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Mathematica Slovaca

DOI: 10.2478/s12175-010-0026-6
Math. Slovaca **60** (2010), No. 4, 471–484

SOME RESULTS ON CERTAIN CLASSES OF MULTIVALENTLY ANALYTIC FUNCTIONS BASED ON DIFFERENTIAL SUBORDINATION INVOLVING A CONVOLUTION STRUCTURE

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(*Communicated by Michal Fečkan*)

ABSTRACT. Making use of the familiar convolution structure of analytic functions, we introduce a general class of multivalently analytic functions and derive various useful properties and characteristics of this function class by using the techniques of differential subordination. Several other results are presented exhibiting relevant connections to some of the results obtained in earlier works.

1. Introduction

Let $\mathscr{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \in \mathbb{N} = \{1, 2, \dots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disk

$$\mathbb{U} = \{z : \ z \in \mathbb{C}, \ |z| < 1\}.$$

Given the functions f and g in $\mathscr{A}_p$, we say that f is subordinate to g in $\mathbb{U}$, and write $f \prec g$, if there exists a function $w(z)$ analytic in $\mathbb{U}$ such that $|w(z)| < 1$, $z \in \mathbb{U}$, and $w(0) = 0$ with $f(z) = g(w(z))$ in $\mathbb{U}$. If f is univalent in $\mathbb{U}$, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

2000 Mathematics Subject Classification: Primary 30C45; Secondary 26A33, 33C20.
Keywords: multivalently analytic functions, Hadamard product (or convolution), differential subordination, Dziok-differintegral operator, hypergeometric functions, Dziok-Srivastava linear operator.

Dziok [5] introduced a differintegral operator $\Omega_\beta^\sigma f(z)$ for the function $f(z) \in \mathscr{A}_p$ which is defined as follows:

$$\Omega_\beta^\sigma f(z) = \begin{cases} \frac{z^{1-\sigma-\beta}}{\Gamma(\sigma)} \int\limits_0^z (z-\xi)^{\sigma-1} \xi^{\beta-1} f(\xi)\,\mathrm{d}\xi & (\Re(\sigma) > 0,\ \Re(\beta) > -p); \\ \frac{z^{1-\sigma-\beta}}{\Gamma(1+\sigma)} \frac{\mathrm{d}}{\mathrm{d}z} \int\limits_0^z (z-\xi)^{\sigma} \xi^{\beta-1} f(\xi)\,\mathrm{d}\xi & (-1 < \Re(\sigma) \le 0,\ \Re(\beta) > -p), \end{cases} \tag{1.2}$$

where the multiplicities of $(z-\xi)^{\sigma-1}$ and $(z-\xi)^{\sigma}$ are removed by requiring that $\log(z-\xi) \in \mathbb{R}$ for $(z-\xi) > 0$. The operator $\Omega_\beta^\sigma f(z)$ on using (1.1) gives

$$\Omega_\beta^\sigma f(z) = \sum_{n=p}^{\infty} \frac{\Gamma(n+\beta)}{\Gamma(n+\beta+\sigma)} a_n z^n \qquad (a_p = 1).$$

For the function $f(z)$ defined by (1.1), we define a linear operator

$$\Delta_\beta^\sigma(f)(z) \colon \mathscr{A}_p \to \mathscr{A}_p \qquad (z \in \mathbb{U})$$

by

$$\begin{aligned} \Delta_\beta^\sigma(f)(z) &= \frac{\Gamma(p+\sigma+\beta)}{\Gamma(p+\beta)} \Omega_\beta^\sigma(f)(z) \\ &= z^p + \frac{\Gamma(p+\sigma+\beta)}{\Gamma(p+\beta)} \sum_{k=p+1}^{\infty} \frac{\Gamma(k+\beta)}{\Gamma(k+\sigma+\beta)} a_k z^k \\ &\qquad (\Re(\sigma+\beta) > 0;\ \ z \in \mathbb{U}). \end{aligned} \tag{1.3}$$

The operator $\Delta_\beta^\sigma(f)(z) \in \mathscr{A}_p$ satisfies the following three-term recurrence relation

$$z\left(\Delta_\beta^\sigma(f)(z)\right)' = (p+\sigma+\beta-1)\Delta_\beta^{\sigma-1}(f)(z) - (\sigma+\beta-1)\Delta_\beta^\sigma(f)(z), \tag{1.4}$$

and from (1.3), one gets

$$\Delta_\beta^1(f)(z) = \mathscr{K}_{p,\beta}(f)(z); \qquad \Delta_\beta^0(f)(z) = f(z),$$

where $\mathscr{K}_{p,\beta}(f)(z)$ denotes the well known generalized Bernardi-Libera-Livingston integral operator ([3]).

