VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-010-0024-8
Math. Slovaca **60** (2010), No. 4, 447–460

RELATIVELY UNIFORM CONVERGENCES IN ARCHIMEDEAN LATTICE ORDERED GROUPS

Ján Jakubík — Štefan Černák

(*Communicated by Jiří Rachůnek*)

ABSTRACT. For an archimedean lattice ordered group G let G^d and $G^\wedge$ be the divisible hull or the Dedekind completion of G, respectively. Put $G^{d\wedge} = X$. Then X is a vector lattice. In the present paper we deal with the relations between the relatively uniform convergence on X and the relatively uniform convergence on G. We also consider the relations between the o-convergence and the relatively uniform convergence on G. For any nonempty class τ of lattice ordered groups we introduce the notion of τ-radical class; we apply this notion by investigating relative uniform convergences.

1. Introduction

For references concerning sequential convergences cf. the expository article [7].

The notion of relatively uniform convergence has been systematically used in the theory of vector lattices; cf. the monographs [2], [12] and [15].

The relatively uniform convergence in archimedean lattice ordered groups was dealt with in [1], [7], [8], [13] and [14]; for related results, cf. also [4], [5].

If G is an ℓ-subgroup of a lattice ordered group H and if α is a convergence on H, then α induces a convergence on G which will be denoted by $\alpha(H, G)$.

2000 Mathematics Subject Classification: Primary 06F20.
Keywords: lattice ordered group, vector lattice, relatively uniform convergence, o-convergence.
Supported by VEGA Agency grants 2/7141/27 and 1/3003/06.
This work was supported by the Slovak Research and Development Agency under the contract No APVV-0071-06.
This work has been partially supported by the Slovak Academy of Sciences via the project Center of Excellence – Physics of Information (grant I/2/2005).

Let G be an archimedean lattice ordered group. We denote by G^d and $G^\wedge$ the divisible hull and the Dedekind completion of G, respectively. We put $G^{d\wedge} = X$. Then X is a vector lattice. Under the natural embedding, G is an ℓ-subgroup of X.

For $(x_n) \in G^{\mathbb{N}}$ and $x \in G$ we write $x_n \to_o x$ if (x_n) o-converges to the element x. Further, we denote by u the relatively uniform convergence on the vector lattice X. The relatively uniform convergence in G will be denoted by α_u.

For any nonempty class τ of lattice ordered groups we introduce the notion of τ-radical class; we apply this notion by investigating relatively uniform convergences. (For the corresponding definitions, cf. Section 5 below.)

Under the above notation we prove the following results:

1) $u(X, G) = \alpha_u$.
2) If G is either divisible or σ-complete, then the relation

$$x_n \to_{\alpha_u} x \implies x_n \to_o x$$

is valid in G.

3) The relatively uniform completion of G under the convergence α_u is equal to the intersection of all $G_i \subseteq X$ such that
   (i) $G \subseteq G_i$, and
   (ii) G_i is a relatively uniformly complete ℓ-subgroup of X under the convergence u.
4) Let $\mathcal{C}$ be the class of all lattice ordered groups satisfying the condition (C) (given in Section 5) and let $\mathcal{K}$ be the collection of all lattice ordered groups G such that G is archimedean and relatively uniformly complete. Then $\mathcal{K}$ is a $\mathcal{C}$-radical class of lattice ordered groups.

2. Relatively uniform convergence

For lattice ordered groups and for vector lattices we apply the notation as in [2].

For the sake of completeness we recall some definitions concerning vector lattices and lattice ordered groups.

Let V be a vector lattice. We say that a sequence (x_n) in V *relatively uniformly converges* to an element $x \in V$ if there exists $a \in V$, $a > 0$, such that for each real $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|x_n - x| \le \varepsilon a$ for all $n \in \mathbb{N}$, $n \ge n_0$.

It is easy to verify that the mentioned definition for V is equivalent with the following one.

Let $(x_n) \in V^{\mathbb{N}}$, $x \in V$. We say that the sequence (x_n) *relatively uniformly converges* to x in the vector lattice V if there exist an element $a > 0$ in V and a sequence of reals (λ_n) such that $\lambda_n \downarrow 0$ and $|x_n - x| \leqq \lambda_n a$ for each $n \in \mathbb{N}$.

