VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-010-0027-5
Math. Slovaca **60** (2010), No. 4, 485–494

ON THE EXISTENCE OF SOLUTIONS FOR SINGULAR BOUNDARY VALUE PROBLEM OF THIRD-ORDER DIFFERENTIAL EQUATIONS

Feng Wang* — Yujun Cui**

(*Communicated by Michal Fečkan*)

ABSTRACT. The singular boundary value problems of third-order differential equations

$$-u'''(t) = h(t)f(t,u(t)), \qquad t \in (0,1),$$
$$u(0) = u'(0) = 0, \quad u'(1) = \alpha u'(\eta)$$

are considered under some conditions concerning the first eigenvalues corresponding to the relevant linear operators, where $h(t)$ is allowed to be singular at both $t = 0$ and $t = 1$, and f is not necessary to be nonnegative. The existence results of nontrivial solutions and positive solutions are given by means of the topological degree theory.

1. Introduction

Many authors are interested in the existence of positive solutions for third-order boundary value problem (see [2]–[12] and references therein). In most work mentioned, they study the existence of positive solutions for third-order boundary value problem by the method of upper and lower solutions, Schauder's fixed point theorem or the fixed point index in cone under some different conditions in which f is nonnegative. In this paper, we consider the following singular boundary value problems:

$$\begin{cases} -u'''(t) = h(t)f(t,u(t)), & t \in (0,1), \\ u(0) = u'(0) = 0, \quad u'(1) = \alpha u'(\eta), \end{cases} \tag{1.1}$$

where $0 < \eta < 1$, $1 < \alpha < \frac{1}{\eta}$ and $h(t)$ is allowed to be singular at $t = 0$ and $t = 1$. In particular, f is not necessary to be nonnegative. We obtain

2000 Mathematics Subject Classification: Primary 34B10, 34B18.
Keywords: singular, nontrivial solutions, positive solutions, topology degree.
Research supported by the National Natural Science Foundation of China.

the existence results of nontrivial solutions, and the existence results of positive solutions for some cases, by means of the topological degree theory under some conditions on $f(t,u)$ concerning the first eigenvalue corresponding to the relevant linear operator. For the concepts and properties about the cone theory and the topological degree we refer to [4], [5].

Our paper is organized as follows. Some preliminaries and lemmas are given in Section 2. In Section 3, the main results are established. Requiring no condition that $f(t,u) \geq 0$ when $u \geq 0$, we prove that the singular boundary value problem (1.1) has at least one nontrivial solution. We also investigate the existence of positive solutions in the case that f is nonnegative.

2. Preliminaries and lemmas

In Banach space $C[0,1]$ in which the norm is defined by $\|u\| = \max\limits_{0\leq t\leq 1} |u(t)|$. We set

$$P = \big\{u \in C[0,1] : \ u(t) \geq 0, \ \ t \in [0,1]\big\}, \tag{2.1}$$

then P is a positive cone in $C[0,1]$. We denote by $B_r = \big\{u \in C[0,1] : \ \|u\| < r\big\}$ $(r > 0)$ the open ball of radius r and use θ to denote the zero function in $C[0,1]$.

Lemma 1. *Let $y(t) \in C[0,1]$, $0 < \eta < 1$, $1 < \alpha < \frac{1}{\eta}$, then the BVP:*

$$\begin{cases} -u'''(t) - y(t) = 0, & t \in (0,1), \\ u(0) = u'(0) = 0, & u'(1) = \alpha u'(\eta) \end{cases}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)y(s)\,\mathrm{d}s,$$

where

$$G(t,s) = \frac{1}{2(1-\alpha\eta)} \begin{cases} (2ts - s^2)(1-\alpha\eta) + t^2 s(\alpha - 1), & s \leq \min\{\eta, t\}, \\ t^2(1-\alpha\eta) + t^2 s(\alpha-1), & t \leq s \leq \eta, \\ (2ts - s^2)(1-\alpha\eta) + t^2(\alpha\eta - s), & \eta \leq s \leq t, \\ t^2(1-s), & \max\{\eta, t\} \leq s \end{cases}$$

is Green's function of the BVP

$$\begin{cases} -u'''(t) = 0, & t \in (0,1), \\ u(0) = u'(0) = 0, & u'(1) = \alpha u'(\eta). \end{cases}$$

