times (r, \infty) \cap F = \emptyset$. Then if $x \in U$, we have $b(x) < y_1 + \varepsilon/4 = b(x_0) + \varepsilon/2$. Also for $x \in U$, $d(x) < d(x_0) + \varepsilon/4$, so that $q_t(x) = td(x) + (1-t)b(x) < t(d(x_0) + \varepsilon/4) + (1-t)(b(x_0) + \varepsilon/2) = td(x_0) + (1-t)b(x_0) + t\varepsilon/4 + (1-t)\varepsilon/2 = q_t(x_0) + \varepsilon/2 - t\varepsilon/4 <$

$q_t(x_0) + \varepsilon/2 = y_0 - \varepsilon + \varepsilon/2 = y_0 - \varepsilon/2$. Since this is true for all $x \in U$, we have $U \times (y_0 - \varepsilon/4, y_0 + \varepsilon/4) \cap F_t = \emptyset$, and thus $U \times (y_0 - \varepsilon/4, y_0 + \varepsilon/4) \cap (F_t)_{\max} = \emptyset$. But $U \subseteq U_0$, which gives us the contradiction; and therefore, $F_t = (F_t)_{\max}$.

Now for each $G \in L^-(X)$, define $f_0(G)$ as follows. If $G \subseteq F_t$ for some $t \in [0,1]$, then take $f_0(G) = \inf\{t \in [0,1] : \ G \subseteq F_t\}$. If $G \not\subseteq F_t$ for all $t \in [0,1]$, then take $f_0(G) = 1$. This defines the function $f_0\colon L^-(X) \to [0,1]$. To show that f_0 is continuous, let $G \in L^-(X)$ and let $\varepsilon > 0$. First, suppose that $f_0(G) > 0$; we may assume that $\varepsilon < f_0(G)$. Let $t \in (f_0(G) - \varepsilon, f_0(G))$, so that $G \not\subseteq F_t$. We know that $F_t = (F_t)_{\max}$, so that F_t^+ is closed in $L^-(X)$ by Lemma 4.2. Define $\mathscr{O}_1 = \mathrm{Ł}^-(X) \setminus F_t^+$, which is a neighborhood of G in $L^-(X)$. If $G' \in \mathscr{O}_1$, then $G' \not\subseteq F_t$, so that $f_0(G') \geq t > f_0(G) - \varepsilon$. Secondly, suppose that $f_0(G) < 1$; we may assume that $\varepsilon < 1 - f_0(G)$. Then there exists a $t \in (f_0(G), f_0(G) + \varepsilon)$ such that $G \subseteq F_t$. Let $s \in (t, f_0(G) + \varepsilon)$, and let $F_s^o = \{\langle x, y\rangle \in X \times \mathbb{R} : \ p_s(x) < y < q_s(x)\}$. Then F_s^o is an open subset of $X \times \mathbb{R}$ by Lemma 3.2. Now $G \subseteq F_t \subseteq F_s^o$, so that $\mathscr{O}_2 = (F_s^o)^+$ is a neighborhood of G in $L^-(X)$. If $G' \in \mathscr{O}_2$, then $G' \subseteq F_s^o \subseteq F_s$, so that $f_0(G') \leq s < f_0(G) + \varepsilon$. Now either $f_0(G) = 1$, or $f_0(G) = 0$, or $0 < f_0(G) < 1$. So either $\mathscr{O}_1$ is a neighborhood of G such that $1 - \varepsilon < f_0(G')$ for all $G' \in \mathscr{O}_1$, or $\mathscr{O}_2$ is a neighborhood of G such that $f_0(G') < \varepsilon$ for all $G' \in \mathscr{O}_2$, or $\mathscr{O}_1 \cap \mathscr{O}_2$ is a neighborhood of G such that $f_0(G) - \varepsilon < f_0(G') < f_0(G) + \varepsilon$ for all $G' \in \mathscr{O}_1 \cap \mathscr{O}_2$. Therefore, f_0 is continuous.