Under these assumptions we write $x_n \to_u x$ (or $x_n \overset{a}{\to}_u x$). If the role of the vector lattice V is to be emphasized then we write also $x_n \to_{u(V)} x$. (Cf. [2].)

Further, we say that the sequence (x_n) *o-converges* to the element x in the vector lattice V and we write $x_n \to_o x$ (or $x_n \to_{o(V)} x$, if the role of V is emphasized) if there exist sequences (u_n) and (v_n) in V such that $u_n \uparrow x$, $v_n \downarrow x$ and $u_n \leqq x_n \leqq v_n$ for each $n \in \mathbb{N}$.

As usual, the notation $u_n \uparrow x$ means that $u_n \leqq u_{n+1}$ for each $n \in \mathbb{N}$ and $\bigvee\limits_{n\in\mathbb{N}} u_n = x$. The meaning of the notation $v_n \downarrow x$ is similar.

Now, let (x_n) be a sequence in a lattice ordered group G and $x \in G$. To avoid the trivial case, we always assume that G has more than one element. Suppose that there exists an element $b > 0$ in G such that for every $p \in \mathbb{N}$ there is $n_0 \in \mathbb{N}$ with

$$p|x_n - x| \leqq b \qquad \text{for each} \quad n \in \mathbb{N}, \quad n \geqq n_0.$$

Then we say that the sequence (x_n) *converges relatively uniformly* to the element x in the lattice ordered group G and we write $x_n \to_{\alpha_u} x$ (or $x_n \to_{\alpha_u(G)} x$). (Cf. [12].)

The o-convergence in the lattice ordered group G is defined analogously as in the case of vector lattices.

If V is a vector lattice, then we denote by V_0 the corresponding lattice ordered group (i.e., when dealing with V_0, the multiplication of elements of V with reals is not taken into account).

The following assertion is easy to verify.

Lemma 2.1. *Let (x_n) be a sequence in a vector lattice V and $x \in V$. Then the following conditions are equivalent:*

(i) *The sequence (x_n) converges relatively uniformly to the element x in the vector lattice V.*

(ii) *The sequence (x_n) converges relatively uniformly to the element x in the lattice ordered group V_0.*

Suppose that G is an archimedean lattice ordered group. Consider the divisible hull G^d of G. Under the natural embedding, G is an ℓ-subgroup of G^d. If $y \in G^d$, $y > 0$, then there exist $n \in \mathbb{N}$ and $x \in G^+$ with $y = \frac{x}{n}$. (Cf. e.g., [10].) The lattice ordered group G^d is archimedean as well.

Lemma 2.2. *Let (x_n) be a sequence in an archimedean lattice ordered group G and let $x \in G$. Then the following conditions are equivalent:*

(i) *The sequence (x_n) converges relatively uniformly to the element x in the lattice ordered group G.*

(ii) *The sequence (x_n) converges relatively uniformly to the element x in the lattice ordered group G^d.*

P r o o f. The implication (i) $\Longrightarrow$ (ii) is obvious, since G is embedded in G^d.

Assume that (ii) is valid. Thus there is an element b_1 in G^d such that for every $p \in \mathbb{N}$ there is $n_0 \in \mathbb{N}$ with

$$p|x_n - x| \leqq b_1 \qquad \text{for each} \quad n \in \mathbb{N}, \quad n \geqq n_0.$$

There are $n_1 \in \mathbb{N}$ and $b_2 \in G^+$ such that

$$b_1 = \frac{b_2}{n_1}.$$

Then $b_1 \leqq b_2$. Hence

$$p|x_n - x| \leqq b_2 \qquad \text{for each} \quad n \in \mathbb{N}, \quad n \geqq n_0.$$

Therefore the condition (i) is satisfied. □

Let G be as above. We put $X = G^{d\wedge}$. In view of [10], X is a vector lattice. Under the natural embeddings, both G and G^d are ℓ-subgroups of the lattice ordered group X_0. (The symbol X_0 has an analogous meaning with respect to X as the symbol V_0 with respect to V.)

