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COLORING OF LATTICES

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Dedicated to Professor N. K. Thakare on the occasion of his seventieth birthday

(*Communicated by Constantin Tsinakis*)

ABSTRACT. The concept of coloring is studied for graphs derived from lattices with 0. It is shown that, if such a graph is derived from an atomic or distributive lattice, then the chromatic number equals the clique number. If this number is finite, then in the case of a distributive lattice, it is determined by the number of minimal prime ideals in the lattice. An estimate for the number of edges in such a graph of a finite lattice is given.

1. Preliminaries

There are many papers which interlink graph theory and lattice theory. The papers of Filipov [12], Gedeonová [13], Duffus and Rival [11] and Bollobas and Rival [8] et. al. discuss the properties of graphs derived from partially ordered sets and lattices. In the work of Filipov [12], the adjacency between two elements is defined through the comparability relation between two elements of a poset, i.e., a, b are adjacent if either $a \leq b$ or $b \leq a$. These graphs are called the *comparability graphs*. On the other hand, Gedeonová [13], Duffus and Rival [11] and Bollobas and Rival [8] use the covering relation between two elements in a lattice to define the adjacency of two elements. Such graphs are called the *covering graphs*. Bollobas [7] uses the covering graph of a lattice to study coloring in lattices.

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Some papers give properties of graphs derived from other algebraic structures. Beck [5] has introduced the notion of coloring in commutative rings. The graphs associated with commutative rings are further investigated by Anderson and Naseer [1], Anderson and Livingston [2]. Similar considerations for commutative semigroups can be found in DeMeyer, McKenzie and Schneider [10], for commutative von Neumann regular rings in Anderson, Levy and Shapiro [4] and for meet-semilattices with 0 in Nimbhorkar, Wasadikar and DeMeyer [16].

Beran [6, p. 334] has introduced a graph on *orthologics* by defining *adjacency* of nonzero a, b as $a \perp b$. In this paper we introduce and study the concept of coloring of a graph derived from a lattice with 0 (the smallest element) on the lines of Nimbhorkar, Wasadikar and DeMeyer [16] and characterize the chromatic number in the case of a graph derived from a distributive lattice. We also generalize some results from Cornish and Stewart [9] to distributive lattices with 0. The undefined terms and notations are from Grätzer [13] and Harary [15].

Let L be a lattice (or a meet-semilattice) with 0 and let $\Gamma(L)$ be the graph in which

(a) the vertices are the elements of L,

(b) two distinct elements x, y are *adjacent* if and only if $x \wedge y = 0$.

We denote this graph by $\Gamma(L)$. Let $\chi(L)$ denote the *chromatic number* of $\Gamma(L)$, i.e., the minimal number of colors which can be assigned to the vertices of $\Gamma(L)$ in such a way that every pair of adjacent vertices have different colors. A subset $C = \{x_1, x_2, \dots\}$ of L is called a *clique* in $\Gamma(L)$, if x_i, x_j are adjacent for all i, j, $i \neq j$, i.e., if $x_i \wedge x_j = 0$ for all $i \neq j$. If $\Gamma(L)$ contains a clique with n elements and every clique has at most n elements, then we say that the *clique number* of $\Gamma(L)$ is n and write $\mathrm{Clique}(L) = n$. If the sizes of the cliques are not bounded, then we define $\mathrm{Clique}(L) = \infty$. We recall that the number of edges incident at a vertex a in a graph G is called the *degree* of a and is denoted by $\deg(a)$.

Remark 1.1. Clearly, a graph which does not have a vertex adjacent to every other vertex cannot be a graph of a lattice with 0. However, the graph shown in Figure 1 is not a graph of a lattice with 0 even though, it has a vertex adjacent to every other vertex. We note that both the chromatic number and the clique number of this graph is 3.

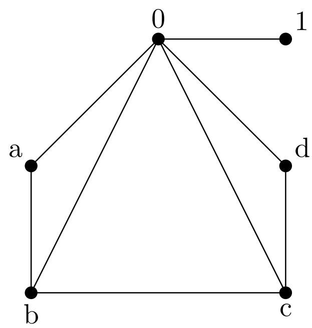

FIGURE 1

We consider the set $S = \{0, a, b, c, d, 1\}$. The graph determines which meets are zero and which are nonzero. Thus $a \wedge b = b \wedge c = c \wedge d = 0$ and $a \wedge c \neq 0$, $a \wedge d \neq 0$, $b \wedge d \neq 0$. In order that S be a lattice, it must be closed under the lattice meet; in particular, we must have $a \wedge d \in S$. We have the following possibilities.

(1) $a \wedge d = a \implies a \wedge c = a \wedge d \wedge c = 0$.

(2) $a \wedge d = b \implies a \wedge d = a \wedge d \wedge a = b \wedge a = 0$.

(3) $a \wedge d = c \implies a \wedge d = a \wedge d \wedge d = c \wedge d = 0$.

(4) $a \wedge d = d \implies b \wedge d = b \wedge a \wedge d = 0$.

Each case is a contradiction. Therefore, $a \wedge d \notin S$ and so S cannot be a lattice.