Proof. The proof of this lemma is easy, and we omit it. □

Remark 1. It is obvious that the Green function $G(t,s)$ is continuous and $G(t,s) \geq 0$ for any $0 \leq t, s \leq 1$. In addition, we also have

$$\max_{0 \leq t \leq 1} G(t,s) \leq g(s),$$

where $g(s) = \frac{1+\alpha}{1-\alpha\eta} s(1-s)$. In fact, for any fixed $s \in [0,1]$, it is easy to see that

$$G_t(t,s) = \frac{1}{1-\alpha\eta} \begin{cases} s(1-\alpha\eta) + ts(\alpha-1), & s \leq \min\{\eta, t\}, \\ t(1-\alpha\eta) + ts(\alpha-1), & t \leq s \leq \eta, \\ s(1-\alpha\eta) + t(\alpha\eta - s), & \eta \leq s \leq t, \\ t(1-s), & \max\{\eta, t\} \leq s. \end{cases}$$

If $s \leq \min\{\eta, t\}$, then

$$\begin{aligned} G_t(t,s) &= \frac{1}{1-\alpha\eta}\big[s(1-\alpha\eta) + ts(\alpha-1)\big] \\ &\leq \frac{1}{1-\alpha\eta}\alpha s(1-\eta) \leq \frac{1}{1-\alpha\eta}\alpha s(1-s) \leq g(s). \end{aligned}$$

If $t \leq s \leq \eta$, then

$$\begin{aligned} G_t(t,s) &= \frac{1}{1-\alpha\eta}\big[t(1-\alpha\eta) + ts(\alpha-1)\big] \\ &\leq \frac{1}{1-\alpha\eta}\alpha s(1-\eta) \leq \frac{1}{1-\alpha\eta}\alpha s(1-s) \leq g(s). \end{aligned}$$

If $\eta \leq s \leq t$, then

$$\begin{aligned} G_t(t,s) &= \frac{1}{1-\alpha\eta}\big[s(1-\alpha\eta) + t(\alpha\eta - s)\big] \\ &= \frac{1}{1-\alpha\eta}\big[s(1-t) + \alpha\eta(t-s)\big] \\ &\leq \frac{1}{1-\alpha\eta}\big[s(1-s) + \alpha s(1-s)\big] = g(s). \end{aligned}$$

If $\max\{\eta, t\} \leq s$, then

$$G_t(t,s) = \frac{1}{1-\alpha\eta} t(1-s) \leq \frac{1+\alpha}{1-\alpha\eta} s(1-s) = g(s).$$

Therefore, $G_t(t,s) \leq g(s)$, $(t,s) \in [0,1] \times [0,1]$. Then for any $(t,s) \in [0,1] \times [0,1]$, we have

$$G(t,s) = \int_0^t G_\tau(\tau,s)\,\mathrm{d}\tau \leq \int_0^t g(s)\,\mathrm{d}\tau = tg(s) \leq g(s).$$

In this paper we suppose that

(H_1) $h\colon (0,1) \to [0,+\infty)$ is continuous, $h(t) \not\equiv 0$ and

$$\int_0^1 g(t)h(t)\,\mathrm{d}t < +\infty. \tag{2.2}$$

(H_2) $f\colon [0,1] \times (-\infty,+\infty) \to (-\infty,+\infty)$ is continuous.

