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ASYMPTOTIC FORMULAS FOR NONOSCILLATORY SOLUTIONS OF CONDITIONALLY OSCILLATORY HALF-LINEAR EQUATIONS

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ABSTRACT. We establish asymptotic formulas for nonoscillatory solutions of a special conditionally oscillatory half-linear second order differential equation, which is seen as a perturbation of a general nonoscillatory half-linear differential equation

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \qquad \Phi(x) = |x|^{p-1}\operatorname{sgn} x, \quad p > 1,$$

where r, c are continuous functions and $r(t) > 0$.

1. Introduction

In this paper we investigate asymptotic properties of nonoscillatory solutions of a special conditionally oscillatory half-linear second order differential equation, which was constructed in [3] as a perturbation of a general half-linear differential equation

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \qquad \Phi(x) = |x|^{p-1}\operatorname{sgn} x, \quad p > 1, \tag{1}$$

where $t \in [t_0, \infty)$, r, c are continuous functions and $r(t) > 0$. In the case $p = 2$, equation (1) reduces to the linear Sturm-Liouville differential equation

$$(r(t)x')' + c(t)x = 0 \tag{2}$$

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and it is well known that the linear oscillation theory of (2) can be naturally extended also to half-linear equation (1). In particular, (1) is called *oscillatory* if its every nontrivial solution has infinitely many zeros tending to infinity and *nonoscillatory* otherwise.

In the whole paper we suppose that (1) is nonoscillatory. Let $d(t)$ be a positive continuous function, we say that the equation

$$(r(t)\Phi(x'))' + [c(t) + \mu d(t)]\Phi(x) = 0 \tag{3}$$

is *conditionally oscillatory* if there exists a constant $\mu_0 > 0$ such that (3) is oscillatory for $\mu > \mu_0$ and nonoscillatory for $\mu < \mu_0$.

Let $h(t)$ be a positive solution of nonoscillatory equation (1) such that $h'(t) \neq 0$ on some interval of the form $[T_0, \infty)$ and denote

$$R(t) := r(t)h^2(t)|h'(t)|^{p-2}, \quad G(t) := r(t)h(t)\Phi(h'(t)). \tag{4}$$

Under the asumptions

$$\int^{\infty} \frac{\mathrm{d}t}{R(t)} = \infty, \quad \liminf_{t\to\infty} |G(t)| > 0,$$

the authors of [3] constructed a conditionally oscillatory equation seen as a perturbation of (1) in the form

$$(r(t)\Phi(x'))' + \left[c(t) + \frac{\mu}{h^p(t)R(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^2}\right]\Phi(x) = 0. \tag{5}$$

The critical oscillation constant of this equation is $\mu_0 = \frac{1}{2q}$, where q is the conjugate number to p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. In [3] it is also shown that (5) has for this constant $\mu = \mu_0$ a solution with the asymptotic formula

$$x(t) = h(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{\frac{1}{p}}\left(1 + O\left(\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{-1}\right)\right) \quad \text{as} \quad t \to \infty. \tag{6}$$

The aim of this paper is to give more precise asymptotic formulas in terms of slowly and regularly varying functions in the case where the constant μ is less than or equal to $\frac{1}{2q}$.

The "perturbation approach", when the studied equation is regarded as a perturbation of another half-linear equation, has been also used in [7]. Here, the asymptotics of nonoscillatory solutions of

$$(\Phi(x'))' + \frac{\gamma_p}{t^p}\Phi(x) + \tilde{c}(t)\Phi(x) = 0, \tag{7}$$

where $\gamma_p = \left(\frac{p-1}{p}\right)^p$, was established under the assumption

$$\lim_{t\to\infty} \log t \int_t^\infty \tilde{c}(s)s^{p-1}\,\mathrm{d}s \in \left(-\infty, \tfrac{1}{2}\left(\tfrac{p-1}{p}\right)^{p-1}\right].$$

Equation (7) has been seen as a perturbation of the half-linear Euler type equation

$$(\Phi(x'))' + \frac{\gamma_p}{t^p}\Phi(x) = 0 \tag{8}$$

with the critical constant γ_p.

