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OSCILLATION OF HIGHER ORDER NEUTRAL FUNCTIONAL DIFFERENCE EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

R. N. Rath* — B. L. S. Barik** — S. K. Rath***

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ABSTRACT. Sufficient conditions are obtained so that every solution of the neutral functional difference equation

$$\Delta^m(y_n - p_n y_{\tau(n)}) + q_n G(y_{\sigma(n)}) - u_n H(y_{\alpha(n)}) = f_n,$$

oscillates or tends to zero or $\pm\infty$ as $n \to \infty$, where Δ is the forward difference operator given by $\Delta x_n = x_{n+1} - x_n$, p_n, q_n, u_n, f_n are infinite sequences of real numbers with $q_n > 0$, $u_n \geq 0$, $G, H \in C(\mathbb{R}, \mathbb{R})$ and $m \geq 2$ is any positive integer. Various ranges of $\{p_n\}$ are considered. The results hold for $G(u) \equiv u$, and $f_n \equiv 0$. This paper corrects, improves and generalizes some recent results.

1. Introduction

In this paper, sufficient conditions are obtained, so that every solution of

$$\Delta^m(y_n - p_n y_{\tau(n)}) + q_n G(y_{\sigma(n)}) - u_n H(y_{\alpha(n)}) = f_n, \tag{1.1}$$

oscillates or tends to zero or $\pm\infty$ as $n \to \infty$, where Δ is the forward difference operator given by $\Delta x_n = x_{n+1} - x_n$, p_n, q_n, u_n and f_n are infinite sequences of real numbers with $q_n > 0$, $u_n \geq 0$, $G, H \in C(\mathbb{R}, \mathbb{R})$. Further, we assume $\{\tau(n)\}$, $\{\sigma(n)\}$, and $\{\alpha(n)\}$ are monotonic increasing and unbounded sequences such that $\tau(n) \leq n$, $\sigma(n) \leq n$ and $\alpha(n) \leq n$ for every n. Different ranges of $\{p_n\}$ are considered. The positive integer $m \geq 2$, can take both odd and even values.

If $[x]$ denotes the greatest integer less than or equal to the real variable x then for any positive integer n, $\tau(n) = [n/3]$ is an non-decreasing and unbounded

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sequence of integers less than n. In this case, note that, there does not exist any positive integer k for which $\tau(n) = n - k$. Hence our results generalize the corresponding results done for (1.1) when $\tau(n) = n - k$. Then think about the inverse τ^{-1} of the function τ. Of course it always exists as a relation, but not necessarily as a function, because it may not be single valued. Hence if we define $\tau_{-1}(n) = \min\{\tau^{-1}(n)\}$ then τ_{-1} is a function such that $\tau(\tau_{-1})(n) = n$. However, $\tau_{-1}(\tau(n)) \neq n$ for every n. Hence, whenever, we require this condition we have to assume $\tau(n)$ is strictly increasing. Then of course τ^{-1} would exist as a function and $\tau^{-1}(\tau(n)) = n$. But in that case the utility of taking $\tau(n)$ in place of $n - k$ is reduced, because, it seems difficult to find an example of a strictly increasing and unbounded function $\tau(n)$ other than of the form $\tau(n) = n - k$ where, k is a positive integer.

In the sequel, we shall need the following conditions.

(H0) G is non-decreasing and $xG(x) > 0$ for $x \neq 0$.

(H1) $\liminf\limits_{n\to\infty} \sigma(n)/n > 0$.

(H2) H is bounded.

(H3) $\liminf\limits_{|v|\to\infty} \frac{G(v)}{v} \geq \delta > 0$.

(H4) $\sum\limits_{n=0}^{\infty} n^{m-2} q_n = \infty$ for $m \geq 2$.

(H5) $\sum\limits_{n=0}^{\infty} n^{m-1} u_n < \infty$.

(H6) $\sum\limits_{n=0}^{\infty} n^{m-1} q_n = \infty$.

(H7) There exists a bounded sequence $\{F_n\}$ such that $\Delta^m F_n = f_n$, and $\lim\limits_{n\to\infty} F_n = 0$.

(H8) There exists a bounded sequence $\{F_n\}$ such that $\Delta^m F_n = f_n$.

We assume that p_n satisfies one of the following conditions in this paper.

(A1) $0 \leq p_n \leq b < 1$.

(A2) $-1 < -b \leq p_n \leq 0$.

(A3) $-b_2 \leq p_n \leq -b_1 < -1$.

(A4) $1 < b_1 \leq p_n \leq b_2 < \infty$.

(A5) $0 \leq p_n \leq b_2 < \infty$.

(A6) $-\infty < -b_2 \leq p_n \leq 0$.

(A7) $1 \leq p_n \leq b_2 < \infty$.

Note that, the parameters b, b_1, b_2 used in the conditions (A1)–(A7), are positive constants. Further note that we do not need the condition "$xH(x) > 0$ for $x \neq 0$"

in the proofs of our results, however, one may assume that for technical reasons, to make (1.1), a neutral equation with positive and negative coefficients.

In recent years, several papers on oscillation of solutions of neutral delay difference equations have appeared; (cf. [1, 2], [11]–[22]) and the references cited therein. In literature we find that (1.1) is very rarely studied. We may note that (1.1) is the discrete analogue of the equation

$$\big(y(t) - p(t)y(r(t))\big)^{(m)} + q(t)G(y(g(t))) - u(t)H(y(h(t))) = f(t). \tag{1.2}$$

We feel, even (1.2) is not studied much for $m > 2$. The equation (1.1) reduces to

$$\Delta^m(y_n - p_n y_{n-k}) + q_n G(y_{n-l}) - u_n G(y_{n-r}) = f_n, \tag{1.3}$$

for $\tau(n) = n - k$, $\sigma(n) = n - l$, $\alpha(n) = n - r$.

Recently, in [14, 15] the authors obtained the oscillation and non-oscillation criteria for oscillation of

$$\Delta(y_n - p_n y_{n-k}) + q_n G(y_{n-l}) - u_n G(y_{n-r}) = f_n. \tag{1.4}$$

The same equation (1.4) with several delay terms, under the restriction $G(u) \equiv u$, is studied in [16]. In [17] the authors have obtained oscillation and non-oscillation results for (1.4) under restrictions $G(u) \equiv u$ and $f_n \equiv 0$. Sufficient conditions for oscillation of

$$\Delta^m(y_n - p_n y_{n-k}) + q_n G(y_{n-l}) = f_n, \tag{1.5}$$

are obtained in [19]. In that paper, p_n is confined to (A2) only and G is restricted with a sublinear condition

$$\left|\int_0^{\pm c} \frac{\mathrm{d}u}{G(u)}\right| < \infty. \tag{1.6}$$

In [21] the authors studied

$$\Delta^m(y_n - p_n y_{n-l}) + q_n y_{n-k}^{\alpha} = 0, \tag{1.7}$$

where $\alpha < 1$, is a quotient of odd integers and p_n satisfies (A1) or (A2). They obtained the sufficient conditions of oscillation of (1.7) under the conditions

$$\sum_{n=n_0}^{\infty} q_n (n-k)^{\alpha(m-1)} = \infty \tag{1.8}$$

and

$$\sum_{n=n_0}^{\infty} q_n (1 + p_{n-k})^{\alpha(m-1)} = \infty, \tag{1.9}$$

and presented the following results.

Theorem 1.1. ([21, Theorem 2.1])

(a) *Let m be even. If $-1 < p_n \leq 0$ and* (1.9) *hold, then all solutions of* (1.7) *are oscillatory.*

(b) *Let m be odd. If* (A2) *and* (1.8) *hold then every solution of* (1.7) *oscillates or tends to zero as $n \to \infty$.*

Theorem 1.2. ([21, Theorem 2.2]) *If* (A1) *and* (1.8) *hold then every solution of* (1.7) *oscillates or tends to zero as $n \to \infty$.*

We may note that, for $m \geq 2$, (H4) implies (H6) and if $\alpha < 1$ then (1.8) implies (H4) for $m \geq \frac{2-\alpha}{1-\alpha}$. Further, all the equations, (1.3)–(1.5) and (1.7) are particular cases of (1.1). The results in [19, 21] do not hold for a class of equations, where G is either linear or super linear, i.e.; for example when $G(u) = u$ or $G(u) = u^3$. Here in this paper an attempt is made to fill this existing gap in literature and obtain sufficient conditions for oscillation of solutions of a more general equation (1.1) under the weaker conditions (H4) or (H6). Moreover, we observe that the existing papers in the literature do not have much to offer when p_n satisfies (A4), (A6) or (A7). In this direction we find that, the authors in [12] have obtained sufficient conditions for the oscillation of solutions of the equation

$$\Delta^m(y_n - p_n y_{n-k}) + q_n G(y_{n-r}) = 0, \tag{1.10}$$

with (A4) or (A7) and have the following results.