If $f \in \mathscr{A}_p$ is given by (1.1) and $g \in \mathscr{A}_p$ is given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, \tag{1.5}$$

then the Hadamard product (or convolution) $f * g$ of f and g is defined (as usual) by

$$(f*g)(z) := z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k. \tag{1.6}$$

We denote by $\mathscr{J}_p^{m,\lambda}(g;\alpha,A,B)$ the class of functions $f(z)$ of the form (1.1) satisfying in terms of subordination the following condition for a given function $g(z)\in\mathscr{A}_p$ (defined by (1.5)):

$$(1-\alpha)\frac{r(p,m)(f*g)^{(m)}(z)}{z^{p-m}}+\alpha\frac{r(p,m+1)(f*g)^{(m+1)}(z)}{z^{p-m-1}}\prec\left(\frac{1+Az}{1+Bz}\right)^{\lambda} \quad (1.7)$$

where $\alpha\geq 0$, $r(p,m)=\frac{(p-m)!}{p!}$, $p>m$ ($m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$), $-1\leq B<A\leq 1$, $0<\lambda\leq 1$, $z\in\mathbb{U}$.

For notational brevity, we put

$$\mathscr{J}_p^{m,1}\left(g;\alpha,1-\frac{2\beta}{p-m},-1\right)=\mathscr{J}_p^{*}(g;\alpha,\beta,m),$$

where $\mathscr{J}_p^{*}(g;\alpha,\beta,m)$ denotes the class of functions $f\in\mathscr{A}_p$ which satisfy the inequality

$$\Re\left((1-\alpha)\frac{r(p,m)(f*g)^{(m)}(z)}{z^{p-m}}+\alpha\frac{r(p,m+1)(f*g)^{(m+1)}(z)}{z^{p-m-1}}\right)>\frac{\beta}{p-m} \quad (1.8)$$

where $\alpha\geq 0$, $0\leq\beta<p-m$, $r(p,m)=\frac{(p-m)!}{p!}$, $p>m$ ($m\in\mathbb{N}_0$), $z\in\mathbb{U}$.

The above function classes $\mathscr{J}_p^{m,\lambda}(g;\alpha,A,B)$ and $\mathscr{J}_p^{*}(g;\alpha,\beta,m)$ contain several known classes of functions which have been studied in the recent past. For example, if the coefficients b_k in (1.5) and the values of m, λ and α in (1.7) are, respectively, set as follows:

$$b_k=\frac{(\alpha_1)_{k-p}\cdots(\alpha_q)_{k-p}}{(\beta_1)_{k-p}\cdots(\beta_s)_{k-p}(k-p)!},\qquad m=0;\quad \lambda=1\quad\text{and}\quad \alpha=\frac{p\delta}{\alpha_i}, \quad (1.9)$$

$\alpha_i>0$ ($i=1,\ldots,q$), $\beta_j>0$ ($j=1,\ldots,s$), $q\leq s+1$; $q,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\delta>0$, where the symbol $(a)_k$ occurring in (1.9) is the familiar Pochhammer symbol defined by

$$(a)_0=1,\qquad (a)_k=a(a+1)\cdots(a+k-1),\quad k\in\mathbb{N},$$

then on using the identity ([8]):

$$z\left(H_s^q[\alpha_i]f\right)'(z)=\alpha_i\left(H_s^q[\alpha_i+1]f(z)\right)-(\alpha_i-p)\left(H_s^q[\alpha_i]f(z)\right)\qquad (i=1,\ldots,q) \quad (1.10)$$

in (1.7), we note that the class $\mathscr{J}_p^{m,\lambda}(g;\alpha,A,B)$ reduces to a known function class studied by Liu [11].

It may be observed here that the linear operator

$$\left(H_s^q[\alpha_1]f\right)(z):=H_s^q(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z)$$

involved in the identity (1.10) is the Dziok-Srivastava linear operator ([8]) (see also [9]) used in many of the recent works, and includes such well known operators as the Hohlov linear operator, Saitoh generalized linear operator, the Carlson-Shaffer linear operator, the Ruscheweyh derivative operator, the

Bernardi-Libera-Livingston operator, and the Srivastave-Owa fractional derivative operator (as well as their various generalized versions). For these various reducible cases of operators (and for further extension) of the Dziok-Srivastava operator, one may refer to the papers [6], [7], [8], [9] and [18]. Aouf and Darwish [2] also studied a class obtainable by suitably specializing the parameters in (1.7) and the sequence b_k in (1.5) and using (1.10). Also, for $m = 0$ and $\lambda = 1$, the class $\mathscr{J}_p^{0,1}(g; \alpha, A, B)$ was very recently investigated in [15] in which different characteristics and properties were studied.