Remark 1.2. There exists a lattice L with 0 for which $\chi(L) = \text{Clique}(L) = \infty$. Consider the set $\mathbb{N}$ of natural numbers. For $a, b \in \mathbb{N}$ we write $a \leq b$ if and only if $a \mid b$, i.e., $b = ac$ for some $c \in \mathbb{N}$. Then $\mathbb{N}$ becomes a lattice with the smallest element 1 and $x \wedge y = \gcd(x, y)$, $x \vee y = \text{lcm}(x, y)$. The meet of any two primes is 1. Since the number of primes is infinite, we get an infinite clique in the graph $\Gamma(\mathbb{N})$. The degree of every prime number is ∞ and $\chi(\mathbb{N}) = \infty$.

The following example shows that nonisomorphic lattices may have the same graph.

Example 1. The distributive lattices shown in Figures 2 and 3 are not isomorphic but their graph is the same; see, Figure 4.

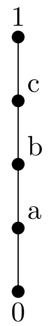

FIGURE 2

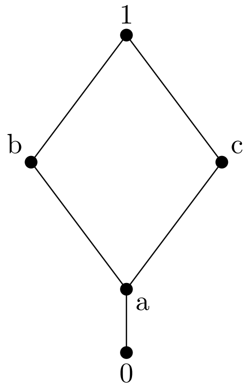

FIGURE 3

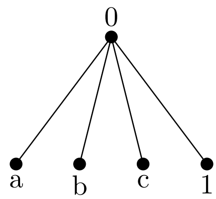

FIGURE 4

Remark 1.3. $\chi(L) = 1$ if and only if $L = \{0\}$.

A graph G is called a *star graph* if it has a vertex adjacent to every other vertex and these are the only adjacency relations. A nonzero element x in a lattice L with 0 is called an *atom*, if there is no $y \in L$ such that $0 < y < x$. We note that the graph of a lattice with only one atom is a star graph.

Remark 1.4. We note in passing that the following statements are equivalent for a lattice L with 0.

(1) The chromatic number of L is 2.

(2) For all $x, y \in L$ we have $x \wedge y = 0$ implies $x = 0$ or $y = 0$.

(3) The graph of L is a star graph.

Remark 1.5. Bollobos [5] proved that, given a natural number k, there is a lattice L whose covering graph is not k-colorable. We note that, given a natural number k, there exists a lattice L whose graph is not k-colorable. If X is a set such that $|X| = n \geq 1$, then the graph of its power set is not n-colorable.

We recall that a graph G is *connected* if there is a path between any two distinct vertices of G, $d(x, y)$ denotes the length of the shortest path from x to y, and $\operatorname{diam}(G) = \sup\{d(x, y) : x, y \in G, \ x \neq y\}$.

LEMMA 1.1. *If L is a lattice with 0, then $\Gamma(L)$ is connected and diameter of $\Gamma(L) \leq 2$.*

Proof. Let $x, y \in \Gamma(L)$. If x, y are adjacent then $d(x, y) = 1$, otherwise x–0–y is a path of length 2. Thus $d(x, y) \leq 2$ for all $x, y \in \Gamma(L)$. $\square$

2. Coloring in atomic and complemented lattices

A lattice L with 0 is called *atomic* if for any nonzero $x \in L$, there exists an atom $a \in L$ such that $a \leq x$.

Theorem 2.1. *The number of atoms in an atomic lattice L is n if and only if* $\mathrm{Clique}(L) = \chi(L) = n + 1$.

Proof.
Case 1:
Suppose L contains a finite number of atoms. Let $A = \{a_1, \ldots, a_n\}$ be the set of atoms in L. Then $A \cup \{0\}$ is a clique with $n+1$ elements. Hence $\chi(L) \geq \mathrm{Clique}(L) \geq n+1$.

We decompose L as follows. Put $A_0 = \{0\}$, $A_1 = \{x \in L : \ x \geq a_1\}, \ldots,$ $A_i = \{x \in L : \ x \geq a_i\} - \bigcup\limits_{j<i} A_j$, for $i = 2, \ldots, n$. Define a coloring f on L by putting $f(0) = 0$ and $f(x) = i$ for $x \in A_i$. We note that if x, y are adjacent, then $x \geq a_i$, $y \geq a_j$ for some distinct atoms a_i, a_j. Hence $f(x) \neq f(y)$. Thus f is a coloring on L, and $\chi(L) \leq n+1$.

Conversely, suppose $\mathrm{Clique}(L) = \chi(L) = n+1$. Let B be the set of atoms in L. Then $B \cup \{0\}$ is a clique in $\Gamma(L)$, which implies $|B| + 1 \leq n+1$. Hence $|B| \leq n$. If $|B| = k$, then as shown above, we conclude that $\chi(L) = k+1$. Thus $n = k$.
Case 2:
If L has an infinite number of atoms then clearly, $\mathrm{Clique}(L) = \chi(L) = \infty$.

Suppose $\mathrm{Clique}(L) = \chi(L) = \infty$. Let $\{x_1, x_2, \ldots\}$ be a clique in $\Gamma(L)$. Since L is atomic, for each i, there exists an atom a_i such that $a_i \leq x_i$. Since x_i, x_j are adjacent for $i \neq j$, it follows that $a_i \nleq x_j$. Thus L cannot have a finite number of atoms. $\square$

The following example shows that the assumption that L is atomic is necessary in Theorem 2.1.