Theorem 1.3. ([12, Theorem 2.6]) *Let p_n satisfy* (A7)*. If the condition*

$$\sum_{n=n_0}^{\infty} q_n = \infty, \tag{1.11}$$

holds, then the following are valid statements.

(i) *Every solution of* (1.10) *oscillates, if m is even.*

(ii) *Every solution of* (1.10) *oscillates or $\liminf_{n\to\infty} y_n = 0$ if m is odd.*

Theorem 1.4. ([12, Theorem 2.7]) *Let p_n satisfy* (A4)*. If* (1.11) *holds then the following statements are true.*

(i) *Every solution of* (1.10) *oscillates for m even.*

(ii) *Every solution of* (1.10) *oscillates or tends to zero as $n \to \infty$ if m is odd.*

Unfortunately, the following example contradicts the above theorems of [12].

Example 1.1. Consider the neutral equation

$$\Delta^m(y_n - 4y_{n-1}) + 4^{\frac{n+1}{3}} y_{n-2}^{\frac{1}{3}} = 0, \tag{1.12}$$

where m may be any odd or even integer. Here, p_n satisfies (A4) and (A7). Clearly, (1.12) satisfies all the conditions of Theorems 1.3 and 1.4. But, (1.12) has an unbounded positive solution $y_n = 2^n$ which tends to ∞ as $n \to \infty$. Thus, this example contradicts Theorems 1.3 and 1.4. Here, $G(u) = u^{1/3}$, is sublinear. This example further establishes that the results of [19, 21] do not hold when p_n is in (A4) or (A7).

The authors of the papers [12, 19, 21] have studied sub-linear equation, and their results do not hold for linear or super linear equations (i.e. (1.5) satisfying (H3) or (1.7) with $\alpha \geq 1$). In this paper we study (1.1) with p_n in all possible ranges and the neutral equation (1.5), as a particular case of (1.1), could be linear or super-linear. Our results hold good for $G(u) \equiv u$, $f_n \equiv 0$ and $u_n \equiv 0$. The last but not the least, this paper corrects, generalizes and improves some of the results of [11, 12, 14, 16, 17, 19, 21].

Let N_1 be a fixed nonnegative integer. Let $N_0 = \min\{\tau(N_1), \sigma(N_1), \alpha(N_1)\}$. By a solution of (1.1) we mean a real sequence $\{y_n\}$ which is defined for all positive integer $n \geq N_0$ and satisfies (1.1) for $n \geq N_1$. Clearly, if the initial condition

$$y_n = a_n \qquad \text{for} \quad N_0 \leq n \leq N_1, \tag{1.13}$$

is given then the equation (1.1) has a unique solution satisfying the given initial condition (1.13). A solution $\{y_n\}$ of (1.1) is said to be oscillatory if for every positive integer $n_0 \geq N_1$, there exists $n \geq n_0$ such that $y_n y_{n+1} \leq 0$, otherwise $\{y_n\}$ is said to be non-oscillatory.

2. Some lemmas

In this section we present some lemmas that would be used for our results in next section. The following lemma which can be easily proved, generalizes [11, Lemma 2.1].

Lemma 2.1. *Let $\{f_n\}$, $\{q_n\}$ and $\{p_n\}$ be sequences of real numbers defined for $n \geq N_0 > 0$ such that*

$$f_n = q_n - p_n q_{\tau(n)}, \qquad n \geq N_1 \geq N_0,$$

where $\tau(n) \leq n$, is member of a monotonic increasing unbounded sequence. Suppose that p_n satisfies one of conditions (A2), (A3) *or* (A5)*. If $q_n > 0$ for $n \geq N_0$, $\liminf_{n\to\infty} q_n = 0$ and $\lim_{n\to\infty} f_n = L$ exists then $L = 0$.*

Lemma 2.2. ([3, 12]) *Let z_n be a real valued function defined for $n \in N(n_0) = \{n_0, n_0+1, \dots\}$, $n_0 \geq 0$ and $z_n > 0$ with $\Delta^m z_n$ of constant sign on $N(n_0)$ and* *not identically zero. Then there exists an integer* p, $0 \le p \le m-1$, *with* $m+p$ *odd for* $\Delta^m z_n \le 0$ *and* $(m+p)$ *even for* $\Delta^m z_n \ge 0$, *such that*

$$\Delta^i z_n > 0 \qquad \textit{for} \quad n \ge n_0, \ 0 \le i \le p,$$

and

$$(-1)^{p+i}\Delta^i z_n > 0, \qquad \textit{for} \quad n \ge n_0, \ p+1 \le i \le m-1.$$

Definition 2.1. Define the factorial function (cf. [8, page 20]) by

$$n^{(k)} := n(n-1)\dots(n-k+1),$$

where $k \le n$ and $n \in \mathbb{Z}$ and $k \in \mathbb{N}$. Note that $n^{(k)} = 0$, if $k > n$.

Then we have

$$\Delta n^{(k)} = kn^{(k-1)}, \tag{2.1}$$

where $n \in \mathbb{Z}$, $k \in \mathbb{N}$ and Δ is the forward difference operator. One can show, by summing up (2.1) that

$$\sum_{i=m}^{n-1} i^{(k)} = \frac{1}{k+1}\left(n^{(k+1)} - m^{(k+1)}\right), \tag{2.2}$$

holds. Now set

$$b_k(n,m) := \begin{cases} 1, & k = 0 \\ \sum\limits_{j=m}^{n} b_{k-1}(n,j), & k \in \mathbb{N}. \end{cases} \tag{2.3}$$

Here, we evaluate b_k by recursion. Clearly, for $k = 1$ in (2.3), we have

$$b_1(n,m) = \sum_{j=m}^{n} b_0(n,j) = \sum_{j=m}^{n} 1 = (n+1-m) = (n+1-m)^{(1)}.$$

By (2.2) and for $k = 2$ in (2.3), we get

$$\begin{aligned} b_2(n,m) &= \sum_{j=m}^{n} b_1(n,j) = \sum_{j=m}^{n} (n+1-j)^{(1)} \\ &= \sum_{i=1}^{n+1-m} i^{(1)} = \frac{1}{2}(n+2-m)^{(2)} - \frac{1}{2}1^{(2)} = \frac{1}{2}(n+2-m)^{(2)}. \end{aligned}$$

Note that $1^{(2)} = 0$. By (2.2) and for $k = 3$ in (2.3), we get

$$\begin{aligned} b_3(n,m) &= \sum_{j=m}^{n} b_2(n,j) = \frac{1}{2}\sum_{j=m}^{n} (n+2-j)^{(2)} \\ &= \frac{1}{2}\sum_{i=2}^{n+2-m} i^{(2)} = \frac{1}{6}\left[(n+3-m)^{(3)} - 2^{(3)}\right] = \frac{1}{3!}(n+3-m)^{(3)}. \end{aligned}$$

Using a simple induction, we obtain

$$b_k(n,m) = \frac{1}{k!}(n+k-m)^{(k)}. \tag{2.4}$$

Lemma 2.3. *Let $p \in \mathbb{N}$ and $x(n)$ be a non oscillatory sequence which is positive for large n. If there exists an integer $p_0 \in \{0,1,\dots,p-1\}$ such that $\Delta^{p_0} w(\infty)$ exits (finite) and $\Delta^i w(\infty) = 0$ for all $i \in \{p_0+1,\dots,p-1\}$. Then*

$$\Delta^p w(n) = -x(n), \tag{2.5}$$

implies

$$\Delta^{p_0} w(n) = \Delta^{p_0} w(\infty) + \frac{(-1)^{p-p_0-1}}{(p-p_0-1)!} \sum_{i=n}^{\infty} (i+p-p_0-1-n)^{(p-p_0-1)} x(i), \tag{2.6}$$

for all sufficiently large n.