This paper studies some useful properties and characteristics of the function classes $\mathscr{J}_p^{m,\lambda}(g; \alpha, A, B)$, $\mathscr{J}_p^{*}(g; \alpha, \beta, m)$ and their associated classes (defined above) by using the differential subordination methods. Special cases of some of the main results with relevances to known results are briefly mentioned.

2. Key lemmas

The following lemmas are required for investigating the function classes defined above.

Lemma 1. (Miller and Mocanu [12]) *Let $h(z)$ be a convex (univalent) function in $\mathbb{U}$ with $h(0) = 1$, and let the function $\phi(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be analytic in $\mathbb{U}$. If*

$$\phi(z) + \frac{z\phi'(z)}{\gamma} \prec h(z) \tag{2.1}$$

for $\gamma \neq 0$ and $\Re(\gamma) \geq 0$, then

$$\phi(z) \prec \psi(z) := \frac{\gamma}{z^\gamma} \int_0^z t^{\gamma-1} h(t)\, \mathrm{d}t \prec h(z) \tag{2.2}$$

and $\psi(z)$ is the best dominant.

The generalized hypergeometric function ${}_pF_q$ is defined by (cf., e.g. [19, p. 333])

$$\begin{aligned} {}_pF_q(z) &\equiv {}_pF_q(\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_s; z) \\ &:= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \cdot \frac{z^n}{n!}, \end{aligned} \tag{2.3}$$

$z \in \mathbb{U}$, $\alpha_j \in \mathbb{C}$ $(j = 1, \dots, q)$, $\beta_j \in \mathbb{C} \backslash \{0, -1, -2, \dots\}$ $(j = 1, \dots, s)$, $q \leq s + 1$, $q, s \in \mathbb{N}_0$.

The following formulas are well known ([10, pp. 556–558]).

Lemma 2. *For real or complex numbers a, b and c $(c \neq 0, -1, -2, \dots)$:*

(i)

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,\mathrm{d}t = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a,b;c;z). \tag{2.4}$$

(ii)

$$F_1(a,b;c;z) = (1-z)^{-a}\,{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right). \tag{2.5}$$

(iii)

$${}_2F_1\left(1,1;3;\frac{az}{az+1}\right) = \frac{2(1+az)}{az}\left(1 - \frac{\ln(1+az)}{az}\right). \tag{2.6}$$

Lemma 3. (see [17]) *Let $q(z)$ be univalent in $\mathbb{U}$ with $q(0) = 1$ and $\psi, \gamma \in \mathbb{C}$. Further, assume that*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} + \frac{\psi}{\gamma}\right) > 0 \qquad (z \in \mathbb{U}).$$

If $p(z)$ is analytic in $\mathbb{U}$, and

$$\psi p(z) + \gamma z p'(z) \prec \psi q(z) + \gamma z q'(z),$$

then $p(z) \prec q(z)$, and $q(z)$ is the best dominant.

Lemma 4. (see [13, p. 2, Lemma 1.1]) *Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ and γ be a complex number satisfying $\Re(\gamma) \geq 0$ $(\gamma \neq 0)$, then*

$$\Re[p(z) + \gamma z p'(z)] > \beta \qquad (0 \leq \beta < 1)$$

implies that

$$\Re\left(p(z)\right) > \beta + (1-\beta)(2\lambda - 1),$$

where λ is given by

$$\lambda = \lambda_{\Re(\gamma)} = \int_0^1 \left(1 + t^{\Re(\gamma)}\right)^{-1}\,\mathrm{d}t.$$

3. Main results

Our first main result is contained in the following:

Theorem 1. *Let $\alpha > 0$, $p > m$ $(m \in \mathbb{N}_0)$, $-1 \leq B < A \leq 1$; $0 < \lambda \leq 1$. If $f(z) \in \mathscr{J}_p^{m,\lambda}(g; \alpha, A, B)$, then*

$$\frac{r(p,m)(f*g)^{(m)}(z)}{z^{p-m}} \prec \mathscr{X}(z) \prec \left(\frac{1+Az}{1+Bz}\right)^{\lambda} \qquad (z \in \mathbb{U}), \tag{3.1}$$

where

$$\mathscr{X}(z) = \begin{cases} \left(\frac{A}{B}\right)^{\lambda} \sum\limits_{i \geq 0} \frac{(-\lambda)_i}{i!} \left(\frac{A-B}{A}\right)^{i} (1+Bz)^{-i} {}_2F_1\left(i, 1; 1+\frac{p-m}{\alpha}; \frac{Bz}{1+Bz}\right) & (B \neq 0); \\ {}_2F_1\left(-\lambda, \frac{p-m}{\alpha}; 1+\frac{p-m}{\alpha}; -Az\right) & (B = 0), \end{cases}$$

and $\mathscr{X}(z)$ *is the best dominant of* (3.1). *Also,*

$$\Re\left(\frac{r(p,m)\,(f*g)^{(m)}(z)}{z^{p-m}}\right) > \mathscr{X}(-1). \tag{3.2}$$