Now momentarily fix $i \in \{1, \ldots, n\}$, and let $\langle x_i, y_i\rangle \in F \cap W_i$. Since $X \times \mathbb{R}$ is completely regular, there exists a continuous function $g_i\colon X \times \mathbb{R} \to [0,1]$ such that $g_i(\langle x_i, y_i\rangle) = 0$ and $g_i(\langle x, y\rangle) = 1$ for all $\langle x, y\rangle \in X \times \mathbb{R} \setminus W_i$. Define $f_i\colon L^-(X) \to [0,1]$ by $f_i(G) = \inf\{t \in [0,1] : \ G \cap g_i^{-1}([0,t)) \neq \emptyset\}$. Then $f_i(F) = 0$, and if $G \notin W_i^-$, we have $f_i(G) = 1$. To show that f_i is continuous, let $G \in L^-(X)$ and let $\varepsilon > 0$. First, suppose that $f_i(G) > 0$; we may assume that $\varepsilon < f_i(G)$. Let $t \in (f_i(G) - \varepsilon, f_i(G))$, so that $G \cap g_i^{-1}([0,t]) = \emptyset$. Then $W_{i,1} = g_i^{-1}((t,1])$ is an open subset of $X \times \mathbb{R}$ containing G, and hence $W_{i,1}^+$ is a neighborhood of G. If $G' \in W_{i,1}^+$, then $G' \cap g_I^{-1}([0,t]) = \emptyset$, and thus $f_i(G') \geq t > f_i(G) - \varepsilon$. Secondly, suppose that $f_i(G) < 1$; we may assume that $\varepsilon < 1 - f_i(G)$. Then there exists a $t \in (f_i(G), f_i(G) + \varepsilon)$ such that $G \cap g_i^{-1}([0,t)) \neq \emptyset$. Then $W_{i,2} = g_i^{-1}([0,t))$ is an open subset of $X \times \mathbb{R}$, and so $W_{i,2}^-$ is a neighborhood of G. If $G' \in W_{i,2}^-$, then $G' \cap g_i^{-1}([0,t)) \neq \emptyset$, and hence $f_i(G') \leq t < f_i(G) + \varepsilon$. Now either $f_i(G) = 1$, or $f_i(G) = 0$, or $0 < f_i(G) < 1$. So either $W_{i,1}^+$ is a neighborhood of G such that $1 - \varepsilon < f_i(G')$ for all $G' \in W_{i,1}^+$, or $W_{i,2}^-$ is a neighborhood of G such that $f_i(G') < \varepsilon$ for all $G' \in W_{i,2}^-$, or $W_{i,1}^+ \cap W_{i,2}^-$ is a neighborhood of G such that $f_i(G) - \varepsilon < f_i(G') < f_i(G) + \varepsilon$ for all $G' \in W_{i,1}^+ \cap W_{i,2}^-$. Therefore, f_i is continuous.

Finally, letting i vary, define the continuous function $f\colon L^-(X) \to [0,1]$ by $f = \max\{f_0, f_1, \ldots, f_n\}$. Then $f(F) = 0$, and if $F \notin W_0^+ \cap W_1^- \cap \cdots \cap W_n^-$, then $F \not\subseteq W_{\max}$ or $F \cap W_i = \emptyset$ for some i. It follows that for such F, $f(F) = 1$, which completes the proof that $L^-(X)$ is completely regular. □

5. Extension Theorem

A general problem is to determine how X and Y are related if $C(X)$ and $C(Y)$ are homeomorphic. It may not be the case that X and Y must be homeomorphic. For example, if X is the space of countable ordinals and Y is its (one point) compactification, it is well-known that $C(X)$ and $C(Y)$ are homeomorphic (see Example II.2.17 in Part II).

This section combined with Part II gives a partial solution to the problem above. In this section, we show that certain homeomorphisms from $C(X)$ onto $C(Y)$ extend to a special kind of homeomorphism from $L^-(X)$ onto $L^-(Y)$, and in the last section of Part II, a factorization is given for this special kind of homeomorphism that shows how X and Y must be related.