As is well known, the singular nonlinear boundary value problem (1.1) can be converted into the equivalent Hammerstein nonlinear integral equation

$$u(t) = \int_0^1 G(t,s)h(s)f(s,u(s))\,\mathrm{d}s, \qquad t \in [0,1]. \tag{2.3}$$

Let

$$(Au)(t) = \int_0^1 G(t,s)h(s)f(s,u(s))\,\mathrm{d}s, \qquad t \in [0,1], \tag{2.4}$$

$$(Tu)(t) = \int_0^1 G(t,s)h(s)u(s)\,\mathrm{d}s, \qquad t \in [0,1]. \tag{2.5}$$

By the method similar to that in [15], we have:

Lemma 2. *Suppose that* (H_1), (H_2) *are satisfied, then* $A\colon C[0,1] \to C[0,1]$ *is a completely continuous operator and* $T\colon C[0,1] \to C[0,1]$ *is a completely continuous linear operator,* $T(P) \subset P$.

Notice that $T(P) \subset P$, it follows that $r(T) \geq 0$. It is obvious that if the operator A has a fixed point u, then u is the solution of (1.1).

Lemma 3. *Suppose that the condition* (H_1) *is satisfied, then for the operator* T *defined by* (2.5), *the spectral radius* $r(T) \neq 0$ *and* T *has a positive eigenfunction corresponding to its first eigenvalue* $\lambda_1 = (r(T))^{-1}$.

We also need the following lemmas in [5].

Lemma 4. *Let* P *be a cone in a real Banach space* E, Ω *a bounded open subset of* E, *and* $A\colon P \cap \overline{\Omega} \to P$ *a completely continuous operator. Assume that there exists a* $u_0 \in P$, $u_0 \neq \theta$, *such that*

$$u - Au \neq \mu u_0$$

for all $u \in P \cap \partial\Omega$ *and* $\mu \geq 0$, *then the fixed point index*

$$i(A, P \cap \Omega, P) = 0.$$

Lemma 5. *Let P be a cone in a real Banach space E, Ω a bounded open subset of E with $\theta \in P \cap \Omega$, and $A\colon P \cap \overline{\Omega} \to P$ a completely continuous operator. If*

$$Au \neq \mu u$$

for all $u \in P \cap \partial\Omega$ and $\mu \geq 1$, then the fixed point index

$$i(A, P \cap \Omega, P) = 1.$$

3. Main results

Theorem 6. *Suppose that the conditions* (H_1), (H_2) *are satisfied. If there exists a constant $b \geq 0$ such that*

$$\min_{t\in[0,1]} f(t,u) \geq -b, \qquad \text{for all} \quad u \in (-\infty, +\infty), \tag{3.1}$$

$$\liminf_{u\to 0} \min_{t\in[0,1]} \frac{f(t,u)}{|u|} > \lambda_1, \tag{3.2}$$

$$\limsup_{u\to+\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u} < \lambda_1, \tag{3.3}$$

where λ_1 is the first eigenvalue of T defined by (2.5)*. Then the singular boundary value problem* (1.1) *has at least one nontrivial solution.*

Proof. It follows from (3.2) that there exists $r_1 > 0$ such that

$$f(t,u) \geq \lambda_1 |u| \qquad \text{for all} \quad |u| \leq r_1. \tag{3.4}$$

For every $u \in \overline{B}_{r_1}$, we have from (3.4) that

$$(Au)(t) \geq \lambda_1 \int_0^1 G(t,s)h(s)|u(s)|\,\mathrm{d}s \geq 0, \qquad t \in [0,1],$$

and thus $A(\overline{B}_{r_1}) \subset P$. For any $u \in \partial B_{r_1} \cap P$, it follows from (3.4) that

$$(Au)(t) \geq \lambda_1 \int_0^1 G(t,s)h(s)u(s)\,\mathrm{d}s = \lambda_1 (Tu)(t), \qquad t \in [0,1]. \tag{3.5}$$

We may suppose that A has no fixed point on ∂B_{r_1} (otherwise, the proof completes). Let u^* be the positive eigenfunction of T corresponding to λ_1, thus $u^* = \lambda_1 T u^*$. Now we show that

$$u - Au \neq \mu u^* \qquad \text{for all} \quad u \in \partial B_{r_1} \cap P, \ \ \mu \geq 0. \tag{3.6}$$