The result (3.2) *is sharp.*

Proof. Let $f(z) \in \mathscr{J}_p^{m,\lambda}(g;\alpha,A,B)$, and put

$$\frac{r(p,m)\,(f*g)^{(m)}(z)}{z^{p-m}} = \theta(z). \tag{3.3}$$

It is observed that the function $\theta(z)$ is of the form

$$\theta(z) = 1 + c_1 z + c_2 z^2 + \cdots,$$

which is analytic in $\mathbb{U}$ with $\theta(0) = 1$. Now differentiating (3.3) with respect to z, we get

$$\frac{r(p,m+1)(f*g)^{(m+1)}(z)}{z^{p-m-1}} = \theta(z) + \frac{1}{p-m} z\theta'(z), \tag{3.4}$$

and applying (1.7), (3.3) and (3.4), we arrive at

$$\begin{aligned} &(1-\alpha)\frac{r(p,m)(f*g)^{(m)}(z)}{z^{p-m}} + \alpha\frac{r(p,m+1)(f*g)^{(m+1)}(z)}{z^{p-m-1}} \\ &= \theta(z) + \frac{\alpha}{p-m} z\theta'(z) \prec \left(\frac{1+Az}{1+Bz}\right)^{\lambda} = h(z) \qquad (z \in \mathbb{U}). \end{aligned}$$

It is easy to verify that the above function $h(z)$ is analytic and convex in $\mathbb{U}$ because

$$\begin{aligned} \Re\left(1 + \frac{zh''(z)}{h'(z)}\right) &= -1 + (1-\lambda)\Re\left(\frac{1}{1+Az}\right) + (1+\lambda)\Re\left(\frac{1}{1+Bz}\right) \\ &> -1 + \frac{1-\lambda}{1+|A|} + \frac{1+\lambda}{1+|B|} \geq 0 \qquad (z \in \mathbb{U}). \end{aligned}$$

Applying Lemma 1, we thus obtain

$$\theta(z) \prec \frac{p-m}{\alpha} z^{-\frac{p-m}{\alpha}} \int_0^z t^{\frac{p-m}{\alpha}-1} \left(\frac{1+At}{1+Bt}\right)^{\lambda} \mathrm{d}t.$$

In order to evaluate the integral, we write the integrand in the form

$$t^{\frac{p-m}{\alpha}-1}\left(\frac{1+At}{1+Bt}\right)^{\lambda} = t^{\frac{p-m}{\alpha}-1}\left(\frac{A}{B}\right)^{\lambda}\left(1 - \frac{A-B}{A(1+Bt)}\right)^{\lambda},$$

and by expanding the binomial expression in the integrand, changing the order of integration and summation (justified on account of the conditions mentioned in Theorem 1), and using Lemma 2, we obtain (after elementary calculations)

$$\theta(z) \prec \begin{cases} \left(\frac{A}{B}\right)^{\lambda} \sum\limits_{i\geq 0} \frac{(-\lambda)_i}{i!} \left(\frac{A-B}{A}\right)^{i} (1+Bz)^{-i}\, {}_2F_1\left(i,1;1+\frac{p-m}{\alpha};\frac{Bz}{1+Bz}\right) & (B\neq 0); \\ {}_2F_1\left(-\lambda,\frac{p-m}{\alpha};1+\frac{p-m}{\alpha};-Az\right) & (B=0). \end{cases}$$
$$= \mathscr{X}(z). \tag{3.5}$$

To establish (3.2), we infer (under the conditions stated with Theorem 1) that

$$\begin{aligned} \Re\left(\frac{r(p,m)\,(f*g)^{(m)}(z)}{z^{p-m}}\right) &= \frac{p-m}{\alpha}\int_0^1 u^{\frac{p-m}{\alpha}-1}\Re\left(\frac{1+Auw(z)}{1+Buw(z)}\right)^{\lambda} du \\ &> \frac{p-m}{\alpha}\int_0^1 u^{\frac{p-m}{\alpha}-1}\left(\frac{1-Au}{1-Bu}\right)^{\lambda} du. \end{aligned}$$