A function $\mu\colon C(X) \to C(Y)$ is *increasing* (respectively, *decreasing*) provided that if $f_1, f_2 \in C(X)$ are such that $f_1 \leq f_2$, then $\mu(f_1) \leq \mu(f_2)$ (respectively, $\mu(f_1) \geq \mu(f_2)$). We say that μ is *monotone* provided that it is either increasing or decreasing.

We now generalize this notion of monotone function by defining a function $\mu\colon C(X) \to C(Y)$ to be *bimonotone* provided that for every $f_1, f_2 \in C(X)$ with $f_1 \leq f_2$ and for every $f \in C(X)$, it is true that $f_1 \leq f \leq f_2$ if and only if $\min\{\mu(f_1), \mu(f_2)\} \leq \mu(f) \leq \max\{\mu(f_1), \mu(f_2)\}$. Then $\mu\colon C(X) \to C(Y)$ is a bimonotone homeomorphism provided that it is a homeomorphism such that both μ and μ^{-1} are bimonotone.

We also define a homeomorphism $M\colon L^-(X) \to L^-(Y)$ to be an *ordered homeomorphism* provided that for each $F_1, F_2 \in L^-(X)$, $M(F_1) \subseteq M(F_2)$ if and only if $F_1 \subseteq F_2$.

If $\mu\colon C(X) \to C(Y)$ is a bimonotone homeomorphism, then for each $F \in L^-(X)$, define

$$\mu^*(F) = \bigcup\{\mu(f) :\ f \in C(X) \text{ with } f \subseteq F\}.$$

Also if $M\colon L^-(X) \to L^-(Y)$ is an ordered homeomorphism, then for each $f \in C(X)$, let $M_*(f) = M(f)$ where the second f is thought of as an element of $L^-(X)$.

Extension Theorem 5.1. *Let X and Y be binormal spaces. If $\mu\colon C(X) \to C(Y)$ is a bimonotone homeomorphism, then $\mu^*\colon L^-(X) \to L^-(Y)$ is a well-defined extension of μ that is an ordered homeomorphism; if $M\colon L^-(X) \to L^-(Y)$ is an ordered homeomorphism, then $M_*\colon C(X) \to C(Y)$ is a well-defined restriction of M that is a bimonotone homeomorphism; and we have $(\mu^*)_* = \mu$ and*

$(M_*)^* = M$. Therefore, there is a natural one-to-one correspondence between the bimonotone homeomorphisms from $C(X)$ onto $C(Y)$ and the ordered homeomorphisms from $L^-(X)$ onto $L^-(Y)$. In particular, a function $\mu\colon C(X) \to C(Y)$ can be extended to an ordered homeomorphism from $L^-(X)$ onto $L^-(Y)$ if and only if μ is a bimonotone homeomorphism.

We will break the proof of the Extension Theorem 5.1 into a number of lemmas. So for the following lemmas in this section, let X and Y be bi-normal spaces, and let us start by working with the ordered homeomorphism $M\colon L^-(X) \to L^-(Y)$.

Lemma 5.2. *Let $F_1, F_2 \in L^-(X)$ be such that $F_1(x) \cap F_2(x) \neq \emptyset$ for all $x \in X$. Then $F_1 \cap F_2 \in L^-(X)$, $M(F_1) \cap M(F_2) \in L^-(Y)$, and $M(F_1 \cap F_2) = M(F_1) \cap M(F_2)$.*