In this paper, we apply the perturbation principle combined with the so called Riccati technique to get our asymptotical results for (5) with $\mu \le \frac{1}{2q}$.

2. Preliminaries

As in the linear oscillation theory, the nonoscillation of equation (1) is equivalent to the solvability of a Riccati type equation (for details see [2]). In particular, if x is an eventually positive or negative solution of the nonoscillatory equation (1) on some interval of the form $[T_0, \infty)$, then $w(t) = r(t)\Phi\left(\frac{x'}{x}\right)$ solves the Riccati type equation

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0. \tag{9}$$

Conversely, having a solution $w(t)$ of (9) for $t \in [T_0, \infty)$, the corresponding solution of (1) can be expressed as

$$x(t) = C\exp\left\{\int^t r^{1-q}(s)\Phi^{-1}(w)\,\mathrm{d}s\right\},$$

where Φ^{-1} is the inverse function of Φ and C a constant.

Using the concept of perturbations it appears useful to deal with the so called modified (or generalized) Riccati equation. Let h be a positive solution of (1) and $w_h(t) = r(t)\Phi\left(\frac{h'}{h}\right)$ be the corresponding solution of the Riccati equation (9). Let us consider another nonoscillatory equation

$$(r(t)\Phi(x'))' + C(t)\Phi(x) = 0 \tag{10}$$

and let $w(t)$ be a solution of the Riccati equation associated with (10). Then $v(t) = (w(t) - w_h(t))h^p(t)$ solves the modified Riccati equation

$$v' + (C(t) - c(t))h^p + pr^{1-q}h^p P(\Phi^{-1}(w_h), w) = 0, \tag{11}$$

where

$$P(u,v) := \frac{|u|^p}{p} - uv + \frac{|v|^q}{q} \geq 0,$$

with the equality $P(u,v) = 0$ if and only if $v = \Phi(u)$. Equation (11), in this form, was derived e.g. in [1]. We deal with this equation in a slightly different, but still equivalent, form

$$v' + (C(t) - c(t))h^p + (p-1)r^{1-q}h^{-q}|G|^q F\left(\frac{v}{G}\right) = 0, \tag{12}$$

where $G(t)$ is defined by (4) and

$$F(u) = |u+1|^q - qu - 1. \tag{13}$$

Regularly and slowly varying functions in the sense of Karamata (see [4], [5] and the references therein) have an important role in half-linear theory, see e.g. [6]. Let us recall their nomenclature.

Let a continuously differentiable function $J(t)\colon [T_0,\infty) \to (0,\infty)$ be such that

$$J'(t) > 0 \quad \text{for } t \geq T_0, \qquad \lim_{t\to\infty} J(t) = \infty$$

and let $g(t), \varepsilon(t)$ be some measurable functions satisfying

$$\lim_{t\to\infty} g(t) = g \in (0,\infty) \quad \text{and} \quad \lim_{t\to\infty} \varepsilon(t) = \varrho \in \mathbb{R}.$$

According to the terminology of the above mentioned papers a positive measurable function $f(t)$, such that $f \circ J^{-1}$ is defined for all large t, and which can be expressed in the form

$$f(t) = g(t) \exp\left\{ \int_{t_0}^{t} \frac{J'(s)\varepsilon(s)}{J(s)}\,\mathrm{d}s \right\}, \qquad t \geq T_0$$

for some $T_0 > t_0$, is called *generalized regularly varying function of index ϱ with respect to J* (the notation $f \in RV_J(\varrho)$ is then used). If $g(t) \equiv g$ (a constant), $f(t)$ is called *normalized regularly varying function.* For $\varrho = 0$ the terminology *(normalized) slowly varying function* is used.

Let us only remark that in [6] this terminology is introduced for $J(t)$ defined on the interval $[0, \infty)$ but since the point is in describing the asymptotic behavior, the interval of existence of the function $f(t)$ is sufficient even for $J(t)$.

3. Main results

In this section we establish asymptotic formulas for nonoscillatory solutions of conditionally oscillatory equation (5) in the cases $\mu < \frac{1}{2q}$ and $\mu = \frac{1}{2q}$, respectively.