The sharpness of the result (3.2) can be established by considering the functions $\mathscr{X}(z)$ defined by (3.5). It is sufficient to show that

$$\inf_{|z|<1}\{\Re(\mathscr{X}(z)\} = \mathscr{X}(-1).$$

We observe from (3.5) that

$$\begin{aligned} \Re\{\mathscr{X}(z)\} \geq \frac{p-m}{\alpha}\int_0^1 u^{\frac{p-m}{\alpha}-1}\Re\left(\frac{1+Aur}{1+Bur}\right)^{\lambda} du = \mathscr{X}(-r) \quad (|z|\leq r \ \ (0<r<1)) \\ \longrightarrow \mathscr{X}(-1) \quad \text{as} \ \ r\to 1-, \end{aligned}$$

and this completes the proof of Theorem 1. □

Remark 1. We deem it appropriate here to point out minor corrections in the main results of [16]. The subordinated function mentioned in [16, Theorem 3.1, p. 131; Corollary 3.1, p. 133; Theorem 3.2, p. 133; Theorem 3.3, p. 134; Theorem 3.4, p. 135] are expressed in terms of a series with summation index from 0 to m. This series, however, should have the same summation index as mentioned above in (3.5).

Remark 2. We observe that if we use the parametric substitutions given by (1.9) and apply the identity (1.10), then Theorem 1 would yield the results given recently by Liu [11, p. 3, Theorem 2.4]. Further, if we set the coefficient b_k in (1.5) and values of the parameters in (1.7) are choosen as follows:

$$b_k = \frac{(\nu+p)_{k-p}}{(k-p)!}, \quad m=0, \quad \lambda=1, \quad \alpha=\frac{\lambda p}{\nu+p} \qquad (\nu>-p\,, \ \ \lambda>0, \ \ p\in\mathbb{N}) \tag{3.6}$$

and making use of the identity [4, p. 124, Eq. (4)], Theorem 1 reduces to a recently established result due to Dinggong and Liu [4, Theorem 1].

If we choose $m = 0$ and $\alpha = 1$ in Theorem 1, we get the following result.

Corollary 1. *Let* $1 \leq B < A \leq 1$ *and* $0 < \lambda \leq 1$. *If*

$$\frac{(f*g)'(z)}{z^{p-1}} \prec p\left(\frac{1+Az}{1+Bz}\right)^{\lambda} \qquad (z \in \mathbb{U}),$$

then

$$\Re\left(\frac{(f*g)(z)}{z^p}\right) >$$
$$> \begin{cases} \left(\frac{A}{B}\right)^{\lambda} \sum\limits_{i\geq 0} \frac{(-\lambda)_i}{i!} \left(\frac{A-B}{A}\right)^{i} (1-B)^{-i} {}_2F_1\left(i,1;p+1;\frac{B}{B-1}\right) & (B \neq 0), \\ {}_2F_1\left(-\lambda,p;p+1;A\right) & (B = 0), \end{cases}$$

and the result is sharp.

Theorem 2. *If* $f(z) \in \mathscr{J}_p^{m,1}(g;\alpha,A,0)$, *then*

$$\left|\frac{z\left(f*g\right)^{(m+1)}(z)}{\left(f*g\right)^{(m)}(z)} - (p-m)\right| < \frac{(p-m)(2\alpha+p-m)A}{\alpha(\alpha+p-m+\alpha A)}. \tag{3.7}$$

P r o o f. If $f(z) \in \mathscr{J}_p^{m,1}(g;\alpha,A,0)$, then

$$(1-\alpha)\frac{r(p,m)(f*g)^{(m)}(z)}{z^{p-m}} + \alpha\frac{r(p,m+1)(f*g)^{(m+1)}(z)}{z^{p-m-1}} \prec 1+Az, \tag{3.8}$$

which implies that

$$\left|(1-\alpha)\frac{r(p,m)(f*g)^{(m)}(z)}{z^{p-m}} + \alpha\frac{r(p,m+1)(f*g)^{(m+1)}(z)}{z^{p-m-1}} - 1\right| < A. \tag{3.9}$$

By virtue of Theorem 1, we obtain

$$\left|\frac{r(p,m)(f*g)^{(m)}(z)}{z^{p-m}} - 1\right| < \frac{\alpha A}{\alpha+p-m}. \tag{3.10}$$

