

SOME PROPERTIES OF EXTENDED REMAINDER OF BINET'S FIRST FORMULA FOR LOGARITHM OF GAMMA FUNCTION

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ABSTRACT. In the paper, we extend Binet's first formula for the logarithm of the gamma function and investigate some properties, including inequalities, star-shaped and sub-additive properties and the complete monotonicity, of the extended remainder of Binet's first formula for the logarithm of the gamma function and related functions.

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1. Introduction

1.1

For positive numbers x and y with $y > x$, let

$$g_{x,y}(t) = \int_x^y u^{t-1} du = \begin{cases} \frac{y^t - x^t}{t}, & t \neq 0, \\ \ln y - \ln x, & t = 0. \end{cases}$$

The reciprocal of $g_{x,y}(t)$ can be rewritten as

$$\frac{1}{g_{e^a, e^b}(t)} = F_{a,b}(t) = \begin{cases} \frac{t}{e^{bt} - e^{at}}, & t \neq 0, \\ \frac{1}{b - a}, & t = 0, \end{cases}$$

where a and b are real numbers with $b > a$.

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It is well-known ([11, p. 11]) that Binet's first formula of $\ln \Gamma(x)$ for $x > 0$ is given by

$$\ln \Gamma(x) = \left(x - \frac{1}{2}\right) \ln x - x + \ln \sqrt{2\pi} + \theta(x), \quad (1)$$

where

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$$

stands for Euler's gamma function and

$$\theta(x) = \int_0^{\infty} \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) \frac{e^{-xt}}{t} dt \quad (2)$$

for $x > 0$ is called the remainder of Binet's first formula for the logarithm of the gamma function.

In [28, 29], some inequalities and completely monotonic properties of the function $g_{x,y}(t)$ were established and applied to construct Steffensen pairs in [4], to refine Gautschi-Kershaw's inequalities in [23, 24], and to study monotonic properties, logarithmic convexities and Schur-convexities of extended mean values $E(r, s; x, y)$ in [13, 14, 15, 16, 25, 30]. See also [17, 18] for related contents.

In recent years, some inequalities and monotonic properties of the function

$$\frac{1}{t^2} - \frac{e^{-t}}{(1 - e^{-t})^2} \quad (3)$$

for $t > 0$ and related ones were researched in [5, 10, 31] and related references therein. These results were used in [2, 8, 26, 27] to consider completely monotonic properties of remainders of Binet's first formula, the psi function and related ones.

Recently, logarithmic convexities of $g_{x,y}(t)$ and $F_{a,b}(t)$ were found in [7, 22]. By virtue of these conclusions, some simple and elegant proofs for the logarithmic convexities, Schur-convexities of extended mean values $E(r, s; x, y)$ were simplified in [6, 19].

1.2

Now it is very natural to ask a question: Are there any relationship between the above studies? Direct computation yields

$$[g_{e^a, e^b}(t)]' = -[\ln F_{a,b}(t)]' = \frac{be^{bt} - ae^{at}}{e^{bt} - e^{at}} - \frac{1}{t} = \frac{b-a}{e^{(b-a)t} - 1} - \frac{1}{t} + b \triangleq \delta_{a,b}(t) \quad (4)$$

for $t \neq 0$. Therefore, if taking $a = -\frac{1}{2}$ and $b = \frac{1}{2}$, then $\delta_{-1/2, 1/2}(t)$ for $t > 0$ equals the integrand in the remainder of Binet's first formula for the logarithm of the gamma function and the first order derivative of $\delta_{-1/2, 1/2}(t)$ for $t > 0$

also equals the function (3). These relationships connect closely the above three seemingly unrelated problems.

If replacing $\delta_{-1/2,1/2}(t)$ by $\delta_{a,b}(t)$ on $t \in (0, \infty)$ for $b > a$ in (2), then a more problem emerges: How to calculate the improper integral

$$\int_0^\infty \left[\frac{b-a}{e^{(b-a)t} - 1} - \frac{1}{t} + b \right] \frac{e^{-tx}}{t} dt \quad (5)$$

for $b > a$? The following Theorem 1 answers this question affirmatively.

THEOREM 1. *Let $b > a$, $\alpha > 0$ and $x \in \mathbb{R}$ be real numbers. Then the improper integral (5) converges if and only if $a + b = 0$ and*

$$\int_0^\infty \left(\frac{\alpha}{e^{\alpha t} - 1} - \frac{1}{t} + \frac{\alpha}{2} \right) \frac{e^{-tx}}{t} dt = \alpha \ln \Gamma\left(\frac{x}{\alpha}\right) - \left(x - \frac{\alpha}{2}\right) \ln \frac{x}{\alpha} + x - \frac{\alpha}{2} \ln(2\pi). \quad (6)$$

Remark 1. It is easy to see that the formula (1) is the special case $\alpha = 1$ of (6). So we call (6) the extended remainder of Binet's first formula for the logarithm of the gamma function Γ .