If, otherwise, there exist $u_1 \in \partial B_{r_1} \cap P$ and $\tau_0 \geq 0$ such that $u_1 - Au_1 = \tau_0 u^*$, then $\tau_0 > 0$ and

$$u_1 = Au_1 + \tau_0 u^* \geq \tau_0 u^*.$$

Put

$$\tau^* = \sup\{\tau : \ u_1 \geq \tau u^*\}. \tag{3.7}$$

It is easy to see that $\tau^* \geq \tau_0 > 0$ and $u_1 \geq \tau^* u^*$. We have from $T(P) \subset P$ that

$$\lambda_1 T u_1 \geq \tau^* \lambda_1 T u^* = \tau^* u^*.$$

Therefore by (3.5),

$$u_1 = Au_1 + \tau_0 u^* \geq \lambda_1 T u_1 + \tau_0 u^* \geq \tau^* u^* + \tau_0 u^*,$$

which contradicts the definition of τ^*. Hence (3.6) is true. Since $A(\overline{B}_{r_1}) \subset P$, we have from the permanence property of fixed point index and Lemma 4 that

$$\deg(I - A, B_{r_1}, \theta) = i(A, B_{r_1} \cap P, P) = 0, \tag{3.8}$$

where deg denotes the topological degree.

Letting $\widetilde{u}(t) = b\int_0^1 G(t,s)h(s)\,\mathrm{d}s$, obviously, $\widetilde{u} \in P$. It easy to see from (3.1) that $A\colon C[0,1] \to P - \widetilde{u}$. Define $\widetilde{A}u = A(u - \widetilde{u}) + \widetilde{u}$, $u \in C[0,1]$, then $\widetilde{A}\colon C[0,1] \to P$.

It follows from (3.3) that there exist $r_2 > r_1 + \|\widetilde{u}\|$ and $0 < \sigma < 1$ such that

$$f(t,u) \leq \sigma \lambda_1 u \qquad \text{for all} \quad u \geq r_2. \tag{3.9}$$

Let $T_1 u = \sigma \lambda_1 T u$, $u \in C[0,1]$. Then $T_1 : C[0,1] \to C[0,1]$ is a bounded linear operator and $T_1(P) \subset P$. Let

$$M = 2\max\Big\{ \sup_{u \in \overline{B}_{r_2}} \int_0^1 G(s,s)h(s)|f(s,u(s))|\,\mathrm{d}s, 2\|\widetilde{u}\| \Big\}. \tag{3.10}$$

It is clear that $M < +\infty$. Let

$$W = \big\{u \in P : \ u = \mu \widetilde{A} u, \ \ 0 \leq \mu \leq 1\big\}. \tag{3.11}$$

In the following, we prove that W is bounded.

For any $u \in W$, set $\widetilde{v}(t) = \min\{u(t) - \widetilde{u}(t), r_2\}$ and denote

$$e(u) = \big\{t \in [0,1] : \ u(t) - \widetilde{u}(t) > r_2\big\}.$$

When $u(t)-\widetilde{u}(t)<0$, $\widetilde{v}(t)=u(t)-\widetilde{u}(t)\geq u(t)-r_2\geq -r_2$, and so $\|\widetilde{v}\|\leq r_2$. Thus for $u\in W$, we have from (3.9)