Lemma 2.3. *Let (x_n) be a sequence in G^d and $x \in G^d$. Then the following conditions are equivalent:*

(i) *The sequence (x_n) converges relatively uniformly to the element x in the lattice ordered group G^d.*

(ii) *The sequence (x_n) converges relatively uniformly to the element x in the vector lattice X.*

P r o o f. Assume that (i) is valid. Since G^d is embedded in X_0, in view of Lemma 2.1 we conclude that (ii) holds.

Further, suppose that (ii) is satisfied. Hence according to Lemma 2.1, (x_n) relatively uniformly converges to x in the lattice ordered group X_0. In view of the construction of Dedekind completion of G^d, for each element $0 < b_1 \in G^{d\wedge}$ there exists an element $b_2 \in G^d$ such that $b_1 < b_2$. Now, applying the analogous argument as in the proof of Lemma 2.2, we conclude that the condition (i) is valid. □

Proposition 2.4. *Let (x_n) be a sequence in an archimedean lattice ordered group G and $x \in G$. Put $G^{d\wedge} = X$. Then the following conditions are equivalent:*

(i) *The sequence (x_n) converges relatively uniformly to the element x in the lattice ordered group G.*

(ii) *The sequence (x_n) converges relatively uniformly to the element x in the vector lattice X.*

Proof. This is a consequence of Lemmas 2.2 and 2.3. □

In other words, we can say that the convergence α_u on the lattice ordered group G is induced by the convergence u on the vector lattice X.

The following example shows that if X is replaced by a vector lattice containing G and different from X, then Proposition 2.4 need not hold.

Example 1. Let $\mathbb{R}$ be the additive group of all reals with the natural linear order. For each $i \in \mathbb{N}$ let $G_i = \mathbb{R}$. We put

$$H = \prod_{i \in \mathbb{N}} G_i.$$

Evidently, H is a vector lattice. For $h \in H$ and $i \in \mathbb{N}$ we denote by $h(i)$ the component of h in the direct factor G_i; further we put $\operatorname{supp}(h) = \big\{i \in \mathbb{N} : h(i) \neq 0\big\}$. Let G be the set of all $h \in H$ such that $\operatorname{supp}(h)$ is finite. Then G is an ℓ-subgroup of H; moreover, G is archimedean.

For $n \in \mathbb{N}$ we define $x_n \in G$ as follows: if $i \in \mathbb{N}$ and $i \neq n$, then the component $x_n(i)$ of x_n in G_i is equal to 0; if $i = n$, then $x_n(i) = \frac{1}{n}$.

There exists $a \in H$ such that $a(i) = 1$ for each $i \in \mathbb{N}$. Put $\lambda_n = \frac{1}{n}$ for each $n \in \mathbb{N}$. Thus

$$x_n = |x_n| \leqq \lambda_n a \qquad \text{for each} \quad n \in \mathbb{N},$$

hence $x_n \to_u 0$ in the vector lattice H. On the other hand, the relation $x_n \to_{\alpha_u} 0$ fails to be valid in the lattice ordered group G.

3. o-convergence

Again, let G be an archimedean lattice ordered group and let G^d, X be as above.

In view of [2, Chap. X, §9], we have:

Lemma 3.1. *Let L be a lattice and let L_1 be the Dedekind completion of L. Then the o-convergence in L is induced by the o-convergence in L_1.*

Hence, in particular, we obtain:

Lemma 3.2. *Let G be an archimedean lattice ordered group. Let (x_n) be a sequence in G and $x \in G$. Then the following conditions are equivalent:*

(i) *The sequence (x_n) o-converges to x in G.*

(ii) *The sequence (x_n) o-converges to x in $G^\wedge$.*

Theorem 3.3. (Cf. [2, Chap. XV, Theorem 19].) *Let (x_n) be a sequence in archimedean vector lattice V and $x \in V$. Assume that (x_n) relatively uniformly converges to x in V. Then (x_n) o-converges to x in V.*

Proposition 3.4. *Assume that G is a divisible lattice ordered group. Let (x_n) be a sequence in G and $x \in G$. Then we have*

$$x_n \to_{\alpha_u(G)} x \implies x_n \to_{o(G)} x.$$

Proof. Since G is divisible, $G^d = G$. Put $G^{d\wedge} = X$. Hence X is a vector lattice and $X = G^\wedge$.