1.3

It is well-known ([8, 20, 21]) that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \geq 0$$

for $x \in I$ and $n \geq 0$, that a function $f(x)$ is said to be star-shaped on $(0, \infty)$ if

$$f(\alpha x) \leq \alpha f(x)$$

for $x \in (0, \infty)$ and $0 < \alpha < 1$, that a function f is said to be super-additive on $(0, \infty)$ if

$$f(x + y) \geq f(x) + f(y)$$

for all $x, y > 0$, and that a function f is said to be sub-additive if $-f$ is super-additive.

The function $\delta_{a,b}(t)$ defined by (4) has the following properties.

THEOREM 2. *Let a and b be real numbers with $a \neq b$, and let $0 < \tau < 1$.*

(1) *The function $\delta_{a,b}(t)$ is increasingly concave on $(0, \infty)$ and convex on $(-\infty, 0)$;*

(2) *The inequality*

$$\delta_{a,b}(\tau t) < \delta_{a,b}(t) \quad (7)$$

is valid on $(0, \infty)$;

if either $a + b \neq 0$ or $ab \neq 0$, then the inequality (7) is sharp;

(3) If $a + b \geq 0$, then

$$\tau\delta_{a,b}(t) < \delta_{a,b}(\tau t) \quad (8)$$

is valid on $(0, \infty)$, i.e., the function $-\delta_{a,b}(t)$ is star-shaped;
 if $\max\{a, b\} \leq 0$, then the inequality (8) is reversed, i.e., the function $\delta_{a,b}(t)$ is star-shaped;
 if $a + b = 0$, then the inequality (8) is sharp.

Remark 2. Some properties of special cases of the function $\delta_{a,b}(t)$ and related ones have been investigated and applied extensively in [5, 6, 7, 8, 10, 22, 28, 29, 31] and related references therein.

1.4

If denoting the extended remainder of Binet's first formula for the logarithm of the gamma function Γ by

$$\theta_\alpha(x) = \int_0^\infty \left(\frac{\alpha}{e^{\alpha t} - 1} - \frac{1}{t} + \frac{\alpha}{2} \right) \frac{e^{-tx}}{t} dt \quad (9)$$

for $\alpha > 0$ and $x > 0$, then formula (6) in Theorem 1 can be simplified as

$$\theta_\alpha(x) = \alpha\theta\left(\frac{x}{\alpha}\right) \quad \text{or} \quad \theta_\alpha(\alpha x) = \alpha\theta(x).$$

This motivates us to study properties of $\theta_\alpha(x)$ and the function

$$f_{p,q;\alpha}(x) = \theta_\alpha(px) - q\theta_\alpha(x)$$

on $(0, \infty)$, where $p > 0$, $\alpha > 0$ and $q \in \mathbb{R}$, which may be concluded as the following theorem.

THEOREM 3. The extended remainder $\theta_\alpha(x)$ of Binet's first formula for the logarithm of the gamma function Γ satisfies

$$\frac{(-1)^k}{(1+\lambda)^k} \theta_\alpha^{(k)}\left(\frac{x}{1+\lambda}\right) > \frac{(-1)^k}{2} \left[\frac{1}{\lambda^k} \theta_\alpha^{(k)}\left(\frac{x}{\lambda}\right) + \theta_\alpha^{(k)}(x) \right] \quad (10)$$

for $x > 0$, $\lambda > 0$ with $\lambda \neq 1$, $k \geq 0$ and $\alpha > 0$.

The function $f_{p,q;\alpha}(x)$ is completely monotonic on $(0, \infty)$ if either $0 < p \leq 1$ and $q \leq 1$ or $p > 1$ and $q \leq \frac{1}{p}$; the function $-f_{p,q;\alpha}(x)$ is completely monotonic on $(0, \infty)$ if $p \geq 1$ and $q \geq 1$.

The function $-\theta_\alpha(x)$ is star-shaped and $\theta_\alpha(x)$ is sub-additive.

Remark 3. If taking $a = -\frac{1}{2}$ and $b = \frac{1}{2}$, then the results obtained in [2, 3, 8, 26, 27] can be deduced directly from Theorem 3.