Proof. It is immediate that $F_1 \cap F_2$ is a locally bounded member of $\mathscr{L}(X)$. By considering almost lsc points, we see that $(F_1 \cap F_2)_{\max} \subseteq (F_1)_{\max} = F_1$, and $(F_1 \cap F_2)_{\max} \subseteq (F_2)_{\max} = F_2$. Therefore, $F_1 \cap F_2 = (F_1 \cap F_2)_{\max}$, so that $F_1 \cap F_2 \in L^-(X)$. Now since M is ordered, we have $M(F_1 \cap F_2) \subseteq M(F_1)$ and $M(F_1 \cap F_2) \subseteq M(F_2)$, and hence $M(F_1 \cap F_2) \subseteq M(F_1) \cap M(F_2)$. Then note that $M(F_1)(y) \cap M(F_2)(y) \neq \emptyset$ for all $y \in Y$, so that by the argument above, $M(F_1) \cap M(F_2) \in L^-(Y)$. Again using the argument above, but with M^{-1}, we have $M^{-1}(M(F_1)) \cap M(F_2)) \subseteq M^{-1}(M(F_1)) \cap M^{-1}(M(F_2)) = F_1 \cap F_2$. It now follows that $M(F_1) \cap M(F_2) = M(M^{-1}(M(F_1) \cap M(F_2))) \subseteq M(F_1 \cap F_2)$, and therefore, $M(F_1 \cap F_2) = M(F_1) \cap M(F_2)$. □

Lemma 5.3. *Let $f_1, f_2 \in C(X)$ with $f_1 \leq f_2$, let $g_1 = \min\{M(f_1), M(f_2)\}$, let $g_2 = \max\{M(f_1), M(f_2)\}$, let $F = \{\langle x,t\rangle \in X \times \mathbb{R} : \ f_1(x) \leq t \leq f_2(x)\}$, and let $G = \{\langle y,t\rangle \in Y \times \mathbb{R} : \ g_1(y) \leq t \leq g_2(y)\}$. Then $F \in L^-(X)$, $G \in L^-(Y)$, and $M(F) = G$.*

Proof. It is immediate that $F \in L^-(X)$ and $G \in L^-(Y)$. Now since $f_1, f_2 \subseteq F$, we have $M(f_1), M(f_2) \subseteq M(F)$. For each $y \in Y$, $M(F)(y)$ is connected, and every element of $G(y)$ lies between $M(f_1)(y)$ and $M(f_2)(y)$, so it follows that $G \subseteq M(F)$. On the other hand, $M(f_1), M(f_2) \subseteq G$, so that $f_1, f_2 \subseteq M^{-1}(G)$. Again, by the connectivity of each $M^{-1}(G)(x)$, we have $F \subseteq M^{-1}(G)$. Therefore, $M(F) \subseteq G$, which finishes the argument that $M(F) = G$. □

Lemma 5.4. *If $M_*\colon C(X) \to C(Y)$ is the restriction of M to $C(X)$, then M_* is a bimonotone homeomorphism.*

Proof. The ordered property of M forces M_* to map members of $C(X)$ to members of $C(Y)$, so it is clear that M_* is a homeomorphism. Let $f_1, f_2 \in C(X)$ with $f_1 \leq f_2$ and let $f \in C(X)$. Define g_1, g_2, F, and G as in Lemma 5.3, and thus $M(F) = G$. Also let $g = M_*(f) = M(f)$. If $f_1 \leq f \leq f_2$, then $f \subseteq F$, so that by the ordered property of M, $g = M(f) \subseteq M(F) = G$, and hence $g_1 \leq g \leq g_2$. Conversely, if $g_1 \leq g \leq g_2$, then $g \subseteq G$, so that $f = M^{-1}(g) \subseteq M^{-1}(G) = F$, and hence $f_1 \leq f \leq f_2$. Since $g = M_*(f)$, $g_1 = \min\{M_*(f_1), M_*(f_2)\}$, and $g_2 = \max\{M_*(f_1), M_*(f_2)\}$, we see that M_* is bimonotone. □

Now let us work on getting the extension by starting with the bimonotone homeomorphism $\mu\colon C(X) \to C(Y)$.