Proof. Summing up (2.5) from n to ∞, we get

$$\Delta^{p-1} w(\infty) - \Delta^{p-1} w(n) = -\sum_{i=n}^{\infty} x(i),$$

or simply

$$\Delta^{p-1} w(n) = \sum_{i=n}^{\infty} x(i) = \sum_{i=n}^{\infty} b_0(i,n)\, x(i). \tag{2.7}$$

Summing up (2.7) from n to ∞, we get

$$\begin{aligned} \Delta^{p-2} w(n) &= \Delta^{p-2} w(\infty) - \sum_{i=n}^{\infty}\sum_{j=i}^{\infty} b_0(j,i)\, x(j) = -\sum_{j=n}^{\infty}\sum_{i=n}^{j} b_0(j,i)\, x(j) \\ &= -\sum_{j=n}^{\infty} b_1(j,n)\, x(j) = -\sum_{i=n}^{\infty} b_1(i,n)\, x(i). \end{aligned} \tag{2.8}$$

Again summing up (2.8) from n to ∞, we obtain

$$\begin{aligned} \Delta^{p-3} w(n) &= \sum_{j=n}^{\infty}\sum_{i=j}^{\infty} b_1(i,j)\, x(i) = \sum_{i=n}^{\infty}\sum_{j=n}^{i} b_1(i,j)\, x(i) \\ &= \sum_{i=n}^{\infty} b_2(i,n)\, x(i). \end{aligned}$$

By the emerging pattern, we have

$$\Delta^j w(n) = (-1)^{p-j-1} \sum_{i=n}^{\infty} b_{p-j-1}(i,n)\, x(i), \quad j \in \{p_0+1, \dots p-1\}.$$

Then by letting $j = p_0 + 1$, we get

$$\Delta^{p_0+1} w(n) = (-1)^{p-p_0-2} \sum_{i=n}^{\infty} b_{p-p_0-2}\,(i,n)\, x\,(i)\,. \tag{2.9}$$

Summing up (2.9) from n to ∞ and arranging we get

$$\Delta^{p_0} w(n) = \Delta^{p_0} w\,(\infty) + (-1)^{p-p_0-1} \sum_{i=n}^{\infty} b_{p-p_0-1}\,(i,n)\, x\,(i)\,. \tag{2.10}$$

From (2.4) and (2.10) it follows that

$$\Delta^{p_0} w(n) = \Delta^{p_0} w\,(\infty) + \frac{(-1)^{p-p_0-1}}{(p-p_0-1)!} \sum_{i=n}^{\infty} (i+p-p_0-1-n)^{(p-p_0-1)}\, x\,(i)\,.$$

Hence the Lemma is proved. □

Lemma 2.4. *If $\{w_n\}$ is a sequence of real numbers such that $\Delta^i w_n > 0$ for $i = 0, 1, 2, \dots p$, and $\Delta^{p+1} w_n < 0$, for $n \geq n_0$, $p \geq 1$, then there exists a scalar $L > 0$ and a positive integer n_2 such that $n \geq n_2$ implies $w_n > L n^{p-1}$.*

Proof. From the given conditions, it is clear that, $\Delta^{p-1} w_n$ is increasing. Hence, we can find $n_1 \geq n_0$ and a scalar $A > 0$ such that $n \geq n_1$ implies

$$\Delta^{p-1} w_n \geq A. \tag{2.11}$$

Choose $k \geq n_1 + 1$. Then summing (2.11) from $n = n_1$ to $k - 1$, we obtain

$$\Delta^{p-2} w_k > A(k - n_1),$$

for $k \geq n_1 + 1$. First taking $n \geq n_1 + 2$ and then summing up the above inequality from $k = n_1 + 1$ to $n - 1$ we obtain

$$\Delta^{p-3} w_n > \frac{A(n-n_1)^{(2)}}{2},$$

for $n \geq n_1 + 2$. Continuing the above iteration $p - 3$ times more and using (2.2), we easily find

$$w_n > \frac{A(n-n_1)^{(p-1)}}{(p-1)!},$$

for $n \geq n_1 + p - 1$. Since $n^{(r)} \geq (n - r + 1)^r$, it follows from the above inequality that

$$w_n > \frac{A(n-n_1-p+2)^{p-1}}{(p-1)!},$$

for $n \geq n_1+p-1$. Clearly, $\lim_{n\to\infty}\left(1-\frac{(n_1+p-2)}{n}\right)^{p-1} = 1$. Hence for any $1 > \varepsilon > 0$, we can find $n_2 \geq n_1 + p - 1$ such that $n \geq n_2$ implies

$$1 - \varepsilon < \left(1 - \frac{(n_1 + p - 2)}{n}\right)^{p-1} < 1 + \varepsilon.$$

Choose $0 < L < \frac{A}{(p-1)!} = B$ such that $\frac{L}{B} < 1 - \varepsilon$. Hence, for $n \geq n_2$ we obtain $w_n > Ln^{p-1}$. □

Remark 2.1. Suppose that $\{w_n\}$ is a real sequence and L is a positive scalar and defined as in Lemma 2.4. If $\{z_n\}$ is a sequence, which satisfies the condition that $z_n \geq w_n - \varepsilon$ for $n \geq n_3 \geq n_2$, where $\varepsilon > 0$ is any preassigned arbitrary positive number, then there exists a positive scalar $C < L$ and a positive integer $n_4 \geq \max\left((\frac{\varepsilon}{L-C})^{\frac{1}{p-1}}, n_3\right)$ such that $n \geq n_4$ implies $z_n \geq Cn^{p-1}$.

Lemma 2.5. ([9]) *If $\sum u_n$ and $\sum v_n$ are two positive term series such that*

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = l,$$

where l is a non-zero finite number, then the two series converge or diverge together. If $l = 0$ then $\sum v_n$ is convergent implies the convergence of $\sum u_n$. If $l = \infty$ then $\sum v_n$ is divergent implies the divergence of $\sum u_n$.

Remark 2.2. Since $(n-r+1)^r < n^{(r)} < n^r$ for $r \leq n$, the following conclusions follow directly from Lemma 2.5.

(i) (H4) holds if and only if $\sum\limits_{n=n_0}^{\infty} (n - n_0 + m - 2)^{(m-2)} q_n = \infty$.

(ii) (H5) holds if and only if $\sum\limits_{n=n_0}^{\infty} (n - n_0 + m - 1)^{(m-1)} u_n < \infty$.

(iii) (H6) holds if and only if $\sum\limits_{n=n_0}^{\infty} (n - n_0 + m - 1)^{(m-1)} q_n = \infty$.

Remark 2.3. If the condition $\left|\sum\limits_{n=n_0}^{\infty} n^{m-1} f_n\right| < \infty$ is satisfied, then (H7) holds. Indeed, using Lemma 2.5 and Remark 2.2, we define

$$F_n = \frac{(-1)^m}{(m-1)!}\sum_{j=n}^{\infty}(j - n + m - 1)^{(m-1)} f_j.$$

Then $\Delta^m F_n = f_n$ and $\lim_{n\to\infty} F_n = 0$. We may observe that, (H7) implies (H8). Further, (H7) implies and is implied by the following condition

there exists a bounded sequence $\{F_n\}$ such that
$\Delta^m F_n = f_n$, and $\lim_{n\to\infty} F_n = \eta$.

In fact, the implies part is obvious. Conversely, if $\lim_{n\to\infty} F_n = \eta \neq 0$, then we may put $L_n = F_n - \eta$. Then $\lim_{n\to\infty} L_n = 0$ and $\Delta^m L_n = f_n$. Hence (H7) holds.

Before we state and prove our last lemma in this section we have to prepare some ground work for the purpose. In order to move in that direction, let $y = y_n$ be an unbounded non-oscillatory solution of (1.1) for $n \geq N_1$. Define for $n \geq n_0$,

$$z_n = y_n - p_n y_{\tau(n)}\,. \tag{2.12}$$

Further, assuming that (H2) and (H5) hold, we define for $n \geq n_0$

$$T_n = \frac{(-1)^{m-1}}{(m-1)!} \sum_{i=n}^{\infty} (i-n+m-1)^{(m-1)} u_i H(y_{\alpha(i)}). \tag{2.13}$$

Then,

$$\Delta^m T_n = -u_n H(y_{\alpha(n)}). \tag{2.14}$$

Set,

$$w_n = z_n + T_n - F_n. \tag{2.15}$$

Now we state our lemma.