Theorem 1. *Suppose that* (1) *is nonoscillatory and posseses a positive solution* $h(t)$ *such that* $h'(t) \neq 0$ *for large* t *and let*

$$\int^{\infty} \frac{\mathrm{d}t}{R(t)} = \infty, \tag{14}$$

and

$$\liminf_{t\to\infty} |G(t)| > 0. \tag{15}$$

If $\mu < \frac{1}{2q}$, *then the conditionally oscillatory equation* (5) *has a pair of solutions given by the asymptotic formula*

$$x_i = h(t) \left(\int^{t} R^{-1}(s)\,\mathrm{d}s \right)^{(q-1)\lambda_i} L_i(t),$$

where λ_i *are zeros of the quadratic equation*

$$\frac{q}{2}\lambda^2 - \lambda + \mu = 0 \tag{16}$$

and $L_i(t)$ *are generalized normalized slowly varying functions of the form* $L_i(t) = \exp\left\{ \int^{t} \frac{\varepsilon_i(s)}{R(s) \int^{s} R^{-1}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s \right\}$ *and* $\varepsilon_i(t) \to 0$ *for* $t \to \infty$.

P r o o f. We are looking for solutions of the modified Riccati equation associated with (5), which reads as

$$v'(t) + \frac{\mu}{R(t)(\int^{t} R^{-1}(s)\,\mathrm{d}s)^2} + (p-1) r^{1-q}(t) h^{-q}(t) |G(t)|^q F\left(\frac{v(t)}{G(t)}\right) = 0, \tag{17}$$

where G is defined in (4) and F in (13).

Assumptions (14) and (15) imply the convergence of the integral $\int^{\infty} r^{1-q}(t)h^{-q}(t)|G(t)|^q F\left(\frac{v(t)}{G(t)}\right) \mathrm{d}t$, from which follows (see [3]) that $v(t) \to 0$ and $\frac{v(t)}{G(t)} \to 0$ for $t \to \infty$.

Let $C_0[T,\infty)$ be the set of all continuous functions on the interval $[T,\infty)$ (concrete T will be specified later) which converge to zero for $t \to \infty$ and let us consider a set of functions

$$V = \{\omega \in C_0[T,\infty) : \ |\omega(t)| < \varepsilon, \ \ t \geq T\},$$

where $\varepsilon > 0$ is so small that

$$\frac{1}{\sqrt{1-2q\mu}}(q+1)\,\varepsilon < \frac{1}{2}. \tag{18}$$

Let us also observe that the fact

$$\frac{1}{\sqrt{1-2q\mu}}\left(\frac{q}{2}+1\right)\varepsilon \leq 1 \tag{19}$$

is implied.

Let us denote the roots of the quadratic equation (16) for $\mu < \frac{1}{2q}$ as

$$\lambda_1 = \frac{1-\sqrt{1-2q\mu}}{q}, \quad \lambda_2 = \frac{1+\sqrt{1-2q\mu}}{q}.$$

We assume that two solutions of the modified Riccati equation (17) are in the form

$$v_i(z,t) := \frac{\lambda_i + z(t)}{\int^t R^{-1}(s)\,\mathrm{d}s},$$

for $t \in [T,\infty)$ and $z \in V$, $i = 1, 2$. Substituting this function and its derivative into (17), we have

$$z'(t) + \frac{\mu - \lambda_i - z(t)}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s} + (p-1)r^{1-q}(t)h^{-q}(t)|G(t)|^q F\left(\frac{v_i(z,t)}{G(t)}\right)\int^t R^{-1}(s)\,\mathrm{d}s = 0$$

which can be rewritten as

$$z'(t) \quad + \quad \frac{(-1+\lambda_i q)z(t)}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s} + \frac{1}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s}E_i(z,t) = 0, \tag{20}$$

where

$$E_i(z,t) := \mu - \lambda_i - \lambda_i q z(t) + (p-1)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^2 G^2(t) F\left(\frac{v_i(z,t)}{G(t)}\right).$$

This means that looking for solutions v_i of the modified Riccati equation (17) is equivalent to looking for solutions z_i of the equation (20). In the next we shall show that two solutions of (20) can be found through the Banach fixed-point theorem used onto suitable integral operators.