Lemma 5.5. *The bimonotone homeomorphism μ has the property that for every $f_1, f_2, f_3 \in C(X)$ with $f_1 < f_2 < f_3$,*

$$\min\{\mu(f_1), \mu(f_3)\} < \mu(f_2) < \max\{\mu(f_1), \mu(f_3)\}.$$

Proof. Let $f_1, f_2, f_3 \in C(X)$ with $f_1 < f_2 < f_3$, and define $g_1 = \min\{\mu(f_1), \mu(f_3)\}$, $g_2 = \mu(f_2)$, and $g_3 = \max\{\mu(f_1), \mu(f_3)\}$. Let $W = \{\langle x, t\rangle \in X \times \mathbb{R} : f_1(x) < t < f_3(x)\}$, so that W^+ is a neighborhood of f_2 in $C(X)$. Since μ^{-1} is continuous, there exists a neighborhood W_0^+ of g_2 in $C(Y)$ such that $\mu^{-1}(W_0^+) \subseteq W^+$ where $W_0 = \{\langle y, t\rangle \in Y \times \mathbb{R} : h_1(y) < t < h_2(y)\}$ for some $h_1, h_2 \in C(Y)$. Let $k_1 = (h_1 + g_2)/2$ and $k_2 = (g_2 + h_2)/2$. Then $k_1 < g_2 < k_2$ and $k_1, k_2 \in W_0^+$. Now $\mu^{-1}(k_1), \mu^{-1}(k_2) \in \mu^{-1}(W_0^+) \subseteq W^+$, and hence $f_1 < \mu^{-1}(k_1) < f_3$ and $f_1 < \mu^{-1}(k_2) < f_3$. But then since μ is bimonotone, we have $g_1 \leq k_1 \leq g_3$ and $g_1 \leq k_2 \leq g_3$, so that $g_1 \leq k_1 < g_2 < k_2 \leq g_3$. □

Lemma 5.6. *For each $F \in L^-(X)$ and each $y \in Y$, $\mu^*(F)(y)$ is connected.*

Proof. Suppose that $\mu^*(F)(y)$ is not connected. Then there exist $r < s < t$ with $r, t \in \mu^*(F)(y)$ and $s \notin \mu^*(F)(y)$. Let $W = Y \times \mathbb{R} \setminus \{\langle y, s\rangle\}$. By the continuity of μ, for each $f \in C(X)$ with $f \subseteq F$, there exists an open subset W_f of $X \times \mathbb{R}$ with each $W_f(x)$ connected and such that $f \in W_f^+$ in $C(X)$ and $\mu(W_f^+) \subseteq W^+$. Define $W_0 = \bigcup\{W_f : f \in C(X) \text{ with } f \subseteq F\}$. Then W_0 is an open subset of $X \times \mathbb{R}$ with each $W_0(x)$ connected and such that $F \subseteq W_0$ and $\mu(W_0^+) \subseteq W^+$. Let $f_r, f_t \in C(X)$ with $f_r, f_t \subseteq F$ and $\mu(f_r)(y) = r$ and $\mu(f_t)(y) = t$. Now $\min\{f_r, f_t\}, \max\{f_r, f_t\} \subseteq F \subseteq W_0$, so there exist $f_1, f_2 \in C(X)$ with $f_1, f_2 \subseteq W_0$ and $f_1 < \min\{f_r, f_t\} \leq \max\{f_r, f_t\} < f_2$. Let $g_1 = \min\{\mu(f_1), \mu(f_2)\}$ and $g_2 = \max\{\mu(f_1), \mu(f_2)\}$. Lemma 5.5 tells us that $g_1 < \mu(f_r) < g_2$ and $g_1 < \mu(f_t) < g_2$. Then $g_1(y) < \mu(f_r)(y) = r$ and $g_2(y) > \mu(f_t)(y) = t$, so that $g_1(y) < s < g_2(y)$. Then there exists a $g \in C(Y)$ with $g_1 < g < g_2$ and $g(y) = s$. Since μ is bimonotone, we have $f_1 \leq \mu^{-1}(g) \leq f_2$, so that $\mu^{-1}(g) \in W_0^+$. Therefore, $g \in \mu(W_0^+) \subseteq W^+$, which contradicts the fact that $g(y) = s$. □

Lemma 5.7. *For each $F \in L^-(X)$ and each $y \in Y$, $\mu^*(F)(y)$ is bounded.*

