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APPLICATION OF FRACTIONAL CALCULUS OPERATORS FOR A NEW CLASS OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY HOHLOV OPERATOR

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Dedicated to my wife HND HEKMAT ABDULLAH on the occasion of the birth of our first baby MALAK

(*Communicated by Michal Zajac*)

ABSTRACT. In this paper, we introduce a new class $W(a,b,c,\gamma,\beta)$ which consists of analytic and univalent functions with negative coefficients in the unit disc defined by Hohlov operator, we obtain distortion theorem using fractional calculus techniques for this class. Also coefficient inequalities and some results for this class are obtained.

1. Introduction

Let A denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic and univalent in $U = \{z : \ |z| < 1\}$. If a function f is given by (1) and g is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{2}$$

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is in A, then Convolution or Hadamard product of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \qquad z \in U. \tag{3}$$

Let W denote the subclass of A consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geq 0). \tag{4}$$

Definition 1. ([3]) The Gaussian hypergeometric function denoted by ${}_2F_1(a, b; c; z)$ and is defined by

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \qquad |z| < 1,$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, $c > b > 0$ and $c > a + b$.

It is well known (see [1]) that under the conditions $c > b > 0$ and $c > a + b$ we have

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{5}$$

Definition 2. Let $f(z) \in W$ be of the form (4), then the Hohlov operator $F(a, b, c)$, $(F(a, b, c) : W \to W)$ ([2]) is defined by means of a Hadamard product below:

$$F(a, b, c)f(z) = (z\,{}_2F_1(a, b; c; z)) * f(z) = z - \sum_{n=2}^{\infty} \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (n-1)!} a_n z^n, \tag{6}$$

$(a, b, c \in \mathbb{N},\ c \neq Z_0^-;\ z \in U)$.

The integral representation of Hohlov operator is given by

$$F(a, b, c)f(z) =$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_0^1 \frac{(1-\sigma)^{c-a-b} \sigma^{b-2}}{\Gamma(c-a-b+1)} {}_2F_1(c-a, 1-a; c-a-b+1; 1-\sigma) f(z)\, \mathrm{d}\sigma,$$

$$(a > 0,\ \ b > 0,\ \ c-a-b+1 > 0,\ \ f \in W,\ \ z \in U)$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} I_{1,2}^{(a-2,b-2),(1-a,c-b)} f(z). \tag{7}$$

Definition 3. A function $f(z)$ in W is in the class $W(a,b,c,\gamma,\beta)$ if it satisfies the condition

$$\left|\frac{z(F(a,b,c)f(z))''}{z(F(a,b,c)f(z))''-2(1-\gamma)(F(a,b,c)f(z))'}\right|<\beta, \tag{8}$$

where $0\le\gamma<1$, $0<\beta\le 1$, $z\in U$.

2. The class $W(a,b,c,\gamma,\beta)$

Theorem 1. *Let the function f be defined by* (4). *Then $f\in W(a,b,c,\gamma,\beta)$ if and only if*

$$\sum_{n=2}^{\infty} n[(n-1)(1-\beta)+2\beta(1-\gamma)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}a_n\le 2\beta(1-\gamma), \tag{9}$$

where $0\le\gamma<1$, $0<\beta\le 1$.

The result (9) *is sharp for the function*

$$f(z)=z-\frac{2\beta(1-\gamma)z^n}{n[(n-1)(1-\beta)+2\beta(1-\gamma)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}},\qquad n\ge 2.$$

P r o o f. Suppose that the inequality (9) holds true and $|z|=1$. Then we obtain

$$\begin{aligned}
&|z(F(a,b,c)f(z))''|-\beta|z(F(a,b,c)f(z))''-2(1-\gamma)(F(a,b,c)f(z))'|\\
=\;&\left|-\sum_{n=2}^{\infty}n(n-1)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}a_nz^{n-1}\right|\\
&-\beta\left|\sum_{n=2}^{\infty}[n(n-1)-2n(1-\gamma)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}a_nz^{n-1}+2(1-\gamma)\right|\\
\le\;&\sum_{n=2}^{\infty}n[(n-1)(1-\beta)+2\beta(1-\gamma)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}a_n-2\beta(1-\gamma)\\
\le\;&0.
\end{aligned}$$

Hence, by maximum modulus principle, $f\in W(a,b,c,\gamma,\beta)$.