Example 2. The set $A = \left\{\frac{1}{n} : \ n \text{ is a positive integer}\right\} \cup \{0\}$ with the usual order is a bounded distributive lattice having no atom. Thus L is nonatomic. However, $\chi(L) = \mathrm{Clique}(L) = 2$.

Theorem 2.2. *Let G be the graph obtained by adjoining a pendant vertex to the complete graph K_{n-1}. Then $G = \Gamma(L)$ if and only if $L = M_n$, where $M_n = \{0, a_1, \ldots, a_{n-2}, 1\}$ is the lattice in which $a_i \wedge a_j = 0$ and $a_i \vee a_j = 1$ for $i \neq j$.*

P r o o f. Clearly, the graph, $\Gamma(M_n)$, of M_n is G.

Conversely, consider G. Label the pendant vertex as 1 and the vertex adjacent to it as 0 and the remaining vertices as a_i, $i = 1, \dots, n-2$. Since a_i is adjacent to a_j for $i \neq j$, we have $a_i \wedge a_j = 0$. Suppose, $i \neq j$. If $a_i \vee a_j = a_k$, then the absorption identity implies $a_i = a_i \wedge (a_i \vee a_j) = a_i \wedge a_k$. Now, if $i \neq k$, it follows that $a_i \wedge a_k = 0$; hence, we may conclude $a_i = 0$ in this case. This is a contradiction. On the other hand if $a_i \vee a_j = a_i$, then $a_j = a_j \wedge (a_i \vee a_j) = a_j \wedge a_k = 0$. This shows that $a_i \vee a_j = 1$. Thus $\{0, a_1, \dots, a_{n-2}, 1\}$ is the lattice M_n. □

We have the following corollary from this lemma.

Corollary 2.1. *The graph G, obtained by adjoining a pendant vertex to the complete graph K_n, for $n \geq 4$, is a graph such that $G \neq \Gamma(L)$ for any distributive lattice L with* 0.

Remark 2.1. We note that for $n \geq 4$, the graph $\Gamma(M_n)$ of the lattice M_n has a subgraph homeomorphic to K_5. Hence by Kuratowski's theorem (see; Harary [15, Theorem 11.13]), $\Gamma(M_n)$ cannot be planar in this case. In fact $\Gamma(M_n)$ is planar if and only if $n \leq 3$.

Let L be a lattice with 0. We say that $\Gamma(L)$ is n-regular, if every nonzero vertex, other than 1, is of degree n. We have the following result.

Theorem 2.3. *The graph $\Gamma(L)$ of a lattice L with n elements is $n-2$-regular if and only if $L = M_n$.*

A nonempty subset I of a lattice L is called an *ideal* of L if I satisfies the conditions

(i) $a, b \in I$ imply $a \vee b \in I$,

(ii) $a \in I$, $x \leq a$ implies $x \in I$.

A nonzero ideal I of a lattice L with 0 is called a *minimal ideal* if there is no nonzero ideal J such that $J \subset I$. We note that I is a minimal ideal of L if and only if $I = (x]$ for some atom $x \in L$. If Y denotes the set of all minimal ideals of L then the graph, $\Gamma(X)$, of the meet-semilattice $X = Y \cup \{(0]\}$ is a complete graph. Let L be a lattice with 0 then the set $\mathrm{Id}(L)$ of ideals of L is a poset under set inclusion. It is known that this poset is an algebraic lattice. Moreover, the subposet $K[\mathrm{Id}(L)]$ of its compact elements is order isomorphic to L under the isomorphism $x \mapsto (x]$; see, Grätzer [14, Theorem 13, p. 106]. The proof of the following two theorems follow by using this observation.

Theorem 2.4. *The complete graph K_n is a subgraph of $\Gamma(X)$ if and only if L has at least $n-1$ atoms.*

Theorem 2.5. *Suppose L is an atomic lattice. The complete graph K_n is a subgraph of $\Gamma(X)$ if and only if K_n is a subgraph of $\Gamma(L)$.*

Remark 2.2. We note that the assumption that L is atomic is necessary in Theorem 2.5. Consider the lattice L in Example 2. It is not atomic and $\mathrm{Clique}(L) = 2$. Thus K_2 is a subgraph of $\Gamma(L)$. However, K_2 is not a subgraph of $\Gamma(X)$ as L does not have a nonzero minimal ideal.

A lattice L with the smallest element 0 and the largest element 1 is called *complemented* if for each $x \in L$, there exists a $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$, we write $y = x'$ and call x', a *complement* of x. It is known that in a distributive lattice an element can have at most one complement.

Lemma 2.1. *If a complemented distributive lattice L contains an infinite increasing chain, then* $\mathrm{Clique}(L) = \infty$.

Proof. Let $a_1 < a_2 < \cdots$ be an increasing chain in L. Put $y_i = a_{i+1} \wedge a_i'$. If $y_i = 0$, then by the distributivity, we get $a_i = a_i \vee (a_{i+1} \wedge a_i') = a_{i+1}$, a contradiction.