$$\begin{aligned}
u(t) &= \mu(\widetilde{A}u)(t)\\
&\leq \int_0^1 G(t,s)h(s)f(s,u(s)-\widetilde{u}(s))\,\mathrm{d}s+\widetilde{u}(t)\\
&= \int_{e(u)} G(t,s)h(s)f(s,u(s)-\widetilde{u}(s))\,\mathrm{d}s\\
&\quad + \int_{[0,1]\setminus e(u)} G(t,s)h(s)f(s,u(s)-\widetilde{u}(s))\,\mathrm{d}s+\widetilde{u}(t)\\
&\leq \sigma\lambda_1\int_0^1 G(t,s)h(s)u(s)\,\mathrm{d}s+\int_0^1 G(t,s)h(s)f(s,\widetilde{v}(s))\,\mathrm{d}s+2\widetilde{u}(t)\\
&\leq \sigma\lambda_1\int_0^1 G(t,s)h(s)u(s)\,\mathrm{d}s+M=(T_1u)(t)+M,
\end{aligned}$$

where M is defined as (3.10). Thus $((I-T_1)u)(t)\leq M$, $t\in[0,1]$.

Since λ_1 is the first eigenvalue of T and $0<\sigma<1$, the first eigenvalue of T_1, $(r(T_1))^{-1}>1$. Therefore, the inverse operator $(I-T_1)^{-1}$ exists and

$$(I-T_1)^{-1}=I+T_1+T_1^2+\cdots+T_1^n+\cdots. \tag{3.12}$$

It follows from $T_1(P)\subset P$ that $(I-T_1)^{-1}(P)\subset P$. So we have $u(t)\leq(I-T_1)^{-1}M$, $t\in[0,1]$ and W is bounded.

Select $r_3>\max\{r_2,\sup W+\|\widetilde{u}\|\}$ and thus $\widetilde{A}$ has no fixed point on ∂B_{r_3}. In fact, if there exists $u_1\in\partial B_{r_3}$ such that $\widetilde{A}u_1=u_1$, then $u_1\in W$ and $\|u_1\|=r_3>\sup W$, which is a contradiction. Then we have from Lemma 5 and the permanence property that

$$\deg(I-\widetilde{A},B_{r_3},\theta)=i(\widetilde{A},B_{r_3}\cap P,P)=1. \tag{3.13}$$

Set the completely continuous homotopy $H(\lambda,u)=A(u-\lambda\widetilde{u})+\lambda\widetilde{u}$, $(\lambda,u)\in[0,1]\times\overline{B}_{r_3}$. If there exists $(\lambda_0,u_2)\in[0,1]\times\partial B_{r_3}$ such that $H(\lambda_0,u_2)=u_2$, then $A(u_2-\lambda_0\widetilde{u})=u_2-\lambda_0\widetilde{u}$ and $\widetilde{A}(u_2-\lambda_0\widetilde{u}+\widetilde{u})=u_2-\lambda_0\widetilde{u}+\widetilde{u}$. Thus $u_2-\lambda_0\widetilde{u}+\widetilde{u}\in W$ and

$$\|u_2-\lambda_0\widetilde{u}+\widetilde{u}\|\geq\|u_2\|-(1-\lambda_0)\|\widetilde{u}\|\geq r_3-\|\widetilde{u}\|>\sup W,$$

a contradiction! From the homotopy invariance of topological degree and (3.13) we have

$$\begin{aligned}\deg(I-A,B_{r_3},\theta)&=\deg(I-H(0,\cdot),B_{r_3},\theta)\\&=\deg(I-H(1,\cdot),B_{r_3},\theta)\\&=\deg(I-\widetilde{A},B_{r_3},\theta)=1.\end{aligned}\tag{3.14}$$

By (3.8) and (3.14) we have that

$$\deg(I-A,B_{r_3}\backslash\overline{B}_{r_1},\theta)=\deg(I-A,B_{r_3},\theta)-\deg(I-A,B_{r_1},\theta)=1,$$

which implies that A has at least one fixed point on $B_{r_3}\backslash\overline{B}_{r_1}$. This means that the singular nonlinear boundary value problem (1.1) has at least one nontrivial solution. □

Remark 2. The condition (3.1) implies that f is not necessary to be nonnegative.