Proof. Suppose, by way of contradiction, that $\mu^*(F)(y)$ is unbounded for some $y \in Y$; say it is unbounded from above. Then for each $n \in \mathbb{N}$, there exits an $f_n \in C(X)$ with $f_n \subseteq F$ and $\mu(f_n) > n$. Since X is binormal and F is locally bounded, there exist $f, f' \in C(X)$ with $f < \inf \overline{F}$ and $\sup \overline{F} < f'$. Then for each n, $f < f_n < f'$. Now μ is bimonotone, so for each n, we have $\mu(f_n) \leq \max\{\mu(f), \mu(f')\}$. If $\mu(f')(y) \leq \mu(f)(y)$, then for each n, $n < \mu(f_n)(y) \leq \mu(f)(y)$; which contradicts the continuity of $\mu(f)$ at y. The other case contradicts the continuity of $\mu(f')$ at y. □

Lemma 5.8. *For each $F \in L^-(X)$, $\mu^*(F) = \mu^*(F)_{\max}$.*

P r o o f. Suppose, by way of contradiction, that there exists a $\langle y, t\rangle \in \mu^*(F)_{\max} \setminus \mu^*(F)$; say $t > d$ where $d = \sup \mu^*(F)(y)$. Note that y can not be an isolated point of Y because otherwise $\langle y, t\rangle$ would not be in the closure of $\mu^*(F)$. By Lemma 3.6 there exists a $g \in C(Y)$ with $g \subseteq \mu^*(F)_{\max}$ and $g(y) = t$. Since $g \not\subseteq \mu^*(F)$, we have $\mu^{-1}(g) \not\subseteq F$. So there exists an $x \in X$ with $\mu^{-1}(g)(x) \notin F(x)$; say $\mu^{-1}(g)(x) > b$ where $b = \sup F(x)$. Let $s = \mu^{-1}(g)(x)$. Since $\langle x, s\rangle \notin F = F_{\max}$, $\langle x, s\rangle$ is not an almost lsc point of F. So there exists a neighborhood O of s with the property that for every neighborhood U of x, there is a nonempty open subset U_0 of U such that $F(x') \cap O = \emptyset$ for all $x' \in U_0$. In particular, let U be a neighborhood of x such that $\mu^{-1}(g)(U) \subseteq O$ and such that for every $z \subseteq U$, $F(z) \cap (b-\varepsilon, b+\varepsilon) \neq \emptyset$, where $\varepsilon = (s-b)/2$; and let U_0 be a nonempty open subset of this U such that $F(x') \cap O = \emptyset$ for all $x' \in U_0$. Choose an $x_0 \in U_0$. Since $\mu^{-1}(g)(x_0) \in O$, there is an $r \in O$ with $r < \mu^{-1}(g)(x_0)$. Since X is binormal and F is locally bounded, there exist $f_1, f_2 \in C(X)$ with $f_1 \leq \inf \overline{F}$ and $\sup \overline{F} \leq f_2$. Since $F(z) \cap (b-\varepsilon, b+\varepsilon) \neq \emptyset$ for all $z \in U$, we have that $\sup F(x') \leq \inf O$ for all $x' \in U_0$, and so we can choose f_2 so that $f_2(x_0) = r$. Now μ is bimonotone, so that since $\mu^{-1}(g) \not\leq f_2$, it follows that either $\min\{\mu(f_1), \mu)f_2)\} \not\leq g$ or $g \not\leq \max\{\mu(f_1), \mu(f_2)\}$. Suppose $g \not\leq \max\{\mu(f_1), \mu(f_2)\}$; the proof in the other case is similar. Then there is a $y_0 \in Y$ with $\max\{\mu(f_1), \mu(f_2)\}(y_0) < g(y_0)$. Since the functions $\max\{\mu(f_1), \mu(f_2)\}$ and g are continuous, there is a neighborhood $W$of y_0 and a $\sigma > 0$ such that $W \times \big(g(y_0) - \sigma, g(y_0) + \sigma\big) \cap \max\{\mu(f_1), \mu(f_2)\} = \emptyset$. Define the set-valued map G by

$$G(y') = \big[\min\{\mu(f_1), \mu(f_2)\}(y'), \max\{\mu(f_1), \mu(f_2)\}(y')\big]$$

for all $y' \in Y$. Then $\mu^*(F) \subseteq G$ and $G \cap W \times \big(g(y_0) - \sigma, g(y_0) + \sigma\big) = \emptyset$, which is a contradiction to the fact that the point $\langle y_0, g(y_0)\rangle$ is an almost lsc point of $\mu^*(F)_{\max}$. □