Noting that

$$\begin{aligned} &\alpha\left|\frac{r(p,m+1)(f*g)^{(m+1)}(z)}{z^{p-m-1}} - \frac{r(p,m)(f*g)^{(m)}(z)}{z^{p-m}}\right| \\ &\leq \left|(1-\alpha)\frac{r(p,m)(f*g)^{(m)}(z)}{z^{p-m}} + \alpha\frac{r(p,m+1)(f*g)^{(m+1)}(z)}{z^{p-m-1}} - 1\right| \\ &\qquad + \left|\frac{r(p,m)(f*g)^{(m)}(z)}{z^{p-m}} - 1\right|, \end{aligned}$$

the inequalities (3.9) and (3.10) yield

$$\alpha\left|\frac{r(p,m+1)(f*g)^{(m+1)}(z)}{z^{p-m-1}}-\frac{r(p,m)(f*g)^{(m)}(z)}{z^{p-m}}\right|<A+\frac{\alpha A}{\alpha+p-m}. \tag{3.11}$$

On using the inequality (3.10) once more, the above inequality (3.11) gives

$$\begin{aligned}&\left|\frac{z\,(f*g)^{(m+1)}(z)}{(f*g)^{(m)}(z)}-(p-m)\right|\\&<\frac{(p-m)(2\alpha+p-m)A}{\alpha(\alpha+p-m)}\left|\frac{r(p,m)\,(f*g)^{(m)}(z)}{z^{p-m}}\right|^{-1}\\&<\frac{(p-m)(2\alpha+p-m)A}{\alpha(\alpha+p-m)}\left(\frac{\alpha+p-m}{\alpha+p-m+\alpha A}\right),\end{aligned}$$

which yields the desired assertion (3.7). □

Remark 3. A similar known result due to Aghalary et. al. [1, Theorem 3.9] is easily obtainable from Theorem 2 when we use the parametric substitutions given by (3.6) and the identity [4, p. 124, Eq. (4)] (in the case when $p=1$).

By setting

$$b_k=\frac{\Gamma(p+\sigma+\beta)\Gamma(k+\beta)}{\Gamma(p+\beta)\Gamma(k+\sigma+\beta)}\qquad(\sigma+\beta>0;\quad k=p+1,p+2,\dots)$$

in (1.6), so that in view of (1.3): $(f*g)(z)=(\Delta_\beta^\sigma f)(z)$, then Theorem 2 in conjunction with (1.7) (when $\alpha=\lambda=1, B=0$) yields the following:

Corollary 2. *Let*

$$\frac{r(p,m+1)(\Delta_\beta^\sigma f)^{(m+1)}(z)}{z^{p-m-1}}\prec 1+Az\qquad(p>m\ \ (m\in\mathbb{N}_0);\quad z\in\mathbb{U}),$$

where Δ_β^σ *is the operator defined by* (1.3), *then*

$$\left|\frac{z(\Delta_\beta^\sigma f)^{(m+1)}(z)}{(\Delta_\beta^\sigma f)^{(m)}(z)}-(p-m)\right|<\frac{(p-m)(2+p-m)A}{1+p-m+A}.$$

Theorem 3. *Let* $f(z)\in\mathscr{A}_p$. *If*

$$(1-\alpha)\frac{(\Delta_\beta^\sigma(f)*g)^{(m)}(z)}{z^{p-m}}+\alpha\frac{(\Delta_\beta^{\sigma-1}(f)*g)^{(m)}(z)}{z^{p-m}}\prec\frac{1}{r(p,m)}\left(\frac{1+Az}{1+Bz}\right)^\lambda$$
$$(\alpha>0,\quad z\in\mathbb{U}), \tag{3.12}$$

where $\Delta_\beta^\sigma(f)$ *is defined by* (1.3)*, then*

$$\frac{r(p,m)(\Delta_\beta^\sigma(f)*g)^{(m)}(z)}{z^(p-m)} \prec \Theta(z) \prec \left(\frac{1+Az}{1+Bz}\right)^\lambda \qquad (z\in\mathbb{U}), \tag{3.13}$$

where

$$\Theta(z)=\begin{cases} \left(\frac{A}{B}\right)^\lambda \sum\limits_{i\geq 0}\frac{(-\lambda)_i}{i!}\left(\frac{A-B}{A}\right)^i(1+Bz)^{-i}\cdot \\ \cdot\, {}_2F_1\left(i,1;1+\frac{p+\beta+\sigma-1}{\alpha};\frac{Bz}{1+Bz}\right) & (B\neq 0), \\ {}_2F_1\left(-\lambda,\frac{p+\beta+\sigma-1}{\alpha};1+\frac{p+\beta+\sigma-1}{\alpha};-Az\right) & (B=0), \end{cases}$$

and $\Theta(z)$ *is the best dominant of* (3.13)*. Also*

$$\Re\left(\frac{r(p,m)(\Delta_\beta^\sigma(f)*g)^{(m)}(z)}{z^{p-m}}\right) > \Theta(-1). \tag{3.14}$$