Assume that the relation $x_n \to_{\alpha_u(G)} x$ is valid. Then according to Proposition 2.4 we have $x_n \to_{u(X)} x$. Thus Theorem 3.3 yields $x_n \to_{o(X)} x$. Applying Lemma 3.2 we obtain $x_n \to_{o(G)} x$. □

Now, let us suppose that G is σ-complete lattice ordered group. Let (x_n) be a sequence in G, $x \in G$, and assume that the relation $x_n \to_{\alpha_u} x$ is valid in G.

We apply the notation as above. Further, we put $|x_n - x| = z_n$. Thus there is $b \in G$, $b > 0$, such that for each $k \in \mathbb{N}$ there exists $n_0(k) \in \mathbb{N}$ such that for each $n \geqq n_0(k)$ we have $kz_n \leqq b$.

Hence for any $n \geqq n_0(1)$ we get $z_n \leqq b$. Since G is σ-complete, for each $n \in \mathbb{N}$ with $n \geqq n_0(1)$ there exists the join

$$t_n = \bigvee_{m \geqq n} z_m$$

in G. We put

$$\overline{t} = t_{n_0(1)}$$

and for $n = 1, 2, \dots, n_0(1) - 1$ we set

$$t_n = z_n \vee z_{n+1} \vee \cdots \vee z_{n_0(1)-1} \vee \bar{t}.$$

Therefore $t_n \geqq t_{n+1}$ and $t_n \geqq z_n \geqq 0$ for each $n \in \mathbb{N}$.

Further, there exists $q \in G$ with

$$q = \bigwedge_{n \in \mathbb{N}} t_n;$$

clearly $q \geqq 0$.

Let $k \in \mathbb{N}$. Take any $n \in \mathbb{N}$ with $n \geqq n_0(k)$; we get

$$kq \leqq kt_n = k\Big(\bigvee_{m \geqq n} z_m\Big) = \bigvee_{m \geqq n} kz_m \leqq b.$$

Since G is archimedean, we conclude that $q = 0$. Therefore we get that the relation

$$z_n \to_o 0$$

is valid in G. Thus we have also $-z_n \to_o 0$. Clearly

$$-z_n \leqq x - x_n \leqq z_n$$

for each $n \in \mathbb{N}$. Thus $x - x_n \to_o 0$ in G. This yields that $x_n \to_o x$ is valid in G.

It is well-known that σ-complete lattice ordered groups are archimedean. Hence, summarizing the above results, we have:

Proposition 3.5. *Let G be a σ-complete lattice ordered group. Let (x_n) be a sequence in G and $x \in G$ such that (x_n) converges relatively uniformly to the element x. Then (x_n) o-converges to the element x.*

4. Relative uniform completion of G

We recall the definitions of a relatively uniformly Cauchy sequence in vector lattices and in lattice ordered groups (cf., e.g., [12] and [6]).

Let V be a vector lattice. A sequence (x_n) in V is called *relatively uniformly Cauchy* if there exists $b \in V$, $b > 0$, such that for every real $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|x_n - x_m| \leq \varepsilon b$ for each $m, n \in \mathbb{N}$, $m, n \geq n_0$.

Let G be a lattice ordered group. It is said that a sequence (x_n) in G is *relatively uniformly Cauchy* if there exists $b \in G$, $b > 0$, such that for every $p \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ with the property $p|x_n - x_m| \leq b$ for each $m, n \in \mathbb{N}$, $m, n \geq n_0$.

If every relatively uniformly Cauchy sequence in a vector lattice V converges relatively uniformly in V then V is called *relatively uniformly complete*. The concept of a relatively uniformly complete lattice ordered group is defined analogously.

A. I. Veksler [14] discussed the notion of a relative uniform completion of vector lattices. This notion can be applied to lattice ordered groups by a slight modification (cf. [1], [6], [12]):

Let H be an archimedean lattice ordered group with the properties

(a) G is an ℓ-subgroup of H.

(b) H is relatively uniformly complete.

(c) If G is an ℓ-subgroup of K, K is an ℓ-subgroup of H and K is relatively uniformly complete, then $K = H$.

Then H is said to be a *relative uniform completion* of G.