Lemma 2.6. *Suppose that p_n satisfies the condition* (A7). *Assume that there exists a positive integer k such that $\tau(n) = n-k$. Let* (H0)–(H3), (H5)–(H7) *hold. Then for every non-oscillatory solution y_n of* (1.1) *with z_n, T_n, and w_n defined as in* (2.12), (2.13) *and* (2.15) *respectively, either $\lim_{n\to\infty} w_n = 0$ or $\lim_{n\to\infty} w_n = -\infty$.*

Proof. Let y_n be an eventually positive solution of (1.1) for $n \geq n_0 \geq N_1$. Then for $n \geq n_0$, using (2.12)–(2.15) in (1.1), we obtain

$$\Delta^m w_n = -q_n G(y_{\sigma(n)}) \leq 0. \tag{2.16}$$

Hence $w_n, \Delta w_n, \Delta^2 w_n, \dots, \Delta^{m-1} w_n$ are monotonic for $n \geq n_1$ and of one sign. From (2.13) it follows, due to (H2), (H5), Lemma 2.5 and Remark 2.2, that

$$T_n \to 0 \qquad \text{as} \quad n \to \infty. \tag{2.17}$$

Consequently,

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} z_n = \lambda, \tag{2.18}$$

where $-\infty \leq \lambda \leq \infty$. By the method of contradiction, we show that $\lambda \neq \infty$. Suppose that $\lambda = \infty$. Then $w_n > 0$ and $\Delta w_n > 0$ for $n \geq n_1$. Due to (2.16) and Lemma 2.2, it follows that there exists $n_2 > n_1$ and an integer p, $0 \leq p \leq m-1$, $m-p$ is odd, such that $n \geq n_2$ implies

$$\begin{aligned} \Delta^i w_n &> 0 \qquad \text{for} \quad i = 0,1,2,\dots,p, \\ (-1)^{m+i-1} \Delta^i w_n &> 0 \qquad \text{for} \quad i = p+1, p+2, \dots, m-1. \end{aligned} \tag{2.19}$$

Hence $\lim_{n\to\infty} \Delta^p w_n = l$, exists and $\lim_{n\to\infty} \Delta^i w_n = 0$ for $i = p+1, p+2, \ldots m-1$. If $p = 0$, then $0 \le \lambda < \infty$, a contradiction. Hence $1 \le p \le m-1$. Applying Lemma 2.3 to (2.16), we obtain for $n \ge n_2$

$$\Delta^p w_n = l + \frac{(-1)^{m-p-1}}{(m-p-1)!} \sum_{i=n}^{\infty} (i-n+m-p-1)^{(m-p-1)} q_i G(y_{\sigma(i)}). \tag{2.20}$$

This implies

$$\sum_{i=n}^{\infty} (i-n+m-p-1)^{(m-p-1)} q_i G(y_{\sigma(i)}) < \infty, \qquad \text{for} \quad n \ge n_2. \tag{2.21}$$

In view of Lemma 2.5 and Remark 2.2, we have

$$\sum_{i=n_3}^{\infty} i^{m-p-1} q_i G(y_{\sigma(i)}) < \infty. \tag{2.22}$$

From this, it follows, due to (H6), that $\liminf_{n\to\infty} (G(y_{\sigma(n)})/n^p) = 0$. Hence $\liminf_{n\to\infty} (y_{\sigma(n)}/n^p) = 0$, by (H0) and (H3). As $\lim_{n\to\infty} \sigma(n) = \infty$ and by (H1), $\sigma(n) > \gamma n$ for large n, we obtain $\liminf_{n\to\infty} (y_n/n^p) = 0$. Due to Lemma 2.4, we can find $M_0 > 0$ such that $w_n > M_0 n^{p-1}$ for $n \ge n_3 \ge n_2$. For any $0 < \varepsilon$, from (2.15) it follows due to (H7) and (2.17) that $z_n \ge w_n - \varepsilon$ for large n. From this, it follows, again by Remark 2.1 that there exists M_1, with $0 < M_1 < M_0$, and $y_n - p_n y_{\tau(n)} > M_1 n^{p-1}$ for $n \ge n_4 > n_3$. That is

$$y_n > y_{\tau(n)} + M_1 n^{p-1}, \qquad n \ge n_4, \tag{2.23}$$

due to (A7). Let,

$$N_0 > \max\left\{ \frac{(p-2)k}{3}, n_4 \right\}, \qquad M = \min\{y_n :\ N_0 \le n \le N_0 + k\}$$

and

$$0 < \beta < \min\left\{ \frac{M}{(N_0+k)^p}, \frac{M_1}{2pk} \right\}.$$

Define, for $n \ge N_0$,

$$A(n) = \begin{cases} (M_1 - p\beta k) n^{p-1} + \beta \sum\limits_{i=2}^{p} (-1)^i \binom{p}{i} k^i n^{p-i}, & p \ge 2 \\ M_1 - \beta k, & p = 1. \end{cases}$$

If p is odd, then we may write

$$\begin{aligned}\sum_{i=2}^{p}(-1)^i\binom{p}{i}k^in^{p-i} =& \left[\binom{p}{2}k^2n^{p-2}-\binom{p}{3}k^3n^{p-3}\right]\\ &+\left[\binom{p}{4}k^4n^{p-4}-\binom{p}{5}k^5n^{p-5}\right]\\ &+\dots+\left[\binom{p}{p-1}k^{p-1}n-\binom{p}{p}k^p\right],\end{aligned}$$

to obtain

$$\sum_{i=2}^{p}(-1)^i\binom{p}{i}k^in^{p-i}>0,$$

because

$$\binom{p}{i}k^in^{p-i}>\binom{p}{i+1}k^{i+1}n^{p-i-1},$$

if and only if

$$n>k\binom{p}{i+1}\Big/\binom{p}{i}=\frac{(p-i)k}{i+1}$$

for $i=2,4,\dots,p-1$. Further, $n\ge N_0$ implies that

$$n\ge N_0>\frac{(p-2)k}{3}>\frac{(p-4)k}{5}\dots>\frac{k}{p}.$$

If p is even then we put the terms in pair as above with the last single positive term $(-1)^p\binom{p}{p}k^p$. Thus $A(n)>0$ for $n\ge N_0$. Since $y_n\ge M$ for $N_0\le n\le N_0+k$ and $\beta(N_0+k)^p<M$, then $y_n>\beta n^p$ for $N_0\le n\le N_0+k$. Since $\tau(n)=n-k$, then $N_0+k\le n\le N_0+2k$ implies $N_0\le\tau(n)\le N_0+k$. Using (2.23), we obtain, for $N_0+k\le n\le N_0+2k$,

$$\begin{aligned}y_n>y_{\tau(n)}+M_1n^{p-1}&>\beta(\tau(n))^p+M_1n^{p-1}\\ &\ge\beta(n-k)^p+M_1n^{p-1}>\beta n^p,\end{aligned}$$

because, for $p\ge 2$,

$$\begin{aligned}\beta n^p<A(n)+\beta n^p&=(M_1-p\beta k)n^{p-1}+\beta\big[(n-k)^p-n^p+pkn^{p-1}\big]+\beta n^p\\ &=M_1n^{p-1}+\beta(n-k)^p,\end{aligned}$$

and for $p=1$, $\beta n<A(n)+\beta n=M_1+\beta(n-k)$. Proceeding as above we have $y_n>\beta n^p$ for $n\ge N_0$. Hence $\liminf\limits_{n\to\infty}[y_n/n^p]\ge\beta>0$, a contradiction. Thus, $\lambda\ne\infty$. If $\lambda\ne-\infty$ then λ is finite. This implies $(-1)^{m+i}\Delta^iw_n<0$

for $i = 1, 2, \ldots, m-1$, and $\lim_{n\to\infty} \Delta^i w_n = 0$, $i = 1, 2, \ldots, m-1$. Then applying Lemma 2.3 to (2.16), we obtain

$$w_n = \lambda + \frac{(-1)^{m-1}}{(m-1)!} \sum_{i=n}^{\infty} (i-n+m-1)^{(m-1)} q_i G(y_{\sigma(i)}), \tag{2.24}$$

for $n \geq n_1$, where n_1 is some large positive integer. Thus,

$$\frac{1}{(m-1)!} \sum_{i=n}^{\infty} (i-n+m-1)^{(m-1)} q_i G(y_{\sigma(i)}) < \infty, \qquad n \geq n_1. \tag{2.25}$$

Using Lemma 2.5 and Remark 2.2 in the above inequality, we obtain

$$\sum_{i=n}^{\infty} i^{m-1} q_i G(y_{\sigma(i)}) < \infty, \qquad n \geq n_1. \tag{2.26}$$

From this, it follows, due to (H6), that $\liminf_{n\to\infty} G(y_n) = 0$, and hence $\liminf_{n\to\infty} y_n = 0$, by (H0). Then application of Lemma 2.1 yields $\lim_{n\to\infty} z_n = 0$. Thus $\lim_{n\to\infty} w_n = 0$, by (2.18). Hence the lemma is proved. The proof for the case when $y_n < 0$ eventually, is similar. □

3. Sufficient conditions

In this section, we present the results to find sufficient conditions so that every solution of (1.1) oscillates or tends to zero as $n \to \infty$.