Suppose $y_i = y_j$. Without loss of generality, we may assume $i < i+1 \leq j$. Then $a_{i+1} \leq a_j$ and we get $y_j = a_{j+1} \wedge a_j' = a_{i+1} \wedge a_i' \wedge a_j' = 0$, a contradiction. Thus the y_i are distinct.

Again as above, $a_{i+1} \wedge a_j' = 0$ implies $y_i \wedge y_j = 0$ for $i \neq j$. Thus $\{y_i : i = 1, 2, \dots\}$ is an infinite clique in L. □

The Examples 3 and 4 given below show that the condition of distributivity and that of complementedness cannot be deleted in Lemma 2.1.

Example 3. An integer a is divisible by an integer b if $a = bc$ for some integer c. Thus 0 is divisible by all integers including 0 itself. Let $L = A \cup \{0, 1, 3\}$, where $A = \{x : x \text{ is a positive even integer not divisible by } 3\}$. Then L is a complemented lattice under the divisibility order with the smallest element 1 and the largest element 0. Clearly, L is not distributive as every element in A is a complement of 3. We note that L contains an infinite increasing chain, namely $2 < 4 < \cdots$, but $\mathrm{Clique}(L) = 3$.

Example 4. The set $L = \left\{\frac{n}{n+1} : n \text{ is a positive integer}\right\} \cup \{0, 1\}$ with the usual order is a bounded distributive lattice but it is not complemented. L contains an infinite increasing chain, namely, $\frac{1}{2} < \frac{2}{3} < \cdots$. We note that $\mathrm{Clique}(L) = 2$.

A nonzero element $x \in L$ is called a *zerodivisor* if there exists a nonzero $y \in L$ such that $x \wedge y = 0$. We denote the set of all zerodivisors in L by $Z(L)$.

Lemma 2.2. *Let L be a lattice with 0, 1. If $1 = \bigvee_{i=1}^{n} a_i$ for some atoms $a_i \in L$, then every nonzero element in L is a zerodivisor.*

Proof. Let $x \in L$, $x \neq 0, 1$. There exists some $a_i \nleq x$. Then $x \wedge a_i = 0$. $\square$

Lemma 2.3. *Let L be a distributive lattice with 0 and 1. Suppose that L contains a nonzero element x such that $x \neq 1$ and x has a complement. Let f be an automorphism of L such that $f(a) = a$ for every $a \in Z(L) \cup \{0\}$. Then $f(a) = a$ for every $a \in L$.*

Proof. For any $a \in L$, we note that $a \wedge x$, $a \wedge x' \in Z(L) \cup \{0\}$, where x' is the complement of x. Since $a = a \wedge 1 = a \wedge (x \vee x') = (a \wedge x) \vee (a \wedge x')$, we conclude $f(a) = a$. $\square$

3. Coloring in distributive lattices

In this section we prove the following result.

Theorem 3.1. *If L is a distributive lattice with 0, then $\chi(L) = \mathrm{Clique}(L)$.*

This will be accomplished through a series of results.

If $\mathrm{Clique}(L) = \infty$, then $\chi(L) \geq \mathrm{Clique}(L)$ implies $\chi(L) = \infty$. Thus it is sufficient to prove the result when $\mathrm{Clique}(L) < \infty$.

A proper ideal I of L is called a *prime ideal* if $x \wedge y \in I$ implies either $x \in I$ or $y \in I$. A prime ideal is called a *minimal prime ideal*, if it does not contain any other prime ideal. For a nonempty $S \subseteq L$, let $\mathrm{Ann}(S) = \{y \in L : x \wedge y = 0 \text{ for each } x \in S\}$. We call $\mathrm{Ann}(S)$ the annihilator of S. In general $\mathrm{Ann}(S)$ is not an ideal of L. However, if L is distributive, then $\mathrm{Ann}(S)$ is an ideal of L. If $S = \{x\}$, we denote $\mathrm{Ann}(S)$ by $\mathrm{Ann}(x)$. $\mathrm{Ann}(S)$ is called a *maximal annihilator ideal* if $\mathrm{Ann}(S) \neq L$ and $\mathrm{Ann}(S) \subseteq \mathrm{Ann}(T)$ for some $T \subseteq L$ implies $\mathrm{Ann}(S) = \mathrm{Ann}(T)$.

Lemma 3.1. *Let L be a distributive lattice with 0. If $\mathrm{Ann}(S)$ is maximal in the set $\{\mathrm{Ann}(T) : T \subset L\}$, then $\mathrm{Ann}(S) = \mathrm{Ann}(x)$ for some $x \in L$, $x \neq 0$.*

Proof. Since $\mathrm{Ann}(S) \neq L$, there exists $x \in S$, $x \neq 0$. Then $\mathrm{Ann}(x) \neq L$. Clearly, $\mathrm{Ann}(S) \subseteq \mathrm{Ann}(x)$. By the maximality, we conclude $\mathrm{Ann}(S) = \mathrm{Ann}(x)$. $\square$

We prove an analogue of [9, Proposition 2.1] of Cornish and Stewart for distributive lattices with 0.