A relative uniform completion exists and it is uniquely determined up to isomorphisms over G (cf. [6]).

Let V_0 and X_0 be as in Section 2. The following result is easy to verify.

Lemma 4.1. *Let (x_n) be a sequence in a vector lattice V. Then the following conditions are equivalent:*

(i) *(x_n) is a relatively uniformly Cauchy sequence in the vector lattice V.*

(ii) *(x_n) is a relatively uniformly Cauchy sequence in the lattice ordered group V_0.*

As a consequence of Lemma 2.1 and Lemma 4.1 we get:

Lemma 4.2. *The following conditions are equivalent:*

(i) *V is a relatively uniformly complete vector lattice.*

(ii) *V_0 is a relatively uniformly complete lattice ordered group.*

Theorem 4.3. *Let G be an archimedean lattice ordered group and $X = G^{d\wedge}$. Assume that $\{G_i : i \in I\}$ is the system of all ℓ-subgroups K of X_0 satisfying the following conditions:*

(i) *G is an ℓ-subgroup of K.*

(ii) *K is a relatively uniformly complete ℓ-subgroup of X_0.*

Then $\bigcap\limits_{i \in I} G_i$ is a relative uniform completion of G.

P r o o f. With respect to [14], X is a relatively uniformly complete vector lattice. By 4.2, X_0 is a relatively uniformly complete lattice ordered group. We have to prove that $H = \bigcap\limits_{i \in I} G_i$ has the properties (a), (b) and (c). We consider the relative uniform convergence in X_0.

(a) Evidently, H is an ℓ-subgroup of X_0 and G is an ℓ-subgroup of H.

(b) Let (x_n) be a sequence in H and let (x_n) be relatively uniformly Cauchy in H. Hence for each $i \in I$, (x_n) is a sequence in G_i and (x_n) is relatively uniformly Cauchy in G_i. According to the assumption (x_n) is relatively uniformly convergent in all G_i. Whence for each $i \in I$ there exist $x_i \in G_i$ and $0 < v_i \in G_i$ with $x_n \overset{v_i}{\rightarrow}_{\alpha_u} x_i$. Archimedeanicity of X_0 implies that limits in X_0 are uniquely determined (cf. [7]). Hence $x_i = x$ for each $i \in I$, so $x \in H$. Let $i \in I$ be fixed. Then $x_n \overset{v_i}{\rightarrow}_{\alpha_u} x$. According to the proofs of 2.2 and 2.3 there exists $v \in G$, $v \geqq v_i$. Therefore $x_n \overset{v}{\rightarrow}_{\alpha_u} x$. By (a), $G \subseteq H$. Hence (x_n) is relative uniformly convergent in H. This implies that H is relatively uniformly complete.

(c) This is clear. □

5. The $\mathcal{C}$-radical class $\mathcal{K}$

For a lattice ordered group G let $c(G)$ be the system of all convex ℓ-subgroups of G. This system is partially ordered by the set-theoretical inclusion. Then $c(G)$ is a complete lattice.

Let $(G_i)_{i \in I}$ be an indexed system of elements of $c(G)$. In the lattice $c(G)$ we have

$$\bigcap_{i \in I} G_i = \bigwedge_{i \in I} G_i.$$

We put $\bigcup\limits_{i \in I} G_i = H_o$. Let H be the set of all elements $h \in G$ which can be written in the form $h = g_1 + g_2 + \cdots + g_n$ with $g_1, \ldots, g_n \in H_o$. Then the relation

$$H = \bigvee_{i \in I} G_i$$

is valid in the lattice $c(G)$.

The notion of radical class of lattice ordered groups has been introduced in [11] and it was investigated in several papers.

We introduce a generalization of this notion as follows.

Let τ be a nonempty class of lattice ordered groups. A class $\mathcal{A}$ of lattice ordered groups is said to be a *τ-radical class* if the following conditions are satisfied:

(i) $\mathcal{A}$ is closed with respect to isomorphisms.

(ii) If $G \in \mathcal{A}$ and $G_1 \in c(G)$ then $G_1 \in \mathcal{A}$.

(iii) If $G \in \tau$ and $\{G_i\}_{i\in I} \subseteq \mathcal{A} \cap c(G)$, $I \neq \emptyset$, then $\bigvee\limits_{i\in I} G_i \in \mathcal{A}$.