Theorem 3.1. *Let $m \geq 2$. Suppose that, p_n satisfies one of the conditions* (A1) *or* (A2). *If* (H0)–(H5) *and* (H8) *hold, then every unbounded solution of* (1.1) *oscillates.*

Proof. Let $y = y_n$ be an unbounded non-oscillatory solution of (1.1) for $n \geq N_1$. Then $y_n > 0$ or $y_n < 0$. Suppose $y_n > 0$ eventually. There exits a positive integer n_0, and $y_n > 0$, $y_{\tau(n)} > 0$, $y_{\sigma(n)} > 0$ and $y_{\alpha(n)} > 0$ for $n \geq n_0 \geq N_1$. Using the assumptions (H2) and (H5), for $n \geq n_0$, we set z_n, T_n, and w_n as in (2.12), (2.13), and (2.15) to obtain (2.14) and (2.16). Hence w_n, $\Delta w_n, \ldots \Delta^{m-1} w_n$ are monotonic and of one sign for $n \geq n_1 \geq n_0$. Then $\lim_{n\to\infty} w_n = \lambda$, where $-\infty \leq \lambda \leq +\infty$. From (2.13) it follows, due to (H2), (H5), Lemma 2.5 and Remark 2.2, that (2.17) holds. Since y_n is unbounded, there exists a subsequence $\{y_{n_k}\}$ such that

$$y_{n_k} \to \infty \qquad \text{as} \quad k \to \infty,$$

and

$$y(n_k) = \max\{y_n : \; n_1 \leq n \leq n_k\}. \tag{3.1}$$

We may choose n_k large enough so that $\tau(n_k) \geq n_1$, $\sigma(n_k) \geq n_1$ and $\alpha(n_k) \geq n_1$. Then from (2.17) and (H8) it follows that, for $0 < \varepsilon$, we can find a positive integer n_2 such that $k \geq n_2 \geq n_1$ implies $|T_{n_k}| < \varepsilon$ and $|F_{n_k}| < \gamma$, for some constant $\gamma > 0$. Hence for $k \geq n_2$, if (A1) holds, then we have

$$w_{n_k} \geq y_{n_k}(1-p) - \varepsilon - \gamma.$$

Similarly, if (A2) holds, then for $k \geq n_2$, we have

$$w_{n_k} \geq y_{n_k} - \varepsilon - \gamma.$$

Taking $k \to \infty$, we find $\lim\limits_{n\to\infty} w_n = \infty$, because of the monotonic nature of w_n. Hence $w_n > 0$, $\Delta w_n > 0$ for $n \geq n_2 \geq n_1$. Since $\Delta^m w_n \not\equiv 0$ and is in negative, it follows from Lemma 2.2 that there exists a positive integer p such that $m-p$ is odd and for $n \geq n_3 \geq n_2$, we have $\Delta^j w_n > 0$ for $j = 0, 1, \dots, p$ and $\Delta^j w_n \Delta^{j+1} w_n < 0$ for $j = p, p+1, \dots, m-2$. Then $\lim\limits_{n\to\infty} \Delta^p w_n = l$ (finite) exists. Hence $p \geq 1$. Applying Lemma 2.3 to (2.16), we obtain (2.20). Consequently (2.21) and then (2.22) follows due to Lemma 2.5 and Remark 2.2. Because of (H4), the inequality (2.22) yields

$$\liminf_{n\to\infty} \frac{G(y_{\sigma(n)})}{n^{p-1}} = 0,$$

for $n \geq n_3$. Then we claim $\liminf\limits_{n\to\infty} \frac{y_{\sigma(n)}}{n^{p-1}} = 0$. Otherwise, there exists $n_4 \geq n_3$ and $\gamma > 0$ such that $n \geq n_4$ implies $y_{\sigma(n)} > \gamma n^{p-1}$. By (H0) and (H3), we obtain $\frac{G(y_{\sigma(n)})}{n^{p-1}} > \gamma\delta > 0$, for $n \geq n_4$, a contradiction. Hence our claim holds. Next, we assert

$$\liminf_{n\to\infty} \frac{y_n}{n^{p-1}} = 0.$$

Otherwise, there exists $n_4 \geq n_3$ and $\gamma > 0$ such that $n \geq n_4$ implies $\frac{y_n}{n^{p-1}} > \gamma > 0$. As $\lim\limits_{n\to\infty} \sigma(n) = \infty$, we can find $n_5 \geq n_4$ such that $\sigma(n) \geq n_4$ for $n \geq n_5$. Then $\frac{y_{\sigma(n)}}{(\sigma_n)^{p-1}} > \gamma$ for $n \geq n_5$. Due to (H1), we find n_6 and a positive scalar μ such that $n \geq n_6 \geq n_5$ implies $\sigma(n) > \mu n$. Consequently, for $n \geq n_6$, we have $y_{\sigma(n)} > \gamma(\mu n)^{p-1}$. Hence $\frac{y_{\sigma(n)}}{n^{p-1}} > \gamma\mu^{p-1} > 0$, for $n \geq n_6$, a contradiction. Thus our assertion that $\liminf\limits_{n\to\infty} \frac{y_n}{n^{p-1}} = 0$, holds. Since $p \geq 1$, due to Lemma 2.4, we can choose $B > 0$, such that

$$w_n > Bn^{p-1} \qquad \text{for} \quad n \geq n_4 \geq n_3 + p - 1.$$

Thus,

$$\liminf_{n\to\infty} \frac{y_n}{w_n} = 0. \tag{3.2}$$

Set, for $n \geq n_4$,

$$p_n^* = p_n \frac{w_{\tau(n)}}{w_n}.$$

It is clear from (H8), (2.17) and $\lim_{n\to\infty} w_n = \infty$, that

$$\lim_{n\to\infty} \frac{(F_n - T_n)}{w_n} = 0.$$

Then we have

$$\begin{aligned} 1 &= \lim_{n\to\infty} \left[\frac{w_n}{w_n}\right] \\ &= \lim_{n\to\infty} \left[\frac{y_n - p_n y_{\tau(n)} - (F_n - T_n)}{w_n}\right] \\ &= \lim_{n\to\infty} \left[\frac{y_n}{w_n} - \frac{p_n^* y_{\tau(n)}}{w_{\tau(n)}} - \frac{(F_n - T_n)}{w_n}\right] \\ &= \lim_{n\to\infty} \left[\frac{y_n}{w_n} - \frac{p_n^* y_n}{w_n}\right]. \end{aligned} \tag{3.3}$$

Since $\{w_n\}$ is a increasing sequence, then $\frac{w_{\tau(n)}}{w_n} < 1$. If p_n is defined as in (A1) then $0 \le p_n^* < p_n \le b < 1$. However, if p_n is defined as in (A2) then $0 \ge p_n^* \ge p_n \ge -b > -1$. Hence it is clear that if p_n satisfies (A1) or (A2) then p_n^* also satisfies (A1) or (A2) accordingly. Hence use of Lemma 2.1 yields, due to (3.2), that

$$\lim_{n\to\infty} \left[\frac{y_n}{w_n} - \frac{p_n^* y_{\tau(n)}}{w_{\tau(n)}}\right] = 0,$$

a contradiction to (3.3). Hence the unbounded solution $\{y_n\}$ cannot be eventually positive. Next, if y_n is an eventually negative solution of (1.1) for large n then we set $x_n = -y_n$ to obtain $x_n > 0$ and then (1.1) reduces to

$$\Delta^m\big(x_n - p_n x_{\tau(n)}\big) + q_n \tilde{G}(x_{\sigma(n)}) - u_n \tilde{H}(x_{\alpha(n)}) = \tilde{f}_n, \tag{3.4}$$

where

$$\tilde{f}_n = -f_n, \quad \tilde{G}(v) = -G(-v) \quad \text{and} \quad \tilde{H}(v) = -H(-v). \tag{3.5}$$

Further,

$$\tilde{F}_n = -F_n \qquad \text{implies} \quad \Delta^m(\tilde{F}_n) = \tilde{f}_n. \tag{3.6}$$

In view of the above facts, it can be easily verified that the following conditions hold.