Lemma 3.2. *If L is a distributive lattice with 0 and $S \subseteq L$, then the following statements are equivalent.*

(1) $\operatorname{Ann}(S)$ *is a maximal annihilator.*

(2) $\operatorname{Ann}(S)$ *is a prime ideal.*

(3) $\operatorname{Ann}(S)$ *is a minimal prime ideal.*

Proof.

(1) $\implies$ (2): By Lemma 3.1 $\operatorname{Ann}(S) = \operatorname{Ann}(x)$ for any $x \in S$, $x \neq 0$. Let $a \wedge b \in \operatorname{Ann}(x)$ and $a \notin \operatorname{Ann}(x)$. Then $a \wedge b \wedge x = 0$, $a \wedge x \neq 0$ and $b \in \operatorname{Ann}(a \wedge x)$. Let $t \in \operatorname{Ann}(x)$, then $t \wedge x = 0$ leads to $t \wedge a \wedge x = 0$, i.e., $t \in \operatorname{Ann}(a \wedge x)$. Hence $\operatorname{Ann}(x) \subseteq \operatorname{Ann}(a \wedge x)$. By the maximality of $\operatorname{Ann}(x)$, we get $\operatorname{Ann}(x) = \operatorname{Ann}(a \wedge x)$ or $\operatorname{Ann}(a \wedge x) = L$. Since $a \wedge x \neq 0$, the second possibility cannot hold. Thus $b \in \operatorname{Ann}(x)$.

(2) $\implies$ (3): Since $\operatorname{Ann}(S)$ is prime, it is a proper ideal of L. Let $y \in S$, $y \neq 0$. Let Q be a prime ideal of L such that $Q \subset \operatorname{Ann}(S)$. Let $x \in \operatorname{Ann}(S) - Q$. Clearly, $y \notin \operatorname{Ann}(S)$ and $x \wedge y = 0$. Then $x \wedge y = 0 \in Q$ implies either $x \in Q$ or $y \in Q$, a contradiction. Hence $Q = \operatorname{Ann}(S)$.

(3) $\implies$ (1): Suppose $\operatorname{Ann}(S) \subset \operatorname{Ann}(T)$, $\operatorname{Ann}(T) \neq L$. There exists $x \in T$ such that $x \neq 0$, which implies that $x \notin \operatorname{Ann}(T)$. Let $y \in \operatorname{Ann}(T) - \operatorname{Ann}(S)$. Now, $x \wedge y = 0$ implies by (3) that either $x \in \operatorname{Ann}(S)$ or $y \in \operatorname{Ann}(S)$, a contradiction. $\square$

A partially ordered set P is said to satisfy the *ascending chain condition* (ACC) provided every strictly ascending chain in P is finite.

Lemma 3.3. *If L is a distributive lattice with 0 such that $\operatorname{Clique}(L) < \infty$, then the set $\{\operatorname{Ann}(x) : \ x \in L, \ \ x \neq 0\}$ satisfies the ascending chain condition.*

Proof. Suppose $\operatorname{Ann}(a_1) \subset \operatorname{Ann}(a_2) \subset \cdots$. Let $x_j \in \operatorname{Ann}(a_j) - \operatorname{Ann}(a_{j-1})$, $j = 2, 3, \ldots$. If we let $y_n = x_n \wedge a_{n-1}$ $(n = 2, 3, \ldots)$, then $y_n \neq 0$. For $i < j$, we have $x_i \in \operatorname{Ann}(a_i) \subseteq \operatorname{Ann}(a_{j-1})$. Thus $x_i \wedge a_{j-1} = 0$, consequently, $y_i \wedge y_j = 0$ for $i \neq j$. Thus the set $\{y_n : \ n = 2, 3, \ldots\}$ is an infinite clique, a contradiction. $\square$

Theorem 3.2. *Let L be a distributive lattice with 0 and $\operatorname{Clique}(L) < \infty$. Then L has only a finite number of distinct minimal prime ideals, P_i, $1 \leq i \leq n$. These ideals satisfy $\bigcap\limits_{i=1}^{n} P_i = (0]$ and $\bigcap\limits_{i \neq j} P_i \neq (0]$ for all j. Further, no element of $L - \bigcup\limits_{i=1}^{n} P_i$ is a zerodivisor.*

P r o o f. By Lemma 3.3 and Lemma 3.1, L has only a finite number of maximal annihilator ideals and these ideals have the form $\mathrm{Ann}(y_i)$, $1 \leq i \leq n$, $y_i \neq 0$. Lemma 3.2 implies that $\mathrm{Ann}(y_i)$ are minimal prime ideals of L. Let $x \in \bigcap_{i=1}^{n} \mathrm{Ann}(y_i)$, $x \neq 0$. Then $x \in \mathrm{Ann}(y_i)$ for each i, i.e., $y_i \in \mathrm{Ann}(x)$ for each i. By the maximality of $\mathrm{Ann}(y_i)$, $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(y_i)$ for some i. This implies $y_i \in \mathrm{Ann}(y_i)$ for some i, a contradiction. Hence $x = 0$.

Let P be a prime ideal of L. Then $\bigcap_{i=1}^{n} \mathrm{Ann}(y_i) = (0]$ implies $\mathrm{Ann}(y_i) \subseteq P$ for some i. Thus $\mathrm{Ann}(y_i)$ are the only minimal prime ideals of L.