Firstly, let us turn our attention to the behavior of the function $F(u)$, which plays an important role in estimating of certain needed terms.

Studying the behavior of $F(u)$ and $F'(u)$ for u in a neighbourhood of 0, we have

$$\begin{aligned} F(u) &= \frac{F''(0)}{2}u^2 + \frac{F'''(\zeta)}{6}u^3 \\ &= \frac{q(q-1)}{2}u^2 + \frac{q(q-1)(q-2)}{6}|1+\zeta|^{q-3}\operatorname{sgn}(1+\zeta)u^3, \end{aligned}$$

where ζ is between 0 and u. For $|u| < \frac{1}{2}$ and hence also $|\zeta| < \frac{1}{2}$ there exists a positive constant M_q such that

$$\left|\frac{q(q-2)}{6}|1+\zeta|^{q-3}\operatorname{sgn}(1+\zeta)\right| \le M_q \qquad \text{for} \quad q > 1.$$

Therefore

$$\left|F(u) - \frac{q(q-1)}{2}u^2\right| \le (q-1)M_q|u|^3. \tag{21}$$

Similarly,

$$F'(u) = F''(0)u + \frac{F'''(\zeta')}{2}u^2 = q(q-1)u + \frac{q(q-1)(q-2)}{2}|1+\zeta'|^{q-3}\operatorname{sgn}(1+\zeta')u^2,$$

where ζ' is between 0 and u. Again, considering $|\zeta'| < \frac{1}{2}$ we have

$$|F'(u) - q(q-1)u| \le 3(q-1)M_q|u|^2. \tag{22}$$

Now, let us denote $J(t) := \int^t R^{-1}(s)\,\mathrm{d}s$, then the estimate of the function $E_i(z,t)$ for $z \in V$, reads as

$$\begin{aligned}
|E_i(z,t)| = &\Bigg|\mu - \lambda_i - \lambda_i q z(t) + \frac{q}{2} J^2(t) v_i^2(z,t) \\
&\quad + (p-1) J^2(t) G^2(t) F\left(\frac{v_i(z,t)}{G(t)}\right) - \frac{q}{2} J^2(t) v_i^2(z,t)\Bigg| \\
\le &\left|\mu - \lambda_i - \lambda_i q z(t) + \frac{q}{2}(\lambda_i + z(t))^2\right| \\
&\quad + \left|(p-1) J^2(t) G^2(t) \left[F\left(\frac{v_i(z,t)}{G(t)}\right) - \frac{q(q-1)}{2}\left(\frac{v_i(z,t)}{G(t)}\right)^2\right]\right| \\
\le &\frac{q}{2}|z(t)|^2 + \frac{M_q |\lambda_i + z(t)|^3}{|G(t)||J(t)|} \\
\le &\frac{q}{2}|z(t)|^2 + \frac{K M_q |\lambda_i + z(t)|^3}{|J(t)|},
\end{aligned}$$

where (21) was used for $u = \frac{v_i}{G}$ and $K := \sup\limits_{t \ge T} \frac{1}{|G(t)|}$ is a finite constant for T sufficiently large because of (15). According to (14) there exists T_1 such that the last term in the previous inequality is less than ε^2 and therefore

$$|E_i(z,t)| \le \frac{q}{2}\varepsilon^2 + \varepsilon^2 \le \varepsilon^2\left(\frac{q}{2} + 1\right) \tag{23}$$

for $t \ge T_1$.