($\bar{\text{H}}$0) $\tilde{G}$ is non-decreasing and $x\tilde{G}(x) > 0$ for $x \neq 0$.

($\bar{\text{H}}$2) $\tilde{H}$ is bounded.

($\bar{\text{H}}$3) $\liminf_{|v|\to\infty} \frac{\tilde{G}(v)}{v} \ge \delta > 0$.

($\bar{\text{H}}$8) There exists a bounded sequence $\{\tilde{F}_n\}$ such that $\Delta^m(\tilde{F}_n) = \tilde{f}_n$.

Proceeding as in the proof for the case $y_n > 0$, we obtain a contradiction. Hence y_n is oscillatory and the proof is complete. □

The following example illustrates the above theorem.

Example 3.1. The neutral equation

$$\Delta^3\Big(y_n - \frac{1}{2}y_{n-1}\Big) + 135y_{n-2} = 0 \tag{3.7}$$

satisfies all the conditions of Theorem 3.1. Hence, all the unbounded solutions are oscillatory. As such, $y_n = (-2)^n$, is an unbounded solution, which oscillates. But the results of [19, 21] cannot be applied to this equation, because $G(u) = u$ is linear.

Theorem 3.2. *Let $m \geq 2$. Suppose that, p_n satisfies one of the conditions* (A1)–(A4)*. If* (H0) *and* (H5)–(H7) *hold, then every bounded solution of* (1.1) *oscillates or tends to zero as $n \to \infty$.*

Proof. Let $y = y_n$ be a bounded solution of (1.1) for $n \geq N_1$. If it oscillates then there is nothing to prove. If it does not oscillate then $y_n > 0$ or $y_n < 0$ eventually. Suppose $y_n > 0$ for large n. There exists a positive integer n_0 and $y_n > 0$, $y_{\tau(n)} > 0$, $y_{\sigma(n)} > 0$ and $y_{\alpha(n)} > 0$ for $n \geq n_0 \geq N_1$. Set z_n, T_n and w_n as in (2.12), (2.13) and (2.15) respectively, to obtain (2.14) and (2.16). Note that T_n is well defined due to the boundedness of y_n and satisfies (2.17). Then $w_n, \Delta w_n, \dots, \Delta^{m-1} w_n$ are monotonic and of one sign for $n \geq n_1 \geq n_0$. Since y_n is bounded, z_n and w_n are bounded. Using (2.17), (H7) and monotonic nature of w_n, we obtain $\lim\limits_{n\to\infty} z_n = \lim\limits_{n\to\infty} w_n = \lambda$, which exists finitely. Then applying Lemma 2.3 to (2.16), we obtain (2.24). Consequently (2.25) and (2.26) hold. The inequality (2.26), due to (H6) yields $\liminf\limits_{n\to\infty} G(y_{\sigma(n)}) = 0$. Since $\lim\limits_{n\to\infty} \sigma(n) = \infty$, it can be easily shown that $\liminf\limits_{n\to\infty} G(y_n) = 0$. This implies due to (H0) that $\liminf\limits_{n\to\infty} y_n = 0$. From Lemma 2.1, it follows that $\lim\limits_{n\to\infty} z_n = 0$. If p_n is in (A1) then

$$\begin{aligned}
0 = \lim_{n\to\infty} z_n &= \limsup_{n\to\infty}(y_n - p_n y_{\tau(n)}) \\
&\geq \limsup_{n\to\infty} y_n + \liminf_{n\to\infty}(-p_n y_{\tau(n)}) \\
&\geq (1-b)\limsup_{n\to\infty} y_n.
\end{aligned}$$

This implies $\limsup\limits_{n\to\infty} y_n = 0$. Hence $y_n \to 0$ as $n \to \infty$. If p_n is in (A2) or (A3) then, since $y_n \leq z_n$, it follows that $y_n \to 0$ as $n \to \infty$. If p_n satisfies (A4), then $z_n \leq y_n - b_2 y_{\tau(n)}$. Hence, it follows that

$$\begin{aligned}
0 = \liminf_{n\to\infty} z_n &\leq \liminf_{n\to\infty}[y_n - b_2 y_{\tau(n)}] \\
&\leq \limsup_{n\to\infty} y_n + \liminf_{n\to\infty}[-b_2 y_{\tau(n)}] \\
&= (1-b_2)\limsup_{n\to\infty} y_n.
\end{aligned}$$

Then $\limsup_{n\to\infty} y_n = 0$. Thus, $\lim_{n\to\infty} y_n = 0$. If y_n is eventually negative for large n, then we may proceed with $x_n = -y_n$ as in the proof of the Theorem 3.1 and note that, x_n is a positive solution of (3.4) with (3.5) and (3.6). Moreover, the condition ($\bar{\text{H}}$0) along with the following one holds.

($\bar{\text{H}}$7) There exists a bounded sequence $\tilde{F}_n$ such that
$\Delta^m \tilde{F}_n = \tilde{f}_n$ and $\lim_{n\to\infty} \tilde{F}_n = 0$.

Then proceeding as above, we prove $\lim_{n\to\infty} y_n = 0$. Thus the theorem is proved. □

Remark 3.1. The above theorem holds when G is linear, super linear, or sublinear.

Next, we give few examples to establish the significance of our results.

Example 3.2. Consider the neutral equation

$$\Delta^m\left(y_n - \frac{1}{2}y_{n-1}\right) + n^{-m}y_{n-2}^{\alpha} = n^{-m}2^{\alpha(2-n)}, \tag{3.8}$$

where $m \geq 2$, α is a positive rational, being the quotient of two odd integers. Here, $p_n = \frac{1}{2}$, satisfies (A1) and $q_n = n^{-m}$, $f_n = n^{-m}2^{\alpha(2-n)}$. It is clear that

$$\sum_{n=n_0}^{\infty} n^{m-1} f_n < \infty.$$

Hence by Remark 2.3, it follows that

$$F_n = \frac{(-1)^m}{(m-1)!}\sum_{j=n}^{\infty}(j-n+m-1)^{(m-1)}j^{-m}2^{\alpha(2-j)}.$$

Obviously, $|F_n| < \infty$. Hence the equation (3.8) satisfies all the conditions of Theorem 3.2. Hence every bounded non-oscillatory solution tends to zero as $n \to \infty$. In particular $y_n = 2^{-n}$ is a solution of (3.8), which tends to zero as $n \to \infty$. If $\alpha \geq 1$, then (3.8) does not come under the purview of the results in [19, 21], hence those results fail to deliver any conclusion. Further, even if $\alpha < 1$ then $m \geq 2$ implies $m - \alpha m + \alpha > 1$. This further implies (1.8) does not hold. Hence Theorem 1.2 cannot be applied to (3.8). Thus Theorem 3.2 along with Theorem 3.1 of this paper improves and generalizes Theorem 1.2.

Example 3.3. Consider the neutral equation

$$\Delta^m\left(y_n + \frac{1}{2}y_{n-1}\right) + n^{-m}y_{n-2}^{\alpha} = (-1)^m 2^{-n-m+1} + n^{-m}2^{\alpha(2-n)}, \tag{3.9}$$

where $m \geq 2$, α is a positive rational, which is the quotient of two odd integers. Here, $p_n = -\frac{1}{2}$, satisfies (A2) and $q_n = n^{-m}$, $f_n = (-1)^m 2^{-n-m+1} +$

$n^{-m}2^{\alpha(2-n)}$. Easily, we can verify that, $\sum\limits_{n=n_0}^{\infty} n^{m-1} f_n < \infty$ and the equation (3.9) satisfies all the conditions of Theorem 3.2 for (A2). Hence $y_n = 2^{-n}$ is a solution of (3.9), which tends to zero as $n \to \infty$. If $\alpha \geq 1$, then results of [19, 21] cannot be applied to (3.9). Further, if $\alpha < 1$ then neither Theorem 1.1(b) nor [19, Corollary 3] can be applied, because (1.8) does not hold. Thus Theorem 3.2 along with Theorem 3.1 of this paper improves and generalizes Theorem 1.1(b) and [19, Corollary 3].