Suppose $\bigcap_{i \neq j} \mathrm{Ann}(y_i) = (0]$ for some j. Let $x_i \in \mathrm{Ann}(y_i) - \mathrm{Ann}(y_j)$. Then $\bigwedge_{i \neq j} x_i \in \bigcap_{i \neq j} \mathrm{Ann}(y_i) = (0]$. Since $\mathrm{Ann}(y_j)$ is prime, this implies $x_i \in \mathrm{Ann}(y_j)$ for some i, a contradiction.

If $y, z \in L$, $z \neq 0$ and $y \wedge z = 0$. Then $y \in \mathrm{Ann}(z) \subseteq \mathrm{Ann}(y_i)$ for some i implies $y \notin L - \bigcup_{i=1}^{n} \mathrm{Ann}(y_i)$. Thus if $y \in L - \bigcup_{i=1}^{n} \mathrm{Ann}(y_i)$, then $y \wedge z \neq 0$ for any nonzero $z \in L$. $\square$

Remark 3.1. This theorem shows that every minimal prime ideal of L has the form $\mathrm{Ann}(x)$ for some $x \in L$.

Lemma 3.4. *If for some $x, y \in L$, $\mathrm{Ann}(x)$ and $\mathrm{Ann}(y)$ are distinct prime ideals then $x \wedge y = 0$.*

P r o o f. Since $\mathrm{Ann}(x) \neq \mathrm{Ann}(y)$, there exists $t \in \mathrm{Ann}(y) - \mathrm{Ann}(x)$ or $t \in \mathrm{Ann}(x) - \mathrm{Ann}(y)$. In the first case, $t \wedge y = 0 \in \mathrm{Ann}(x)$ implies $y \in \mathrm{Ann}(x)$, by the primeness of $\mathrm{Ann}(x)$. Thus $x \wedge y = 0$. Similarly, in the second case we get $x \wedge y = 0$. $\square$

Lemma 3.5. *Suppose L is a distributive lattice with 0 and $\mathrm{Clique}(L) < \infty$. Let $a \in \Gamma(L)$ be a nonzero element such that $\deg(a) = n$ and there is no nonzero element $b \in \Gamma(L)$ such that $\deg(b) > n$. Then $\mathrm{Ann}(a)$ is a prime ideal and no two elements in $L - \mathrm{Ann}(a)$ are adjacent to each other in $\Gamma(L)$.*

P r o o f. Since L is a distributive lattice, $\mathrm{Ann}(a)$ is an ideal. Suppose that $x \wedge y \in \mathrm{Ann}(a)$, $x \notin \mathrm{Ann}(a)$, $y \notin \mathrm{Ann}(a)$. Then $t = x \wedge a \neq 0$, $t \wedge y = 0$ and $\mathrm{Ann}(x) \cup \mathrm{Ann}(a) \subseteq \mathrm{Ann}(t)$ shows that $\deg(t) \geq n + 1$, a contradiction. Thus $x \in \mathrm{Ann}(a)$. This also shows that $x \wedge y \neq 0$, for $x, y \in L - \mathrm{Ann}(a)$, i.e., x, y cannot be adjacent to each other in $\Gamma(L)$. $\square$

Now we give a characterization of the chromatic number of a distributive lattice.

Theorem 3.3. *For a distributive lattice L with 0, the following statements are equivalent.*

(1) *$\chi(L)$ is finite.*

(2) *$\mathrm{Clique}(L)$ is finite.*

(3) *The ideal $(0]$ of L is the intersection of a finite number of prime ideals.*

Proof.

(1) $\implies$ (2): For any graph L, it is known that $\mathrm{Clique}(L) \leq \chi(L)$.

(2) $\implies$ (3): Follows from Theorem 3.2.

(3) $\implies$ (1): Let $(0] = P_1 \cap \cdots \cap P_n$, where P_i , $i = 1, \ldots, n$ are prime ideals. Define a coloring f on L by putting $f(0) = 0$ and $f(x) = \min\{i : \ x \notin P_i\}$ for $x \neq 0$. If x, y are two nonzero adjacent elements, then $x \notin P_i$ and $y \notin P_j$ for some prime ideals P_i and P_j. Since $x \wedge y = 0$, we conclude $y \in P_i$ and $x \in P_j$. Thus $f(x) \neq f(y)$ and so f is a coloring on L. This implies $\chi(L) \leq n + 1$. □

In the next result, we give a relationship between the chromatic number and the number of minimal prime ideals of L.

Theorem 3.4. *Let L be a distributive lattice with 0. If $\chi(L)$ is finite, then L has only a finite number of minimal prime ideals. If n is this number, then $\chi(L) = \mathrm{Clique}(L) = n + 1$.*

Proof. It follows from Theorem 3.2 that L has only a finite number of minimal prime ideals, say P_i, $1 \leq i \leq n$, $\bigcap_{i=1}^{n} P_i = (0]$ and $\bigcap_{i \neq j} P_i \neq (0]$. As in the proof of Theorem 3.3, we can show $\chi(L) \leq n + 1$. Choose nonzero $x_1 \in \bigcap_{i \neq 1} P_i$, $x_2 \in \bigcap_{i \neq 2} P_i$, $\ldots$, $x_n \in \bigcap_{i \neq n} P_i$. The set $\{0, x_1, \ldots, x_n\}$ is a clique in L. Thus $n + 1 \leq \mathrm{Clique}(L)$. This implies the result. □

Since an ideal of a distributive lattice is a distributive lattice, the proof of the following corollary is immediate.