2. Proofs of theorems

Proof of Theorem 1. By transformations of integral variables, it easily follows that

$$\begin{aligned}
 & \int_{\varepsilon}^{\infty} \left[\frac{b-a}{e^{(b-a)t}-1} - \frac{1}{t} + b \right] \frac{e^{-tx}}{t} dt \\
 &= \int_{(b-a)\varepsilon}^{\infty} \left(\frac{b-a}{e^u-1} - \frac{b-a}{u} + b \right) \frac{e^{-ux/(b-a)}}{u} du \\
 &= (b-a) \int_{(b-a)\varepsilon}^{\infty} \left(\frac{1}{e^u-1} - \frac{1}{u} + \frac{1}{2} \right) \frac{e^{-ux/(b-a)}}{u} du \\
 & \quad + \frac{a+b}{2} \int_{\varepsilon}^{\infty} \frac{e^{-ux/(b-a)}}{u} du \tag{11} \\
 &= (b-a) \int_{(b-a)\varepsilon}^{\infty} \left(\frac{1}{e^u-1} - \frac{1}{u} + \frac{1}{2} \right) \frac{e^{-ux/(b-a)}}{u} du \\
 & \quad + \frac{a+b}{2} \int_{(b-a)\varepsilon}^{\infty} \frac{e^{-ux}}{u} du,
 \end{aligned}$$

where $\varepsilon > 0$.

By virtue of Binet's first formula for $\ln \Gamma(z)$ in (2), the integral in the first term of (11) may be calculated as

$$\begin{aligned}
 & \lim_{\varepsilon \rightarrow 0^+} \int_{(b-a)\varepsilon}^{\infty} \left(\frac{1}{e^u-1} - \frac{1}{u} + \frac{1}{2} \right) \frac{e^{-ux/(b-a)}}{u} du \\
 &= \int_0^{\infty} \left(\frac{1}{e^u-1} - \frac{1}{u} + \frac{1}{2} \right) \frac{e^{-ux/(b-a)}}{u} du \tag{12} \\
 &= \ln \Gamma\left(\frac{x}{b-a}\right) - \left(\frac{x}{b-a} - \frac{1}{2}\right) \ln \frac{x}{b-a} + \frac{x}{b-a} - \frac{1}{2} \ln(2\pi).
 \end{aligned}$$

Furthermore, the second integral in (11) satisfies

$$\lim_{\varepsilon \rightarrow 0^+} \int_{(b-a)\varepsilon}^{\infty} \frac{e^{-ux}}{u} du = \lim_{\varepsilon \rightarrow 0^+} \int_{(b-a)x\varepsilon}^{\infty} t^{-1} e^{-t} dt = \int_0^{\infty} t^{-1} e^{-t} dt$$

which is divergent. As a result, the improper integral (5) is convergent if and only if $a + b = 0$.

Taking $a = -b$ in (5) and (12) and simplifying yield formula (6). The proof of Theorem 1 is complete. \square

Proof of Theorem 2. Straightforward computation gives

$$\begin{aligned} \delta'_{a,b}(t) &= \frac{1}{t^2} - \frac{(a-b)^2 e^{(a+b)t}}{(e^{at} - e^{bt})^2}, \\ \delta''_{a,b}(t) &= \frac{(a-b)^3 e^{(a+b)t} (e^{at} + e^{bt})}{(e^{at} - e^{bt})^3} - \frac{2}{t^3} \\ &= \frac{2e^{3(a+b)t/2}}{t^3} \left(\frac{at-bt}{e^{at}-e^{bt}} \right)^3 \left\{ \frac{e^{(a-b)t/2} + e^{(b-a)t/2}}{2} - \left[\frac{e^{(a-b)t/2} - e^{(b-a)t/2}}{(a-b)t} \right]^3 \right\} \\ &\triangleq \frac{2e^{3(a+b)t/2}}{t^3} \left(\frac{at-bt}{e^{at}-e^{bt}} \right)^3 Q\left(\frac{a-b}{2}t\right). \end{aligned}$$

Lazarević's inequality in [1, p. 131] and [9, p. 300] tells us that

$$Q(t) = \frac{e^{-t} + e^t}{2} - \left(\frac{e^t - e^{-t}}{2t} \right)^3 = \cosh t - \left(\frac{\sinh t}{t} \right)^3 < 0$$

for $t \in \mathbb{R}$ with $t \neq 0$. Hence $\delta''_{a,b}(t) < 0$ on $(0, \infty)$ and $\delta''_{a,b}(t) > 0$ on $(-\infty, 0)$. The convexity and concavity of $\delta_{a,b}(t)$ are proved.