Lemma 5.9. *For each $F \in L^-(X)$, $\mu^*(F)$ is locally bounded.*

P r o o f. Since F is locally bounded, there exist $f_1, f_2 \in C(X)$ with $f_1 \leq \inf \overline{F}$ and $\sup \overline{F} \leq f_2$. Then for every $f \in C(X)$ with $f \subseteq F$, we have $f_1 \leq f \leq f_2$, and hence $\min\{\mu(f_1), \mu(f_2)\} \leq \mu(f) \leq \max\{\mu(f_1), \mu(f_2)\}$. Therefore,

$\min\{\mu(f_1), \mu(f_2)\} \leq \inf \overline{\mu^*(F)}$ and $\sup \overline{\mu^*(F)} \leq \max\{\mu(f_1), \mu(f_2)\}$, showing that $\mu^*(F)$ is locally bounded. □

Lemmas 5.6 through 5.9 show that $\mu^*\colon L^-(X) \to L^-(Y)$ is a well-defined function. The same arguments used on $(\mu^{-1})^*\colon L^-(Y) \to L^-(X)$ show that it is also a well-defined function.

Lemma 5.10. *For each $F \in L^-(X)$, $(\mu^{-1})^*(\mu^*(F)) = F$.*

Proof. If $f \in C(X)$ with $f \subseteq F$, then $\mu(f) \subseteq \mu^*(F)$, so that $f = \mu^{-1}(\mu(f)) \subseteq (\mu^{-1})^*(\mu^*(F))$. This shows that $F \subseteq (\mu^{-1})^*(\mu^*(F))$. For the reverse containment, let $f \in C(X)$ with $f \subseteq (\mu^{-1})^*(\mu^*(F))$. Suppose, by way of contradiction, that $f \not\subseteq F$. Then $f(x) \notin F(x)$ for some $x \in X$; say $f(x) > b$ where $b = \sup F(x)$. Now there exist $f_1, f_2 \in C(X)$ such that $f_1 \leq \inf \overline{F}$ and $\sup \overline{F} \leq f_2$. We can choose f_2 so that $f_2(z) < f(z)$ for some $z \in X$. Since $f \not\leq f_1$, we have either $\min\{\mu(f_1), \mu(f_2)\} \not\leq \mu(f)$ or $\mu(f) \not\leq \max\{\mu(f_1), \mu(f_2)\}$. Now $\min\{\mu(f_1), \mu(f_2)\} \leq \inf \mu^*(F)$ and $\sup \mu^*(F) \leq \max\{\mu(f_1), \mu(f_2)\}$, so that $\mu(f) \not\subseteq \mu^*(F)$. We can repeat this argument to show that $f = \mu^{-1}(\mu(f)) \not\subseteq (\mu^{-1})^*(\mu^*(F))$, which is a contradiction. □

Lemma 5.10 says that $(\mu^{-1})^*\mu^*$ is the identity map on $L^-(X)$. The same argument shows that $\mu^*(\mu^{-1})^*$ is the identity map on $L^-(Y)$. Therefore, $\mu^*\colon L^-(X) \to L^-(Y)$ is a bijection.