Obviously, a class $\mathcal{A}$ of lattice ordered groups is a radical class if and only if it is a $\mathcal{G}$-radical class, where $\mathcal{G}$ is the class of all lattice ordered groups.

We denote by $\mathcal{C}$ the class of all lattice ordered groups H fulfilling the condition

(C) If $K \in c(H)$ and (x_n) is a sequence in K which is relatively uniformly Cauchy in H, then it is relatively uniformly Cauchy in K.

The condition (C) was applied in [8]. Some lattice ordered groups fulfil the condition (C) (e.g., the lattice ordered group R) and some do not (cf. [8, Example 4.9]).

Let $\mathcal{K}$ be the class of all archimedean lattice ordered groups G such that G is relatively uniformly complete. In the present section we prove that $\mathcal{K}$ is a $\mathcal{C}$-radical class.

For the classical Riesz Decomposition Theorem concerning lattice ordered groups, different formulations (and differenct proofs) have been applied in the literature. Let us quote a rather simple formulation (together with the proof) as given in [3, Introduction, Section 10].

Theorem 5.1. (Cf. [3].) *Let G be a lattice ordered group. If $0 < x \leqq d_1 + \dots + d_n$ where $d_i \in G^+$ then $x = c_1 + \dots + c_n$ where $0 \leqq c_i \leqq d_i$.*

P r o o f. Let $c_1 = x \wedge d_1$ and $c = -c_1 + x$. Then $0 \leqq c_1 \leqq d_1$ and

$0 \leqq c = -c_1 + x = -(x \wedge d_1) + x = (-x \vee -d_1) + x = 0 \vee (-d_1 + x) \leqq d_2 + \dots + d_n$.

Thus by induction $c = c_2 + \dots + c_n$ where $0 \leqq c_i \leqq d_i$, whence $x = c_1 + c_2 + \dots + c_n$ where $0 \leqq c_i \leqq d_i$. □

We will apply the idea of this proof below.

In what follows we assume that G is an archimedean lattice ordered group.

Lemma 5.2. *Assume that $x, y, b_1 \in G^+$ and $x \leqq y$. Put*

$$x \wedge b_1 = x_1, \quad y \wedge b_1 = y_1, \quad x - x_1 = x_1', \quad y - y_1 = y_1'.$$

Then $y_1 - x_1 \leqq y - x$ and $x_1' \leqq y_1'$.

P r o o f. (Cf. Fig. 1.) We put $x \vee y_1 = z$. Then we have $x \wedge y_1 = x_1$, hence $z - x = y_1 - x_1$ and $z - y_1 = x - x_1$. Since $x \leqq z \leqq y$, we obtain $z - x \leqq y - x$, thus $y_1 - x_1 \leqq y - x$.

Further, $y_1' \geqq z - y_1 = x - x_1 = x_1'$. □

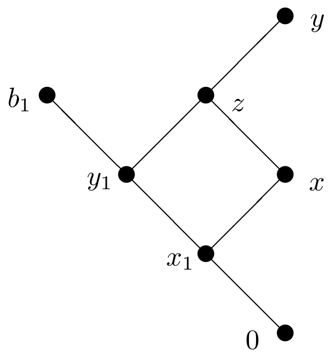

FIGURE 1

Applying Theorem 5.1 (together with the method of its proof) and using Lemma 5.2 we obtain:

LEMMA 5.3. *Assume that* $p, q, d_1, \dots, d_m \in G^+$, $d = d_1 + \dots + d_m$, $p \leqq q \leqq d$. *Then there are* $c_i, c_i' \in G$ $(i = 1, 2, \dots, m)$ *such that*

$$p = c_1 + \dots + c_m,$$
$$q = c_1' + \dots + c_m',$$

$0 \leqq c_i \leqq d_i$, $0 \leqq c_i' \leqq d_i$, $0 \leqq c_i' - c_i \leqq q - p$ *for* $i = 1, 2, \dots, m$.