The result (3.14) *is sharp.*

P r o o f. It follows from (1.3) that

$$\begin{aligned} &z(\Delta_\beta^\sigma(f)*g)^{(m+1)}(z) \\ &=(p+\beta+\sigma-1)(\Delta_\beta^{\sigma-1}(f)*g)^{(m)}(z)-(\beta+\sigma+m-1)(\Delta_\beta^\sigma(f)*g)^{(m)}(z). \end{aligned} \tag{3.15}$$

Assume that

$$\frac{r(p,m)(\Delta_\beta^\sigma(f)*g)^{(m)}(z)}{z^{p-m}}=\psi(z).$$

Differentiating with respect to z and using (3.12) and (3.15), we get

$$\psi(z)+\frac{\alpha z\psi'(z)}{p+\beta+\sigma-1}\prec\left(\frac{1+Az}{1+Bz}\right)^\lambda \qquad (z\in\mathbb{U}),$$

and following similar steps as in the proof of Theorem 1, we get the desired result. $\square$

Putting $\lambda=\alpha=1$, $m=0$ in Theorem 3, and observing that

$$(\Delta_\beta^\sigma(f)*g)(z)=\frac{\Gamma(p+\beta+\sigma)}{\Gamma(p+\beta)\Gamma(\sigma)}\int_0^1(1-t)^{\sigma-1}t^{\beta-1}(f*g)(zt)\,\mathrm{d}t$$

$$(f\in\mathscr{A}_p,\quad \Re(\sigma)>0,\quad \Re(\beta)>-p;\quad z\in\mathbb{U}),$$

we get the following:

Corollary 3. *If $f(z) \in \mathscr{A}_p$, such that*

$$\frac{(\Delta_\beta^{\sigma-1}(f) * g)(z)}{z^p} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}), \tag{3.16}$$

then

$$\Re\left(\frac{\Gamma(p+\beta+\sigma)}{\Gamma(p+\beta)\Gamma(\sigma)}\int_0^1 (1-t)^{\sigma-1}t^{\beta-1}(f*g)(zt)\,\mathrm{d}t\right) > \xi \qquad (z \in \mathbb{U}), \tag{3.17}$$

where

$$\xi = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1-B)^{-1}{}_2F_1\left(1,1;p+\beta+\sigma;\frac{B}{B-1}\right) & (B \neq 0); \\ 1-\left(\frac{p+\beta+\sigma-1}{p+\beta+\sigma}\right)A & (B=0). \end{cases} \tag{3.18}$$

The result (3.17) *it is best possible.*

A special case of Corollary 3 (when $\sigma = 1$) gives the known result [16, p. 135, Corollary 3.2]. A further special case of Corollary 3 when $A = 1-2\delta$ $(0 \le \delta < 1)$, $B = -1$, $p = 1$ and $g(z) = z/(1-z)$ would immediately yield the following result.

Corollary 4. *If $f(z) \in \mathscr{A}$ $(\mathscr{A}_1 = \mathscr{A})$, $\Re(\sigma) > 0$, $\Re(\beta) > -1$ and*

$$\Re\left(\frac{\Delta_\beta^{\sigma-1}f(z)}{z}\right) > \delta \qquad (0 \le \delta < 1; \quad z \in \mathbb{U}), \tag{3.19}$$

then

$$\Re\left(\frac{\Gamma(1+\beta+\sigma)}{\Gamma(1+\beta)\Gamma(\sigma)}\int_0^1 (1-t)^{\sigma-1}t^{\beta-1}f(zt)\,\mathrm{d}t\right) > \tau, \tag{3.20}$$

where

$$\tau = \delta + (1-\delta)\left({}_2F_1\left(1,1;1+\beta+\sigma;\frac{1}{2}\right)-1\right).$$

Remark 4. Corollary 4 provides a new variation of a similar result of Obradovic [14, Theorem 1, p. 677]. Also, on putting $\sigma = \beta = 1$ and using the result (iii) of Lemma 2, we can easily deduce a much simpler form from Corollary 4.