Lemma 5.11. *The bijection $\mu^*\colon L^-(X) \to L^-(Y)$ is continuous when $L^-(X)$ and $L^-(Y)$ have the upper Vietoris topology.*

Proof. Let $F \in L^-(X)$ and let W^+ be a basic neighborhood of $\mu^*(F)$ where W is a locally bounded element of $\mathscr{L}(Y)$. Then there exists a $W' \in \mathscr{L}(Y)$ with $\mu^*(F) \subseteq W'$ and $W'_{\max} \subseteq W$. Note that $W'_{\max} \in L^-(Y)$. Define W_0 to be the interior of $(\mu^{-1})^*(W'_{\max})$ in $X \times \mathbb{R}$. To show that $F \subseteq W_0$, let $\langle x, t\rangle \in F$. By Lemma 3.5, there exists an $f \in C(X)$ with $f \subseteq F$ and $f(x) = t$. Since $\mu^*(F) \subseteq W'$, we have $\mu(f) \subseteq W'$. Then there exist $g_1, g_2 \in C(Y)$ with $g_1 < \mu(f) < g_2$ and $g_1, g_2 \subseteq W' \subseteq W'_{\max}$. Let $f_1 = \min\{\mu^{-1}(g_1), \mu^{-1}(g_2)\}$ and $f_2 = \max\{\mu^{-1}(g_1), \mu^{-1}(g_2)\}$. Then by Lemma 5.5 applied to μ^{-1}, $f_1 < f < f_2$ and $\mu^{-1}(g_1), \mu^{-1}(g_2) \subseteq (\mu^{-1})^*(W'_{\max})$. We have $f_1, f_2 \subseteq (\mu^{-1})^*(W'_{\max})$, so that $\{\langle x', t'\rangle \in X \times \mathbb{R} : \ f_1(x') < t' < f_2(x')\}$ is an open subset of $X \times \mathbb{R}$ containing f and contained in $(\mu^{-1})^*(W'_{\max})$, and thus contained in W_0. Therefore, $\langle x, t\rangle \in W_0$, showing that W_0^+ is a neighborhood of F in $L^-(X)$. Finally, if $F' \in L^-(X)$ with $F' \in W_0^+$, then $F' \subseteq W_0 \subseteq (\mu^{-1})^*(W'_{\max})$, so that $\mu^*(F') \subseteq W'_{\max} \subseteq W$, and hence $\mu^*(F') \in W^+$. □

The same argument as in Lemma 5.11 shows that $(\mu^{-1})^*\colon L^-(Y) \to L^-(X)$ is continuous when $L^-(X)$ and $L^-(Y)$ have the upper Vietoris topology, so that $\mu^*\colon L^-(X) \to L^-(Y)$ is a homeomorphism with respect to the upper Vietoris topology. Now the fact that μ^* is also a homeomorphism with respect to the Vietoris topology follows from Proposition II.3.6 in Part II. That μ^* is an ordered homeomorphism comes directly from the definition of μ^*.

Lemma 5.12. *For $\mu\colon C(X) \to C(Y)$, we have $(\mu^*)_* = \mu$; and for $M\colon L^-(X) \to L^-(Y)$, we have $(M_*)^* = M$.*

Proof. For $(\mu^*)_* = \mu$, observe that for each $f \in C(X)$, $\mu^*(f) = \mu(f)$, so that $(\mu^*)_*(f) = \mu(f)$. For $(M_*)^* = M$, let $F \in L^-(X)$. Now $(M_*)^*(F) = \bigcup\{M(f) :\ f \in C(X) \text{ with } f \subseteq F\} \subseteq M(F)$ since M is ordered. To show that $M(F) \subseteq \bigcup\{M(f) :\ f \in C(X) \text{ with } f \subseteq F\}$, let $\langle y,t\rangle \in M(F)$. Then, by Lemma 3.5, there exists a $g \in C(Y)$ with $\langle y,t\rangle \in g \subseteq M(F)$. Now $M^{-1}(g) \subseteq M^{-1}(M(F)) = F$, so that $\langle y,t\rangle \in g \subseteq \bigcup\{M(f) :\ f \in C(X) \text{ with } f \subseteq F\}$. $\square$

This finishes the proof of the Extension Theorem 5.1. The related Factorization Theorem II.3.3 is proved in the following Part II.

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