LEMMA 5.4. *Assume that* $x, y, d_1, \dots, d_m \in G^+$, $d = d_1 + \dots + d_m$, $x \leqq d$, $y \leqq d$. *Then there are* $x_i, y_i \in G^+$ $(i = 1, 2, \dots, m)$ *such that*

$$x = x_1 + \dots + x_m, \qquad y = y_1 + \dots + y_m,$$
$$0 \leqq x_i \leqq d_i, \qquad 0 \leqq y_i \leqq d_i.$$

Further we have

$$|x_i - y_i| \leqq |x - y| \qquad \text{for} \quad i = 1, 2, \dots, m.$$

Proof. The first part of the assertion of the lemma is a consequence of Theorem 5.1. For proving the assertion concerning $|x_i - y_i|$ and $|x - y|$ it suffices to consider the elements $p = x \wedge y$, $q = x \vee y$ and to apply Lemma 5.3. We have clearly $|x - y| = q - p$ and analogously for $|x_i - y_i|$. □

LEMMA 5.5. (Cf. [8].) *Suppose that G is relatively uniformly complete and that G_1 is a convex ℓ-subgroup of G. Then G_1 is relatively uniformly complete.*

Lemma 5.6. *The following conditions are equivalent:*

(i) *G is relatively uniformly complete.*

(ii) *If (x_n) is a relatively uniformly Cauchy sequence in G such that $x_n \geqq 0$ for each $n \in \mathbb{N}$, then (x_n) is relatively uniformly convergent in G.*

P r o o f. The implication (i) $\Longrightarrow$ (ii) is obvious. Assume that the condition (ii) is valid. Suppose that (y_n) is a relatively uniformly Cauchy sequence in G. Then (y_n) is bounded in G, hence there is $b \in G$ such that $b \leqq y_n$ for each $n \in \mathbb{N}$. Put $y_n - b = x_n$; hence $0 \leqq x_n$ for each $n \in \mathbb{N}$. Also, (x_n) is a relatively uniformly Cauchy sequence in G. In view of (ii), there exists $x \in G$ such that (x_n) relatively uniformly converges to x in G. Thus (y_n) relatively uniformly converges to $x + b$ in G. Hence the condition (i) is satisfied. □

The validity of the following assertion is obvious.

Lemma 5.7. *Let (x_n) and (y_n) be sequences in a lattice ordered group H. Suppose that (y_n) is a relatively uniformly Cauchy sequence in H and that $|x_n - x_m| \leqq |y_n - y_m|$ for each $n, m \in \mathbb{N}$. Then (x_n) is a relatively uniformly Cauchy sequence in H.*

Lemma 5.8. *Let H be a lattice ordered group satisfying the condition* (C) *and let $\{G_i\}_{i\in I} \subseteq c(H)$, $\bigvee_{i\in I} G_i = H$. Assume that all G_i are archimedean and relatively uniformly complete. Then H is archimedean and relatively uniformly complete.*

P r o o f. It is well-known that the collection of all archimedean lattice ordered groups is a radical class. Hence H is an archimedean lattice ordered group.

Let (x_n) be a relatively uniformly Cauchy sequence in H such that $0 \leqq x_n$ for each $n \in \mathbb{N}$. Then (x_n) is upper-bounded in H. Hence there is $b \in H$ such that $b \geqq x_n$ for each $n \in \mathbb{N}$.

In view of the relation $H = \bigvee_{i\in I} G_i$ there exist $b_1, \dots, b_m \in \bigcup_{i\in I} G_i$ such that $b = b_1 + \dots + b_m$. Without loss of generality we can suppose that $b_1 \geqq 0, \dots, b_m \geqq 0$.

There exist $i(1), \dots, i(m) \in I$ such that $b_1 \in G_{i(1)}$, $\dots$, $b_m \in G_{i(m)}$. Let $n \in \mathbb{N}$. Then $0 \leqq x_n \leqq b_1 + \dots + b_m$. Hence according to Theorem 5.1 there are elements $x_{n1}, \dots, x_{nm}$ in H such that $0 \leqq x_{nk} \leqq b_k$ for each $k \in \{1, 2, \dots, m\}$ and

$$x_n = x_{n1} + \dots + x_{nm}.$$

If $s, n \in \mathbb{N}$, then in view of Lemma 5.4 we have

$$|x_{nj} - x_{sj}| \leqq |x_n - x_s|$$

for each $j \in \{1, 2, \dots, m\}$. In view of Lemma 5.7 we obtain that (x_{nj}) is a relatively uniformly Cauchy sequence in the lattice ordered group H.