Furthermore, for $z_1, z_2 \in V$ we have

$$\begin{aligned}
&|E_i(z_1,t) - E_i(z_2,t)| \\
&= \left|-\lambda_i q (z_1 - z_2) + (p-1) J^2(t) G^2(t)\left[F\left(\frac{v_i(z_1,t)}{G(t)}\right) - F\left(\frac{v_i(z_2,t)}{G(t)}\right)\right]\right|,
\end{aligned}$$

which, by the mean value theorem with a suitable $z(t) \in V$, becomes

$$\begin{aligned}
= &\Bigg|-\lambda_i q (z_1 - z_2) + q J(t) v_i(z,t)(z_1 - z_2) \\
&\quad + (p-1) J(t) G(t) F'\left(\frac{v_i(z,t)}{G(t)}\right)(z_1 - z_2) - q J(t) v_i(z,t)(z_1 - z_2)\Bigg| \\
\le &\Bigg|-\lambda_i q + q J(t) v_i(z,t) \\
&\quad + (p-1) J(t) G(t)\left(F'\left(\frac{v_i(z,t)}{G(t)}\right) - q(q-1)\frac{v_i(z,t)}{G(t)}\right)\Bigg| \cdot \|z_1 - z_2\|
\end{aligned}$$

$$\leq \left(q|z(t)| + \left| \frac{3KM_q(\lambda_i + z(t))^2}{J(t)} \right| \right) \cdot \|z_1 - z_2\|,$$

where (22) was used. Similarly as in the previous estimate, there exists T_2 such that the middle term in the last row of the inequality is less than ε and hence

$$|E_i(z_1, t) - E_i(z_2, t)| \leq \varepsilon(q+1) \cdot \|z_1 - z_2\| \tag{24}$$

for $t \in [T_2, \infty)$.

Now, let us consider the pair of functions

$$r_i(t) := \exp\left\{ \int^t \frac{-1 + \lambda_i q}{R(s) \int^s R^{-1}(\tau)\, \mathrm{d}\tau} \, \mathrm{d}s \right\}, \qquad i = 1, 2.$$

Then equation (20) is equivalent to

$$(r_i(t) z(t))' + r_i(t) \frac{1}{R(t) \int^t R^{-1}(s)\, \mathrm{d}s} E_i(z, t) = 0. \tag{25}$$

For $i = 1$, we have a function

$$r_1(t) = \exp\left\{ \int^t \frac{-1 + \lambda_1 q}{R(s) \int^s R^{-1}(\tau)\, \mathrm{d}\tau} \, \mathrm{d}s \right\} = \exp\left\{ \int^t \frac{-\sqrt{1 - 2q\mu}}{R(s) \int^s R^{-1}(\tau)\, \mathrm{d}\tau} \, \mathrm{d}s \right\}$$

and it is easy to see that $r_1(t) \to 0$ for $t \to \infty$.

Finally, let us define an integral operator F_1 on the set of functions V by

$$(F_1 z)(t) = \frac{1}{r_1(t)} \int_t^\infty \frac{r_1(s)}{R(s) \int^s R^{-1}(\tau)\, \mathrm{d}\tau} E_1(z, s)\, \mathrm{d}s.$$

We observe that

$$\int_t^\infty \frac{r_1(s)}{R(s) \int^s R^{-1}(\tau)\, \mathrm{d}\tau} \, \mathrm{d}s = \frac{r_1(t)}{\sqrt{1 - 2q\mu}}.$$

Taking $T = \max\{T_1, T_2\}$, by (23) and (18) we have

$$\begin{aligned} |(F_1 z)(t)| &\leq \frac{1}{r_1(t)} \int_t^\infty \frac{r_1(s)}{R(s) \int^s R^{-1}(\tau)\, \mathrm{d}\tau} |E_1(z, s)|\, \mathrm{d}s \\ &\leq \frac{1}{\sqrt{1 - 2q\mu}} \left(\frac{q}{2} + 1 \right) \varepsilon^2 \leq \varepsilon, \end{aligned}$$

which means that F_1 maps the set V into itself, and by (24) and (19) we see that

$$|(F_1z_1)(t)-(F_1z_2)(t)| \le \frac{1}{r_1(t)}\int_t^\infty \frac{r_1(s)}{R(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau}|E_1(z_1,s)-E_1(z_2,s)|\,\mathrm{d}s$$
$$\le \|z_1-z_2\|\frac{1}{\sqrt{1-2q\mu}}\varepsilon(q+1) < \frac{1}{2}\|z_1-z_2\|,$$

which implies that F_1 is a contraction. Using the Banach fixed-point theorem we can find a function $z_1(t)$, that satisfies $z_1 = F_1z_1$. That means that $z_1(t)$ is a solution of (25) and also of (20) and $v_1(t) = \frac{\lambda_1+z_1(t)}{\int^t R^{-1}\,\mathrm{d}s}$ is a solution of (17).