Corollary 3.1. *Let L be a distributive lattice with 0. For any ideal I of L, $\mathrm{Clique}(I) = \chi(I)$.*

The following example shows that the condition of distributivity in Theorem 3.4 is needed.

Example 5. The lattice shown in Figure 6 is not distributive and $\{0, a, b, c\}$ is a clique. Thus $\mathrm{Clique}(L) = 4 = \chi(L)$. However, the number of minimal prime ideals is 0. Thus $\chi(L)$ cannot be determined by the number of minimal prime ideals in the case of a nondistributive lattice.

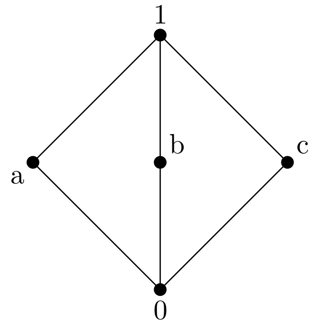

FIGURE 6

4. Complemented graphs

Anderson et al [4] define and obtain some results on complemented zero divisor graphs on commutative von Neumann regular rings. We introduce these concepts in graphs derived from distributive lattices. Let L be a distributive lattice with 0. For $a, b \in L$, we write $a \perp b$ if and only if a and b are adjacent in $\Gamma(L)$ and there is no nonzero vertex which is adjacent to both a and b. Thus we have $a \perp b$ if and only if $a \wedge b = 0$ and $\operatorname{Ann}(a) \cap \operatorname{Ann}(b) \subseteq \{0, a, b\}$. We say that a graph G is *complemented* if for each $a \in G$, $a \neq 0$, $a \neq 1$, there exists $b \in G$, $b \neq 0$, called a *complement* of a such that $a \perp b$. For $a, b \in G$, we write $a \sim b$ provided the following conditions are met:

(i) The vertices a, b are not adjacent in G,

(ii) for all $x \in G$, we have x is adjacent to a if and only if x is adjacent to b.

A complemented graph G is called *uniquely complemented*, if $a \perp b$, $a \perp c$ imply, $a \sim c$. We say that an element $a \in L$ is *regular* if $a \wedge b = 0$ implies $b = 0$ for any $b \in L$.

LEMMA 4.1. *For a, b in a distributive lattice L, the following statements are equivalent.*

(1) *$a \perp b$.*

(2) *$a \wedge b = 0$ and $a \vee b$ is regular.*

P r o o f.

(1) $\implies$ (2): Clearly, $a \wedge b = 0$. Suppose $(a \vee b) \wedge x = 0$. By distributivity, $(a \wedge x) \vee (b \wedge x) = 0$ and so $a \wedge x = b \wedge x = 0$ leads to $x = 0$.

(2) $\implies$ (1): Suppose $a \wedge x = b \wedge x = 0$. Then distributivity implies $(a \vee b) \wedge x = 0$ and so $x = 0$. Thus $a \perp b$. $\square$

Lemma 4.2. *Let a, b, c be nonzero elements in a distributive lattice L with* 0. *If $a \perp b$ and $a \perp c$, then* $\mathrm{Ann}(b) = \mathrm{Ann}(c)$.

P r o o f. If $b \wedge c = 0$ then $a \perp b$ implies $c = 0$, a contradiction. Let $d \in \mathrm{Ann}(b)$. Then $d \wedge b = 0$ and so $d \wedge c \wedge b = 0$. Similarly, $a \perp c$ implies $d \wedge a \wedge c = 0$. By $a \perp b$ we get $d \wedge c = 0$. Thus $d \in \mathrm{Ann}(c)$. Similarly, we get $\mathrm{Ann}(c) \subseteq \mathrm{Ann}(b)$. $\square$

In view of Lemma 4.2 we conclude that the graph of a distributive lattice with 0 is uniquely complemented if and only if it is complemented.

Examples 6.

1) Star graph is not complemented.
2) The graph of a Boolean lattice L is complemented. For $x \in \Gamma(L)$, its complement x' satisfies $x \perp x'$.

5. Combinatorial results

In this section all the lattices under consideration are finite.

Bollobas and Rival in [8] have shown that the number of edges in the covering graph of a lattice with n elements is less than $3n^{\frac{3}{2}}$ edges.

In the next theorem we give an estimate for the number of edges in $\Gamma(L)$, for a finite lattice L.