Corollary 1. *Suppose that the conditions* (H_1), (H_2) *are satisfied. If there exists a constant $b^* \geq 0$ such that*

$$\min_{t\in[0,1]} f(t,u)\geq -\frac{b^*}{M}\qquad \textit{for all}\quad u\geq -b^*,\tag{3.15}$$

where $M=\max\limits_{t\in[0,1]}\int_0^1 G(t,s)h(s)\,\mathrm{d}s$ and in addition, (3.2) *and* (3.3) *hold, then the singular boundary value problem* (1.1) *has at least one nontrivial solution.*

P r o o f. Denote

$$f_1(t,u)=\begin{cases} f(t,u), & u\geq -b^*,\\ f(t,-b^*), & u<-b^*.\end{cases}\tag{3.16}$$

Define

$$(A_1u)(t)=\int_0^1 G(t,s)h(s)f_1(s,u(s))\,\mathrm{d}s,\qquad t\in[0,1].$$

By Theorem 1 we know that A_1 has at least one nonzero fixed point $\widetilde{u}$. Then

$$\widetilde{u}(t)=\int_0^1 G(t,s)h(s)f_1(s,\widetilde{u}(s))\,\mathrm{d}s\geq -\frac{b^*}{M}\int_0^1 G(t,s)h(s)\,\mathrm{d}s\geq -b^*.$$

From (3.16) we have that $f_1(t,\widetilde{u}(t))=f(t,\widetilde{u}(t))$, $t\in[0,1]$, then

$$\widetilde{u}(t)=\int_0^1 G(t,s)h(s)f_1(s,\widetilde{u}(s))\,\mathrm{d}s=\int_0^1 G(t,s)h(s)f(s,\widetilde{u}(s))\,\mathrm{d}s.$$

Thus $\widetilde{u}$ is the nontrivial solution of singular boundary value problem (1.1). □

Theorem 7. *Suppose that the conditions* (H_1), (H_2) *are satisfied. If*

$$uf(t,u) \geq 0 \qquad \text{for all} \quad t \in [0,1], \quad u \in (-\infty,+\infty), \tag{3.17}$$

$$\liminf_{u\to 0} \min_{t\in[0,1]} \frac{f(t,u)}{u} > \lambda_1, \tag{3.18}$$

$$\limsup_{|u|\to+\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u} < \lambda_1, \tag{3.19}$$

where λ_1 is the first eigenvalue of T defined by (2.5). *Then the singular boundary value problem* (1.1) *has at least one positive solution and one negative solution.*

Proof. From (3.17) we have that $A(P) \subset P$. Similar to the proof of Theorem 1 in which $b = 0$, we have by Lemmas 4 and 5 that there exist $0 < \widetilde{r}_1 < \widetilde{r}_2$ such that

$$i(A, B_{\widetilde{r}_1} \cap P, \theta) = 1, \qquad i(A, B_{\widetilde{r}_2} \cap P, \theta) = 0. \tag{3.20}$$

Then

$$i(A, (B_{\widetilde{r}_2} \cap P)\backslash(\overline{B}_{\widetilde{r}_1} \cap P), \theta) = i(A, B_{\widetilde{r}_2} \cap P, \theta) - i(A, B_{\widetilde{r}_1} \cap P, \theta) = -1.$$

So A has a fixed point in $(B_{\widetilde{r}_2} \cap P)\backslash(\overline{B}_{\widetilde{r}_1} \cap P)$ and (1.1) has at least one positive solution.

Denote $f_2(t,u) = -f(t,-u)$ for all $t \in [0,1]$, $u \in (-\infty,+\infty)$, and define

$$(A_2u)(t) = \int_0^1 G(t,s)h(s)f_2(s,u(s))\,\mathrm{d}s, \qquad t \in [0,1].$$

Then $A_2(P) \subset P$ and A_2 has a fixed point $\widetilde{u} \in P\backslash\{\theta\}$, i.e., $A_2\widetilde{u} = \widetilde{u}$.