Theorem 4. *Let $q(z)$ be univalent in $\mathbb{U}$ and $q(z)$ satisfy the inequality*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right) + \frac{p-m}{\alpha} > 0 \qquad (z \in \mathbb{U}). \tag{3.21}$$

If $f \in \mathscr{A}_p$ satisfies the subordination condition

$$(1-\alpha)\frac{r(p,m)\,(f*g)^{(m)}(z)}{z^{p-m}} + \alpha\frac{r(p,m+1)\,(f*g)^{(m+1)}(z)}{z^{p-m-1}} \prec q(z) + \frac{\alpha}{p-m}zq'(z), \tag{3.22}$$

then

$$\frac{r(p,m)\,(f*g)^{(m)}(z)}{z^{p-m}} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. If we set

$$\theta(z) = \frac{r(p,m)\,(f*g)^{(m)}(z)}{z^{p-m}},$$

then it readily follows that

$$\theta(z) + \frac{\alpha}{p-m} z\theta'(z) \prec q(z) + \frac{\alpha}{p-m} zq'(z).$$

Applying now Lemma 3 (when $\gamma = \frac{\alpha}{p-m}$ and $\psi = 1$ therein), we infer that $\theta(z) \prec q(z)$, which proves Theorem 4.

If we put

$$q(z) = \frac{1+Az}{1+Bz} \qquad (-1 \leq B < A \leq 1),$$

Theorem 4 yields the following result. $\square$

Corollary 5. *Let $-1 \leq B < A \leq 1$ and*

$$\Re\left(\frac{1-Bz}{1+Bz}\right) + \frac{p-m}{\alpha} > 0 \qquad (z \in \mathbb{U}).$$

If $f \in \mathscr{A}_p$ satisfies the following subordination:

$$(1-\alpha)\frac{r(p,m)\,(f*g)^{(m)}(z)}{z^{p-m}} + \alpha\frac{r(p,m+1)\,(f*g)^{(m+1)}(z)}{z^{p-m-1}} \prec \frac{\alpha(A-B)z}{(p-m)(1+Bz)^2} + \frac{1+Az}{1+Bz},$$

then

$$\frac{r(p,m)\,(f*g)^{(m)}(z)}{z^{p-m}} \prec \frac{1+Az}{1+Bz}.$$

Theorem 5. *If $f(z) \in \mathscr{I}_p^*(g;\alpha,\beta,m)$, then*

$$\Re\left(\frac{(f*g)^{(m)}(z)}{z^{p-m}}\right) > \frac{\beta + (p-\beta-m)(2\lambda-1)}{(p-m)r(p,m)} \qquad (z \in \mathbb{U};\quad 0 \leq \beta < p-m), \tag{3.23}$$

where

$$\lambda = \int_0^1 \left(1 + t^{\frac{\alpha}{p-m}}\right)^{-1} \mathrm{d}t. \tag{3.24}$$

Proof. Let $f(z) \in \mathscr{J}_p^*(g;\alpha,\beta,m)$. Following same steps as in the proof of Theorem 1, we have

$$\Re\left((1-\alpha)\frac{r(p,m)\,(f*g)^{(m)}(z)}{z^{p-m}}+\alpha\frac{r(p,m+1)\,(f*g)^{(m+1)}(z)}{z^{p-m-1}}\right)$$
$$=\Re\left(\theta(z)+\frac{\alpha}{p-m}z\theta'(z)\right)>\frac{\beta}{p-m}\qquad(0\le\beta<p-m).$$

The assertion (3.23) of Theorem 5 follows by applying Lemma 4 (when $\gamma = \frac{\alpha}{p-m}$), which proves Theorem 5. □

To cite here an application of Theorem 5, we put

$$b_k=\frac{(\alpha_1)_{k-p}\cdots(\alpha_q)_{k-p}}{(\beta_1)_{k-p}\cdots(\beta_s)_{k-p}(k-p)!}$$

$(\alpha_i>0\ (i=1,\dots,q),\ \beta_j>0\ (j=1,\dots,s),\ q\le s+1;\ q,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ in (1.6), and noting that

$$(f*g)(z)=(H_s^q[\alpha_1]f)(z):=H_s^q(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s)f(z),$$

where $(H_s^q[\alpha_1]f)(z)$ is the Dziok-Srivastava linear operator [8], and setting $p=\alpha=1$ and $m=0$ in Theorem 5, and using (1.8), we get the following result.

Corollary 6. *If*

$$\Re\left\{z\,(H_s^q[\alpha_1]f)'(z)\right\}>\beta\qquad(z\in U;\ \ 0\le\beta<1),$$

then

$$\Re\left\{\frac{(H_s^q[\alpha_1]f)(z)}{z}\right\}>\beta+(1-\beta)(2\log 2-1).$$

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Received 3. 9. 2008
Accepted 3. 12. 2008

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