For $i = 2$ we have

$$r_2(t) = \exp\left\{\int^t \frac{-1+\lambda_2 q}{R(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\right\} = \exp\left\{\int^t \frac{\sqrt{1-2q\mu}}{R(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\right\}$$

and we define an integral operator F_2 by

$$(F_2z)(t) = -\frac{1}{r_2(t)}\int^t \frac{r_2(s)}{R(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau}E_2(z,s)\,\mathrm{d}s.$$

Since

$$\int^t \frac{r_2(s)}{R(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s = \frac{r_2(t)-c}{\sqrt{1-2q\mu}},$$

where c is a positive suitable constant, the inequality

$$\frac{1}{r_2(t)}\int^t \frac{r_2(s)}{R(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s \le \frac{1}{\sqrt{1-2q\mu}}$$

holds for t sufficiently large, as $r_2(t)\to\infty$ for $t\to\infty$. Taking $T=\max\{T_1,T_2\}$, the estimates for the operator F_2 are the same as in the previous case and we can find a fixed point $z_2(t)$ satisfying $F_2z_2 = z_2$. Thus $z_2(t)$ solves (25) and $v_2(t) = \frac{\lambda_2+z_2(t)}{\int^t R^{-1}\,\mathrm{d}s}$ solves the modified Riccati equation (17).

Expressing the solutions of the standard Riccati equation for (5) corresponding to the solutions $v_i(z_i,t)$ of the modified Riccati equation, we have

$$\begin{aligned} w_i(t) &= h^{-p}(t)v_i(z_i,t)+w_h(t)=w_h(t)\left(1+\frac{v_i(z_i,t)}{h^p(t)w_h(t)}\right)\\ &= w_h(t)\left(1+\frac{\lambda_i+z_i(t)}{h^p(t)w_h(t)\int^t R^{-1}\,\mathrm{d}s}\right)=w_h(t)\left(1+\frac{\lambda_i+z_i(t)}{G(t)\int^t R^{-1}(s)\,\mathrm{d}s}\right). \end{aligned}$$

Since solutions of (5) are given by the formula $x(t)=\exp\left\{\int^t r^{1-q}(s)\Phi^{-1}(w)\,\mathrm{d}s\right\}$, we need to express

$$\begin{aligned} &r^{1-q}(t)\Phi^{-1}(w_i)\\ &=\frac{h'(t)}{h(t)}\left(1+\frac{\lambda_i+z_i(t)}{G(t)\int^t R^{-1}(s)\,\mathrm{d}s}\right)^{q-1}\\ &=\frac{h'(t)}{h(t)}\left(1+(q-1)\frac{\lambda_i+z_i(t)}{G(t)\int^t R^{-1}(s)\,\mathrm{d}s}+o\left(\frac{\lambda_i+z_i(t)}{G(t)\int^t R^{-1}(s)\,\mathrm{d}s}\right)\right)\\ &=\frac{h'(t)}{h(t)}+\frac{(q-1)\lambda_i}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s}+\frac{(q-1)z_i(t)}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s}+o\left(\frac{\lambda_i+z_i(t)}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s}\right). \end{aligned}$$

Because

$$\begin{aligned} o\left(\frac{\lambda_i+z_i(t)}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s}\right) &= \frac{R(t)\int^t R^{-1}(s)\,\mathrm{d}s\; o\left(\frac{\lambda_i+z_i(t)}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s}\right)}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s}\\ &= \frac{o(\lambda_i+z_i(t))}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s} \end{aligned}$$

holds for large t, the pair of solutions of (5) for $i=1,2$ is in the form

$$\begin{aligned} x_i(t)=\exp\Bigg\{\log h(t)+\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{(q-1)\lambda_i}\\ +\int^t\frac{(q-1)z_i(s)+o(\lambda_i+z_i(s))}{R(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau}\Bigg\}. \end{aligned}$$

As $z_i \in V$ and hence $z_i(t) \to 0$ for $t \to \infty$, the statement of the theorem holds for $\varepsilon_i(t) = (q-1)z_i(t) + o(\lambda_i + z_i(t))$. □

Now let us present the asymptotic formula in case $\mu = \frac{1}{2q}$, which gives an improved version of (6).