Since $\delta''_{a,b}(t) < 0$ on $(0, \infty)$, the derivative $\delta'_{a,b}(t)$ is decreasing on $(0, \infty)$ for all real numbers a and b with $a \neq b$. Since

$$\delta'_{a,b}(t) = \frac{1}{t^2} - \frac{(a-b)^2 e^{(b-a)t}}{[1 - e^{(b-a)t}]^2} = \frac{1}{t^2} - \frac{(a-b)^2 e^{(a-b)t}}{[1 - e^{(a-b)t}]^2},$$

it is easy to obtain that

$$\lim_{t \rightarrow \infty} \delta'_{a,b}(t) = 0.$$

Consequently, the function $\delta'_{a,b}(t)$ is positive, and so $\delta_{a,b}(t)$ is increasing, on $(0, \infty)$. This means that inequality (7) holds for $0 < \tau < 1$ and $t > 0$.

From

$$\delta_{a,b}(t) = \frac{be^{bt} - ae^{at}}{e^{bt} - e^{at}} - \frac{1}{t} = \frac{be^{(b-a)t} - a}{e^{(b-a)t} - 1} - \frac{1}{t} = \frac{b - ae^{(a-b)t}}{1 - e^{(a-b)t}} - \frac{1}{t},$$

it follows easily that

$$\lim_{t \rightarrow \infty} \delta_{a,b}(t) = \max\{a, b\}.$$

L'Hôpital's rule gives

$$\begin{aligned}\lim_{t \rightarrow 0^+} \delta_{a,b}(t) &= \lim_{t \rightarrow 0^+} \frac{t(be^{bt} - ae^{at}) - e^{bt} + e^{at}}{t(e^{bt} - e^{at})} \\ &= \lim_{t \rightarrow 0^+} \frac{b^2 e^{bt} - a^2 e^{at}}{(be^{bt} - ae^{at}) + (e^{bt} - e^{at})/t} \\ &= \frac{a+b}{2}.\end{aligned}$$

For $0 < \tau < 1$, let

$$h_{a,b}(t) = \delta_{a,b}(\tau t) - \tau \delta_{a,b}(t)$$

for $t > 0$. It is obvious that

$$\lim_{t \rightarrow 0^+} h_{a,b}(t) = \frac{(1-\tau)(a+b)}{2} \quad \text{and} \quad \lim_{t \rightarrow \infty} h_{a,b}(t) = (1-\tau) \max\{a, b\}.$$

Since $\delta'_{a,b}(t)$ is decreasing, then

$$h'_{a,b}(t) = \tau [\delta'_{a,b}(\tau t) - \delta'_{a,b}(t)] > 0,$$

and so $h_{a,b}(t)$ is strictly increasing. If $a+b \geq 0$, then $h_{a,b}(t) > 0$ on $(0, \infty)$ and inequality (8) is valid. If $\max\{a, b\} \leq 0$, then inequality (8) is reversed.

It is apparent that

$$\lim_{t \rightarrow \infty} \frac{\delta_{a,b}(\tau t)}{\delta_{a,b}(t)} = 1$$

if $ab \neq 0$, which implies that inequality (7) is sharp. If $a+b \neq 0$, then it is clear that

$$\lim_{t \rightarrow 0^+} \frac{\delta_{a,b}(\tau t)}{\delta_{a,b}(t)} = 1,$$

which also implies that inequality (7) is sharp.

By L'Hôpital's rule, it is not difficult to obtain that

$$\lim_{t \rightarrow 0^+} \delta'_{a,b}(t) = \frac{(a-b)^2}{12}.$$

If $a+b=0$ and $a \neq b$, then

$$\lim_{t \rightarrow 0^+} \frac{\delta_{a,b}(\tau t)}{\delta_{a,b}(t)} = \lim_{t \rightarrow 0^+} \frac{\tau \delta'_{a,b}(\tau t)}{\delta'_{a,b}(t)} = \tau,$$

which means that inequality (8) is sharp. The proof of Theorem 2 is complete. \square