Example 3.4. Consider the equation

$$\Delta^4(y_n) + \frac{1}{n^4} y_{n-1}^{\alpha} = 2^{-n-4} + \frac{2^{(-n+1)\alpha}}{n^4}, \tag{3.10}$$

where α is a positive rational, being the quotient of two odd integers. Here, $p_n = 0$, satisfies (A1) and (A2) and $q_n = \frac{1}{n^4}$, $f_n = 2^{-n-4} + \frac{2^{(-n+1)\alpha}}{n^4}$. It is easy to verify that $\sum\limits_{n=n_0}^{\infty} n^3 f_n < \infty$ and equation (3.10) satisfies all the conditions of Theorem 3.2 for (A2). Hence $y_n = 2^{-n}$ is a solution of (3.10), which tends to zero as $n \to \infty$. If $\alpha \geq 1$, then the results of the papers [19, 21] cannot be applied to (3.10). Further, even if $\alpha < 1$ then Theorem 1.1(a) cannot be applied, because (1.9) doesnot hold. Thus Theorem 3.2 along with Theorem 3.1 of this paper improves and generalizes Theorem 1.1(a) and [19, Corollary 3].

Theorem 3.3. *Suppose that $m \geq 2$, and that* (A6) *holds. Assume that* $\liminf\limits_{n\to\infty} \frac{\tau(n)}{n} > 0$ *and* $\sigma(\tau(n)) = \tau(\sigma(n))$. *Let* (H0)–(H3), (H5) *and* (H7) *hold. Further assume that*

- (H9) $G(-u) = -G(u)$.
- (H10) *For $u > 0$, $v > 0$, there exists a scalar $\beta > 0$ such that* $G(u)G(v) \geq G(uv)$ *and* $G(u) + G(v) \geq \beta G(u + v)$.
- (H11) $\sum\limits_{n_0}^{\infty} n^{m-2} q_n^* = \infty$, *where* $q_n^* = \min[q_n, q_{\tau(n)}]$.

Then every solution of (1.1) *oscillates or tends to zero as $n \to \infty$.*

P r o o f. Let $y = \{y_n\}$ be an eventually positive solution of (1.1) for $n \geq n_0 \geq N_1$. Then set z_n, T_n, and w_n as in (2.12), (2.13) and (2.15) respectively to get (2.16) for $n > n_1 \geq n_0$. Hence w_n, Δw_n, $\Delta^2 w_n, \dots, \Delta^{m-1} w_n$ are monotonic and of one sign for $n \geq n_1$. As (2.17) holds, from (H7), it follows that (2.18) holds i.e.;

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} z_n = \lambda, \qquad \text{where} \quad -\infty \leq \lambda \leq \infty.$$

If $\lambda < 0$, then $z_n < 0$, for large n, a contradiction. If $\lambda = 0$, then $y_n \leq z_n$, implies $\lim\limits_{n\to\infty} y_n = 0$. If $\lambda > 0$, then $w_n > 0$ for $n \geq n_2$. Then from Lemma 2.2, it follows that, there exists an integer p, $0 \leq p \leq m-1$, such that $m-p$ is odd, and for $n \geq n_3 \geq n_2$, we have $\Delta^j w_n > 0$ for $j = 0, 1, \ldots, p$ and $(-1)^{m+j-1}\Delta^j w_n > 0$ for $j = p+1, p+2, \ldots, m-1$. Hence $\lim_{n\to\infty} \Delta^p w_n = l$ exists and $\lim_{n\to\infty} \Delta^i w_n = 0$ for $i = p+1, p+2, \ldots, m-1$. Note that, $0 < \lambda < \infty$ implies $p = 0$, but $\lambda = \infty$ implies $p > 0$ such that $m-p$ is odd. Applying Lemma 2.3 to (2.16), we obtain(2.20) and consequently (2.21) follows. In view of Lemma 2.5 and Remark 2.2, we obtain for $N_2 \geq n_3$,

$$\sum_{i=N_2}^{\infty} i^{m-p-1} q_i G(y_{\sigma(i)}) < \infty. \tag{3.11}$$

Note that, since $\tau(n)$ is monotonic increasing, its inverse function $\tau^{-1}(n)$ exists, such that $\tau(\tau^{-1}(n)) = n$. Since $q_i > q^*_{\tau^{-1}(i)}$, it follows that

$$\sum_{i=N_2}^{\infty} i^{m-p-1} q^*_{\tau^{-1}(i)} G(y_{\sigma(i)}) < \infty.$$

Then replacing i by $\tau(i)$ in the above inequality and multiplying by the scalar $G(b_2)$, we obtain

$$G(b_2) \sum_{i=N_3}^{\infty} (\tau(i))^{m-p-1} q^*_i G(y_{\sigma(\tau(i))}) < \infty,$$

where $N_3 \geq \tau^{-1}(N_2)$. Since $\liminf_{n\to\infty} \tau(n)/n > 0$ implies $\tau(n)/n > a > 0$ for $n \geq N_4 \geq N_3$, and $p_n \geq -b_2$, then due to (H0), we obtain

$$\sum_{i=N_4}^{\infty} i^{m-p-1} q^*_i G(-p_{\sigma(i)}) G(y_{\sigma(\tau(i))}) < \infty.$$

This with the use of (H10) yields

$$\sum_{i=N_4}^{\infty} i^{m-p-1} q^*_i G(-p_{\sigma(i)} y_{\sigma(\tau(i))}) < \infty.$$

Since $\sigma(\tau(i)) = \tau(\sigma(i))$, the above inequality takes the form

$$\sum_{i=N_4}^{\infty} i^{m-p-1} q^*_i G(-p_{\sigma(i)} y_{\tau(\sigma(i))}) < \infty. \tag{3.12}$$

From (3.11) and the fact that $q_n \geq q^*_n$, we obtain

$$\sum_{i=N_4}^{\infty} i^{m-p-1} q^*_i G(y_{\sigma(i)}) < \infty. \tag{3.13}$$

Further, using (H10), (3.12) and (3.13), one may get

$$\beta \sum_{i=N_4}^{\infty} i^{m-p-1} q_i^* G(z_{\sigma(i)}) < \infty. \tag{3.14}$$

If $p = 0$ then (H11) and (3.14) implies $\liminf_{n\to\infty} nG(z_{\sigma(n)}) = 0$. Applying the assumption $\lim_{n\to\infty} \sigma(n) = \infty$ and (H0), we obtain $\lim_{n\to\infty} z_n = 0$, a contradiction. If $p > 0$ then by Lemma 2.4, there exists $A > 0$ such that $w_n > An^{p-1}$ for $n \ge N_5 \ge N_4$. For any $\varepsilon > 0$, using (H7) and (2.17), we obtain $z_n \ge w_n - \varepsilon$, for $n \ge N_6 \ge N_5$. Thus, due to Remark 2.1, we can find $0 < B < A$ such that

$$z_n > Bn^{p-1} \qquad \text{for} \quad n \ge N_7 \ge N_6. \tag{3.15}$$

By (H1), we have $\sigma(n)/n > b > 0$ for $n \ge N_8 \ge N_7$. Then further use of (3.15) and (H3) yields

$$\begin{aligned}\sum_{i=N_8}^{\infty} i^{m-p-1} q_i^* G(z_{\sigma(i)}) &\ge B\delta \sum_{i=N_8}^{\infty} i^{m-p-1} q_i^* (\sigma(i))^{p-1} \\ &\ge \delta B b^{p-1} \sum_{i=N_8}^{\infty} i^{m-2} q_i^* = \infty,\end{aligned}$$

by (H11), a contradiction due to (3.14). Hence the proof for the case $y_n > 0$ is complete. If $y_n < 0$, eventually for large n, then we may proceed with $x_n = -y_n$ as in the proof of the Theorem 3.1 and note that, x_n is a positive solution of (3.4) with (3.5) and (3.6). Further, we note that, (H9) implies $G = \tilde{G}$. In view of this, it is easy to verify that the conditions ($\bar{\text{H}}$0) and ($\bar{\text{H}}$3) hold along with the the following two conditions.

($\bar{\text{H}}$9) $\tilde{G}(-u) = -\tilde{G}(u)$.