Theorem 2. *Let the assumptions of the previous theorem be satisfied and let $\mu = \frac{1}{2q}$. Then equation (5) has a solution of the form*

$$x = h(t) \left(\int^t R^{-1}(s)\,\mathrm{d}s \right)^{\frac{1}{p}} L(t), \tag{26}$$

where $L(t)$ is a generalized normalized slowly varying function of the form $L(t) = \exp\left\{ \int^t \frac{\varepsilon(s)}{R(s) \int^s R^{-1}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s \right\}$ and $\varepsilon(t) \to 0$ for $t \to \infty$.

P r o o f. For $\mu = \frac{1}{2q}$ the quadratic equation (16) has a double root $\lambda = \frac{1}{q}$. We assume the solution of modified Riccati equation to be in the form (for $z \in V$)

$$v(z,t) = \frac{\frac{1}{q} + z(t)}{\int^t R^{-1}(s)\,\mathrm{d}s},$$

which gives, after substituting into the modified Riccati equation (17) for $\mu = \frac{1}{2q}$,

$$z'(t) + \frac{-z(t) - \frac{1}{2q}}{R(t) \int^t R^{-1}(s)\,\mathrm{d}s} + (p-1) r^{1-q}(t) h^{-q}(t) |G(t)|^q F\left(\frac{v(z,t)}{G(t)} \right) \int^t R^{-1}(s)\,\mathrm{d}s = 0.$$

Let us denote

$$E(z,t) = -z(t) - \frac{1}{2q} + (p-1) \left(\int^t R^{-1}(s)\,\mathrm{d}s \right)^2 G^2(t) F\left(\frac{v(z,t)}{G(t)} \right)$$

and let us consider an integral operator F_3

$$(F_3 z)(t) = \int_t^\infty \frac{1}{R(s) \int^s R^{-1}(\tau)\,\mathrm{d}\tau} E(z,s)\,\mathrm{d}s$$

on a set of continuous functions

$$V = \{\omega \in C_0[T,\infty) : \ |\omega(t)| < \varepsilon, \ t \geq T\},$$

where T and ε are to be established similarly as in the proof of the previous theorem. Then the solution of modifed Riccati equation and also the solution of the studied equation can be found in almost the same manner as for the previous statement. $\square$

Remark 1. If $r(t) \equiv 1, c(t) = \gamma_p t^{-p}$ and $h(t) = t^{\frac{p-1}{p}}$ then the conditionally oscillatory equation (5) with $\mu = \frac{1}{2q}$, seen as a perturbation of the Euler equation (8), becomes the Euler-Weber (or alternatively Riemann-Weber) half-linear differential equation

$$(\Phi(x'))' + \left[\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t}\right] \Phi(x) = 0$$

with the so-called critical coefficient $\mu_p = \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}$. The asymptotic formula (26) then reduces to the formula given in [7, Theorem 2].

Remark 2. For the Euler-Weber half linear equation also the asymptotic formula for its second linearly independent solution is known (see [8]). An open question remains whether the second linearly independent solution of (5) with $\mu = \frac{1}{2q}$ could be found in a similar form

$$x_2(t) = h(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{\frac{1}{p}} \left(\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\right)^{\frac{2}{p}} L_2(t),$$

where

$$L_2(t) = \exp\left\{\int^t \frac{\varepsilon_2(s)}{R(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau \log\left(\int^s R^{-1}(\tau)\,\mathrm{d}\tau\right)}\,\mathrm{d}s\right\}$$

and $\varepsilon_2(t) \to 0$ for $t \to \infty$.

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