Theorem 5.1. *Suppose $L = \{0, a_1, \dots, a_m, 1\}$ is a finite lattice. If n is the number of distinct edges in $\Gamma(L)$, then*

$$m + 1 \le n \le \frac{m(m+1)}{2} + 1. \tag{1}$$

P r o o f. We note that 0 is adjacent to each element. Hence the minimum number of edges in $\Gamma(L)$ is $m + 1$. It is known that the number of edges in the complete graph K_n on n vertices is $\frac{n(n-1)}{2}$ (see [15, p. 16] from Harary). If the set $\{0, a_1, \dots, a_m\}$ forms a K_{m+1}, then the number of edges is $\frac{m(m+1)}{2}$. The element 1 is adjacent to 0 only. Thus the maximum number of distinct edges in $\Gamma(L)$ is $1 + \frac{m(m+1)}{2}$. Thus $m + 1 \le n \le 1 + \frac{m(m+1)}{2}$. $\square$

We note the following.

Remark 5.1. If L has only one atom, then the number of edges in $\Gamma(L)$, is $m+1$. Thus equality holds on the left hand side inequality of (1).

Remark 5.2. If each a_i in $L = \{0, a_1, \dots, a_m, 1\}$ is an atom then each a_i is adjacent to each a_j, $i \neq j$. Thus the total number of edges is $1 + \frac{m(m+1)}{2}$. Thus in this case equality holds on the right hand side inequality of (1).

Lemma 5.1. *If L has more than one atom and L is nonmodular, then strict inequality occurs at both the places in* (1).

Proof. Let $L = \{0, a_1, \dots, a_m, 1\}$ Suppose that a_1, a_2 are the two atoms. Then a_1, a_2 are adjacent to each other and so $\{a_1, a_2\}$ is an edge in $\Gamma(L)$. If the number of edges in $\Gamma(L)$ is n, then $m+1 < n$. Since L is nonmodular, by Dedekind's modularity criterion (see [14, Theorem 2, p. 80] from Grätzer), L has a sublattice isomorphic to the lattice n_5, shown in Figure 7. Thus L has two elements a_i, a_j comparable with each other. Thus $\{a_i, a_j\}$ cannot be an edge in $\Gamma(L)$. Therefore, $n < 1 + \frac{m(m+1)}{2}$. □

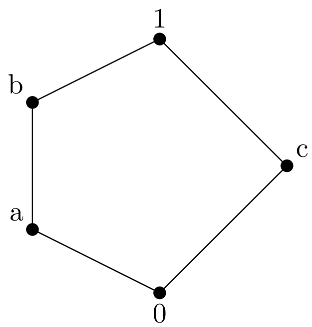

Figure 7

Remark 5.3. Equalities hold in (1) if and only if either $L = \{0\}$ or $L = \{0, 1\}$, i.e., if and only if either $m = 0$ or 1 with the convention that $a_0 = 0$ and $a_1 = 1$.

In the following theorems we estimate the clique number and the chromatic number of direct products of lattices.

Remark 5.4. If L_1 and L_2 are two lattices, then in general, $\mathrm{Clique}(L_1 \times L_2) \neq \mathrm{Clique}(L_1) \times \mathrm{Clique}(L_2)$ and $\chi(L_1 \times L_2) \neq \chi(L_1) \times \chi(L_2)$, for example take $L_1 = \{0, a\}$ and $L_2 = \{0, b\}$.

We have the following theorem.

Theorem 5.2. *Let L_i, $i = 1, \dots, n$, be lattices with* 0 *and with* $\mathrm{Clique}(L_i) = m_i$. *Let $L = L_1 \times \cdots \times L_n$. Then* $\mathrm{Clique}(L) = \sum_{i=1}^{n} m_i - n + 1$.

Proof. Let $\{0, a_{i,1}, \dots, a_{i,m_i-1}\}$ be a clique in L_i, $i = 1, \dots, n$. We note that $B = \big\{(0, \dots, 0), (a_{1,1}, 0, \dots, 0), \dots (a_{1,m_1-1}, 0, \dots, 0), \dots, (0, \dots, 0, a_{n,m_n-1})\big\}$ is a clique in L with $\sum_{i=1}^{n}(m_i - 1) + 1 = 1 - n + \sum_{i=1}^{n} m_i = t$ elements. Hence $\mathrm{Clique}(L) \geq t$.

Suppose that $\{a_1, \dots, a_k\}$ is a clique in L. Let $a_i = (b_{i,1}, \dots, b_{i,n})$, $i = 1, \dots, k$. The set $\{b_{1,1}, \dots, b_{k,1}\}$ is a subset of L_1 such that $b_{i,1} \wedge b_{j,1} = 0$ for all i, j, $i \neq j$. Since $\mathrm{Clique}(L_1) = m_1$, we conclude $k \leq m_1$. More generally, we get $k \leq m_i$ for each i. This implies $nk \leq \sum_{i=1}^{n} m_i$, consequently, $n(k-1) \leq \sum_{i=1}^{n} m_i - n$. Since, n and $k - 1$ are positive integers, we get $k - 1 \leq \sum_{i=1}^{n} m_i - n$ i.e. $k \leq \sum_{i=1}^{n} m_i - n + 1 = t$. Thus $\mathrm{Clique}(L) = t$. □

Using similar techniques, we get the following theorem.

Theorem 5.3. *Let L_i, $i = 1, \dots, n$, be lattices with* 0 *with* $\chi(L_i) = m_i$. *If $L = L_1 \times \cdots \times L_n$, then* $\chi(L) = \sum_{i=1}^{n} m_i - n + 1$.

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