Proof of Theorem 3. From the concavity of $\delta_{a,b}(t)$ on $(0, \infty)$, it follows that

$$\delta_{-\alpha/2, \alpha/2}\left(\frac{(1+\lambda)t}{2}\right) > \frac{\delta_{-\alpha/2, \alpha/2}(\lambda t) + \delta_{-\alpha/2, \alpha/2}(t)}{2}$$

for $t > 0$ and positive numbers α and $\lambda \neq 1$. Multiplying by the factor $t^{k-1}e^{-tx}$ for any nonnegative integer $k \geq 0$ and integrating from 0 to ∞ on both sides of the above inequality yields

$$\begin{aligned} & \int_0^\infty \left[\frac{\alpha}{e^{\alpha(1+\lambda)t/2} - 1} - \frac{2}{(1+\lambda)t} + \frac{\alpha}{2} \right] t^{k-1} e^{-tx} dt \\ & > \frac{1}{2} \left[\int_0^\infty \left(\frac{\alpha}{e^{\alpha\lambda t} - 1} - \frac{1}{\lambda t} + \frac{\alpha}{2} \right) t^{k-1} e^{-tx} dt + \int_0^\infty \left(\frac{\alpha}{e^{\alpha t} - 1} - \frac{1}{t} + \frac{\alpha}{2} \right) t^{k-1} e^{-tx} dt \right] \end{aligned}$$

which can be rewritten by transformations of integral variables as

$$\begin{aligned} & \int_0^\infty \left(\frac{\alpha}{e^{\alpha u} - 1} - \frac{1}{u} + \frac{\alpha}{2} \right) \frac{u^k}{(1+\lambda)^k} \cdot \frac{e^{-xu/(1+\lambda)}}{u} du \\ & > \frac{1}{2} \left[\int_0^\infty \left(\frac{\alpha}{e^{\alpha u} - 1} - \frac{1}{u} + \frac{\alpha}{2} \right) \frac{u^k}{\lambda^k} \cdot \frac{e^{-xu/\lambda}}{u} du + \int_0^\infty \left(\frac{\alpha}{e^{\alpha t} - 1} - \frac{1}{t} + \frac{\alpha}{2} \right) t^k \frac{e^{-tx}}{t} dt \right]. \end{aligned}$$

Substituting formula (9) and its derivatives into the above inequalities leads to

$$(-1)^k \left[\theta_\alpha \left(\frac{x}{1+\lambda} \right) \right]^{(k)} > \frac{1}{2} \left\{ (-1)^k \left[\theta_\alpha \left(\frac{x}{\lambda} \right) \right]^{(k)} + (-1)^k \theta_\alpha^{(k)}(x) \right\}.$$

As a result, inequalities in (10) follow.

Easy calculation yields

$$\begin{aligned} f_{p,q;\alpha}(x) &= \int_0^\infty \delta_{-\alpha/2,\alpha/2}(t) \frac{e^{-pxt}}{t} dt - q \int_0^\infty \delta_{-\alpha/2,\alpha/2}(t) \frac{e^{-xt}}{t} dt \\ &= \int_0^\infty \delta_{-\alpha/2,\alpha/2} \left(\frac{t}{p} \right) \frac{e^{-xt}}{t} dt - q \int_0^\infty \delta_{-\alpha/2,\alpha/2}(t) \frac{e^{-xt}}{t} dt \\ &= \int_0^\infty \left[\delta_{-\alpha/2,\alpha/2} \left(\frac{t}{p} \right) - q \delta_{-\alpha/2,\alpha/2}(t) \right] \frac{e^{-xt}}{t} dt \\ &\triangleq \int_0^\infty h_{p,q;\alpha}(t) \frac{e^{-xt}}{t} dt. \end{aligned}$$

By virtue of properties of $\delta_{a,b}(t)$ obtained in Theorem 2, it follows by standard arguments that

- (1) $h_{p,q;\alpha}(t) \geq 0$ if either $0 < p \leq 1$ and $q \leq 1$, or $p > 1$ and $q \leq \frac{1}{p}$, or $0 < q < 1$ and $q \leq \frac{1}{p}$;
 (2) $h_{p,q;\alpha}(t) \leq 0$ if $p \geq 1$ and $q \geq 1$.

It is clear that

- (1) if $h_{p,q;\alpha}(t) \geq 0$ then $f_{p,q;\alpha}(x)$ is completely monotonic on $(0, \infty)$;
 (2) if $h_{p,q;\alpha}(t) \leq 0$ then $-f_{p,q;\alpha}(x)$ is completely monotonic on $(0, \infty)$.

As a result, the completely monotonic properties of $f_{p,q;\alpha}(x)$ is proved.

It is easy to see that the star-shaped properties of the function $\theta_\alpha(x)$ follow from those of the function $\delta_{-\alpha/2, \alpha/2}(t)$ and formula (9).

In [12, p. 453], it was presented that a star-shaped function must be super-additive, therefore the function $\theta_\alpha(x)$ is also sub-additive. The proof of Theorem 3 is complete. \square

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