Since $f_2(t,\widetilde{u}(t)) = -f(t,-\widetilde{u}(t))$ for all $t \in [0,1]$, we have

$$-\widetilde{u}(t) = \int_0^1 G(t,s)h(s)f(s,-\widetilde{u}(s))\,\mathrm{d}s = (A(-\widetilde{u}))(t) \qquad \text{for all} \quad t \in [0,1].$$

So $-\widetilde{u}$ is the negative solution of (1.1). $\square$

Remark 3. In Theorem 2, f is not required to be bounded below, and in particular, the existence of positive solutions is obtained in Theorem 2 though A may be not a cone mapping.

4. An example

In this section, we give the following example to illustrate the application of our main result obtained in Section 3.

Let

$$h(t) = \frac{1}{\sqrt{t(1-t)}}, \qquad f(t,u) = \frac{1-u^2}{1+u^2}. \tag{4.1}$$

Then h is singular at $t = 0, 1$ and f is unbounded from below and sign-changing for $u \geq 0$. It is easy to prove that all the conditions in Theorem 1 are satisfied. As a result, BVP (1.1) with $h(t)$ and $f(t, u)$ given by (4.1) has at least one nontrivial solution.

Acknowledgement. The authors express their sincere gratitude to the referee for very valuable comments.

REFERENCES

[1] ANDERSON, D.: *Green's function for a third-order generalized right focal problem*, J. Math. Anal. Appl. **288** (2003), 1–14.

[2] CABADA, A.: *The method of lower and upper solutions for second, third, fourth and higher order boundary value problems*, J. Math. Anal. Appl. **185** (1994), 302–320.

[3] CHU, J.—ZHOU, Z.: *Positive solutions for singular non-linear third-order periodic boundary value problems*, Nonlinear Anal. **64** (2006), 1528–1542.

[4] DEIMLING, K.: *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[5] GUO, D.—LAKSHMIKANTHAM, V.: *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.

[6] KONG, L.—WANG, S.—WANG, J.: *Positive solution of a singular nonlinear third-order periodic boundary value problem*, J. Comput. Appl. Math. **132** (2001), 247–253.

[7] LI, S.: *Positive solutions of nonlinear singular third-order two-point boundary value problem*, J. Math. Anal. Appl. **323** (2006), 413–425.

[8] LIU, Z.—UME, J.—KANG, S.: *Positive solutions of a singular nonlinear third order two-point boundary value problem*, J. Math. Anal. Appl. **326** (2007), 589–601.

[9] SUN, J.—LIU, Y.: *Multiple positive solutions of singular third-order periodic boundary value problem*, Acta Math. Sci. Ser. B Engl. Ed. **25** (2005), 81–88.

[10] SUN, Y.: *Positive solutions of singular third-order three-point boundary value problem*, J. Math. Anal. Appl. **306** (2005), 589–603.

[11] SUN, Y.—LIU, L.: *Positive solutions of nonlinear third-order three point boundary value problem*, Far East J. Appl. Math. **12** (2003), 79–91.

[12] YAO, G.: *Successive iteration of positive solution for a discontinuous third-order boundary value problem*, Comput. Math. Appl. **53** (2007), 741–749.

[13] YAO, Q.: *The existence and multiplicity of positive solutions for a third-order three-point boundary value problem*, Acta Math. Appl. Sinica (English Ser.) **19** (2003), 117–122.

[14] YAO, Q.: *The existence of solution for a third-order two-point boundary value problem*, Appl. Math. Lett. **15** (2002), 227–232.

[15] ZHANG, G.—SUN, J.: *Positive solutions of m-point boundary value problems*, J. Math. Anal. Appl. **291** (2004), 406–418.

Received 15. 9. 2008
Accepted 19. 11. 2009

* *School of Mathematics and Physics*
Changzhou University
Changzhou 213164
P.R. CHINA
E-mail: fengwang188@163.com

** *Department of Applied Mathematics*
Shandong University of Science and Technology
Qingdao 266510
P.R. CHINA