According to the condition (C), (x_{nj}) is a relatively uniformly Cauchy sequence in $G_{i(j)}$ $(j = 1, 2, \dots, m)$. Since $G_{i(j)}$ is relatively uniformly complete, the sequence (x_{nj}) relatively uniformly converges in $G_{i(j)}$ to some element x^j. From the fact that $G_{i(j)} \subseteq H$ we conclude that (x_{nj}) relatively uniformly converges to x^j also in the lattice ordered group H. Therefore the sequence (x_n) relatively uniformly converges in H to the element $x^1 + \dots + x^m$.

For finishing the proof it suffices to apply Lemma 5.6. □

In view of Lemma 5.5, Lemma 5.8 and the fact that if G satisfies the condition (C), so does each convex ℓ-subgroup of G, we get:

Theorem 5.9. *The collection $\mathcal{K}$ is a $\mathcal{C}$-radical class of lattice ordered groups.*

REFERENCES

[1] BALL, R. N.—HAGER, A. W.: *Algebraic extensions of an archimedean lattice ordered group, II*, J. Pure Appl. Algebra **138** (1999), 197–204.

[2] BIRKHOFF, G.: *Lattice Theory* (3rd ed.), Amer. Math. Soc., Providence, RI, 1967.

[3] CONRAD, P.: *Lattice Ordered Groups*, Tulane University, New Orleans, 1970.

[4] ČERNÁK, Š.: *Convergence with a fixed regulator in archimedean lattice ordered groups*, Math. Slovaca **56** (2006), 167–180.

[5] ČERNÁK, Š.: *Convergence with a fixed regulator in lattice ordered groups and applications to MV-algebras*, Soft Comput. **12** (2008), 453–462.

[6] ČERNÁK, Š.—LIHOVÁ, J.: *Convergence with a regulator in lattice ordered groups*, Tatra Mt. Math. Publ. **39** (2005), 35–45.

[7] ČERNÁK, Š.—LIHOVÁ, J.: *Relatively uniform convergence in lattice ordered groups*. In: Selected Questions of Algebra, Collection of Papers Dedicated to the Memory of N. Ja. Medvedev, Altai State Univ. Barnaul, Barnaul, 2007, pp. 218–241.

[8] ČERNÁK, Š.—LIHOVÁ, J.: *On a relative uniform completion of an archimedean lattice ordered group* (To appear).

[9] FRIČ, R.—KOUTNÍK, V.: *Recent development in sequential convergence*. In: Convergence Structures and Applications II. Abh. Akad. Wiss. DDR, Abt. Math.-Naturwiss.-Technik, Akademie Verlag, Berlin, 1984, pp. 37–46.

[10] JAKUBÍK, J.: *Representations and extensions of ℓ-groups*, Czechoslovak Math. J. **13** (1963), 267–283 (Russian).

[11] JAKUBÍK, J.: *Radical classes and radical mappings of lattice ordered groups*. In: Symposia Math. 31, Academic Press, New York-London, 1977, pp. 451–477.

[12] LUXEMBURG, M.—ZAANEN, A.: *Riesz Spaces, Vol. I*, North Holland Publ., Amsterdam-London, 1971.

[13] MARTINEZ, J.: *Polar functions, III. On irreducible maps vs. essential extensions of archimedean ℓ-groups with unit*, Tatra Mt. Math. Publ. **27** (2003), 189–211.

[14] VEKSLER, A. I.: *On a new construction of Dedekind completion of vector lattices and of ℓ-groups with division*, Sibirsk. Mat. Zh. **10** (1969), 1206–1213 (Russian).

[15] VULIKH, B. Z.: *Introduction to the Theory of Partially Ordered Spaces*, Wolters-Nordhoff Sci. Publ., Groningen, 1967.

Received 17. 6. 2008
Accepted 3. 1. 2009

Mathematical Institute
Slovak Academy of Sciences
Grešákova 6
SK–040 01 Košice
SLOVAKIA

E-mail: kstefan@saske.sk
Stefan.Cernak@tuke.sk