($\bar{\text{H}}$10) For $u > 0, v > 0$, there exists a scalar $\beta > 0$ such that $\tilde{G}(u)\tilde{G}(v) \ge \tilde{G}(uv)$ and $\tilde{G}(u) + \tilde{G}(v) \ge \beta\tilde{G}(u+v)$.

Also, it is not difficult to see that ($\bar{\text{H}}$2) and ($\bar{\text{H}}$7) hold. Then proceeding as above, in the proof for the case $y_n > 0$, we prove that $\lim_{n\to\infty} y_n = 0$ and complete the proof of the theorem. □

Remark 3.2. The prototype of the function G satisfying (H0), (H3), (H9) and (H10) is $G(u) = (\beta + |u|^{\mu})|u|^{\lambda} \operatorname{sgn} u$, where $\lambda > 0$, $\mu > 0$, $\lambda + \mu \ge 1$, $\beta \ge 1$. For verification we may take help of the well known inequality (see [7, p. 292])

$$u^p + v^p \ge \begin{cases} (u+v)^p, & 0 \le p < 1, \\ 2^{1-p}(u+v)^p, & p \ge 1. \end{cases}$$

For our next result we need the following hypothesis.

(H12) Suppose that for every sub-sequence $\{q_{n_j}\}$ of $\{q_n\}$, we have

$$\sum_{j=0}^{\infty}(n_j)^{m-1}q_{n_j} = \infty.$$

Theorem 3.4. *Suppose that p_n satisfies the condition* (A7). *Further assume that there exists a positive integer k such that $\tau(n) = n - k$. Let* (H0)–(H3), (H5), (H7) *and* (H12) *hold. Then*

(i) *every bounded solution of* (1.1) *oscillates or tends to zero as $n \to \infty$.*

(ii) *every unbounded solution of* (1.1) *oscillates or tends to $\pm\infty$ as $n \to \infty$.*

Proof. Clearly (H12) implies (H6). Now, let us prove (i) and assume $y = \{y_n\}$ be any non-oscillatory positive solution of (1.1) which is bounded. We have to prove $\lim\limits_{n\to\infty} y_n = 0$. Set z_n, T_n and w_n as in (2.12), (2.13), (2.15) respectively to get (2.16). Since (H12) implies (H6), we apply Lemma 2.6 to get $\lim\limits_{n\to\infty} w_n = 0$ or $\lim\limits_{n\to\infty} w_n = -\infty$. Since y_n is bounded, w_n is bounded, and hence $\lim\limits_{n\to\infty} w_n = -\infty$ is not possible. Thus $\lim\limits_{n\to\infty} w_n = 0$. Then we apply Lemma 2.3 to (2.16) to get (2.24). Consequently (2.25) and (2.26) follows. Then we apply (H6) to get $\liminf\limits_{n\to\infty} G(y_{\sigma(n)}) = 0$. This implies $\liminf\limits_{n\to\infty} y_{\sigma(n)} = 0$, because of (H0). Then applying the condition, $\lim\limits_{n\to\infty} \sigma(n) = \infty$, we obtain $\liminf\limits_{n\to\infty} y_n = 0$. Suppose $\limsup\limits_{n\to\infty} y_n = \omega > 0$. Then we can find a subsequence such that $y_{\tau(n_j)} > \eta > 0$, for $j \geq n_1$. Hence

$$\sum_{j=n_1}^{\infty}(n_j)^{n-1}q_{n_j}G(y_{\tau(n_j)}) > G(\eta)\sum_{j=n_1}^{\infty}(n_j)^{n-1}q_{n_j} = \infty,$$

a contradiction to (2.26). The proof for the case $y_n < 0$ for large n is similar.

Next, let us prove (ii) and assume $y = \{y_n\}$ be an unbounded positive solution of (1.1). Then we procced as in case (i) above, apply Lemma 2.6 to obtain $\lim\limits_{n\to\infty} w_n = 0$ or $\lim\limits_{n\to\infty} w_n = -\infty$. In this case we claim $\lim\limits_{n\to\infty} w_n = 0$ cannot hold. Otherwise, as in the proof for the case (i) we prove (2.26) holds. Since y_n is unbounded then we can find a subsequence such that $y_{\tau(n_j)} > \zeta > 0$, for $j > n_1$. Hence

$$\sum_{j=n_1}^{\infty}(n_j)^{n-1}q_{n_j}G(y_{\tau(n_j)}) > G(\zeta)\sum_{j=n_1}^{\infty}(n_j)^{n-1}q_{n_j} = \infty,$$

a contradiction to (2.26). Thus $\lim\limits_{n\to\infty} w_n = -\infty$. We observe that (2.18) holds because of (H7) and (2.17). Hence $\lim\limits_{n\to\infty} z_n = -\infty$. From (A7) and (2.12) it follows that, $y_{\tau(n)} \geq \frac{-z_n}{b_2}$. This implies $\lim\limits_{n\to\infty} y_n = +\infty$. The proof for the case, $y_n < 0$ for large n, is similar. □

Remark 3.3. For $m \geq 2$, the condition

$$\sum_{n=N_1}^{\infty} q_n^* = \infty, \tag{3.16}$$

implies (H11). Further the condition (3.16) implies (1.11). However, if q_n is monotonic then both (3.16) and (1.11) are equivalent. Indeed, if q_n is decreasing then $q_n^* = q_n$. Hence the equivalence of (3.16) and (1.11) is immediate. On the other hand if q_n is increasing then assume that (1.11) holds. Then $q_n^* = q_{\tau(n)}$. Hence $\sum\limits_{n=N_1}^{\infty} q_n^* = \sum\limits_{n=N_1}^{\infty} q_{\tau(n)} = \sum\limits_{k=\tau(N_1)}^{\infty} q_k = \infty$. Thus, (3.16) and (1.11) are equivalent, when q_n is monotonic. Now we quote a result from [12], which uses the condition (3.16).

Theorem 3.5. ([12, Theorem 2.10]) *Let p_n be in* (A5) *and $r \geq k$. If* (1.6) *and* (3.16) *hold then every solution of* (1.10) *oscillates.*

Remark 3.4. First of all, we note that the above theorem holds for sublinear equations. It does not hold for linear or super linear equations. However, Theorem 3.4 holds for linear and super linear equations, to complement Theorem 3.5. It is important to note that our Example 1.1 contradicts the above theorem, because, from the neutral equation (1.12), we find $q_n = 4^{(n+1)/3}$, which is monotonic. Clearly, (1.11) holds, which is equivalent to (3.16). Thus the neutral equation (1.12) satisfies all the conditions of the Theorem 3.5, but it has a solution $y_n = 2^n$, which does not oscillate. Thus the Theorem 3.5 is contradicted. Hence one may find a result similar to Theorem 3.5 for sublinear equations.

Remark 3.5. Using [5, Krasnoselskii's Fixed Point Theorem] and proceeding as in the proofs of the results of [15], one may easily establish that under the conditions (H5), (H8) and with any one of the conditions (A1)–(A4), if every solution of (1.1) oscillates or tends to zero as $n \to \infty$ then (H6) holds. This result would obviously hold, even if q_n changes sign. In that case we have to replace q_n by $|q_n|$ in (H6). Further, this result would improve [12, Theorems 4.1, 4.2], where there are restrictions on m and on the bounds of F_n. Further it would generalize and extend the necessary part of [11, Theorem 2.3], and [14, Theorem 2.4]. In all these results of [11, 12, 14] the authors require (H0) and the condition that G is Lispchitzian in intervals of the form $[a, b]$.

We conclude this paper with two open problems which may be helpful for further research.

Problem 3.1. Can we do the Theorem 3.3 under a condition weaker than (H11)?

Problem 3.2. Can we do Theorem 3.4 with the assumption (H6) in place of (H12)?
Or with any other condition weaker than (H12)?

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* *Corresponding author:*
Department of Mathematics
Veer Surendra Sai University of Technology
BURLA, 768018
Sambalpur, District Orissa
INDIA
E-mail: radhanathmath@yahoo.co.in

** *Department of Mathematics*
K. I. S. T
Bhubaneswar, Orissa
INDIA
E-mail: bls_barikmath@yahoo.co.in

*** *Department of M. C. A*
Gandhi Eng. College
Bhubaneswar, Orissa
INDIA
E-mail: rath_subhendu@gmail.com