Now suppose that $f\in W(a,b,c,\gamma,\beta)$ so that

$$\left|\frac{z(F(a,b,c)f(z))''}{z(F(a,b,c)f(z))''-2(1-\gamma)(F(a,b,c)f(z))'}\right|<\beta,\qquad z\in U,$$

then

$$|z(F(a,b,c)f(z))''|<\beta|z(F(a,b,c)f(z))''-2(1-\gamma)(F(a,b,c)f(z))'|;$$

we get

$$\begin{aligned}&\left|-\sum_{n=2}^{\infty} n(n-1)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}a_n z^{n-1}\right|\\ < \quad & \beta\left|\sum_{n=2}^{\infty}[n(n-1)-2n(1-\gamma)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}a_n z^{n-1}+2(1-\gamma)\right|,\end{aligned}$$

thus

$$\sum_{n=2}^{\infty} n[(n-1)(1-\beta)+2\beta(1-\gamma)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}a_n \le 2\beta(1-\gamma),$$

and the proof is complete. □

3. Application of the fractional calculus

Various operators of fractional calculus (that is, fractional derivative and fractional integral) have been rather extensively studied by many researchers (c.f. [5], [6], [7]). However, we try to restrict ourselves to the following definition given by Owa [4] for convenience.

Definition 4 (**Fractional integral operator**). The fractional integral of order λ is defined, for a function $f(z)$, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}}\, \mathrm{d}t \qquad (\lambda > 0), \tag{10}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real, when $(z-t) > 0$.

Definition 5 (**Fractional derivative operator**). The fractional derivative of order λ is defined, for a function $f(z)$ by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{\mathrm{d}}{\mathrm{d}z} \int_0^z \frac{f(t)}{(z-t)^{\lambda}}\, \mathrm{d}t \qquad (0 \le \lambda < 1), \tag{11}$$

where $f(z)$ is constrained, and the multiplicity of $(z-t)^{-\lambda}$ is removed, as in Definition 4.

Definition 6. Under the hypothesis of Definition 5, the fractional derivative of order $k+\lambda$ is defined, for a function $f(z)$, by

$$D_z^{k+\lambda} f(z) = \frac{\mathrm{d}^k}{\mathrm{d}z^k} D_z^{\lambda} f(z) \qquad (0 \le \lambda < 1, \quad k \in \mathbb{N}_0). \tag{12}$$

Next, we state the following definition of fractional integral operator given by Srivastava et. al. [8].

Definition 7. For real numbers $\alpha > 0$, η and δ, the fractional operator, $I_{0,z}^{\alpha,\eta,\delta}$ is defined by

$$I_{0,z}^{\alpha,\eta,\delta} f(z) = \frac{z^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} F\big(\alpha+\eta, -\delta; \alpha; 1-\tfrac{t}{2}\big) f(t)\,\mathrm{d}t, \tag{13}$$

where $f(z)$ is analytic function in a simply connected region of the z-plane containing the origin with order

$$f(z) = O(|z|^{\varepsilon}), \qquad z \to 0,$$

where $\varepsilon > \max(0, \eta-\delta) - 1$,

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z$$

and $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N}) \end{cases}$$

and the multiplicity of $(z-t)^{\alpha-1}$ is removed by requiring $\log(z-t)$ to be real, when $z-t>0$.

In order to prove our result concerning the fractional integral operator, we recall here the following lemma due to Srivastava et. al. [8].

Lemma 1. *Let $\alpha > 0$ and $n > \eta - \delta - 1$. Then*

$$I_{0,z}^{\alpha,\eta,\delta} z^n = \frac{\Gamma(n+1)\Gamma(n-\eta+\delta+1)}{\Gamma(n-\eta+1)\Gamma(n+\alpha+\delta+1)} z^{n-\eta}.$$

Now making use of above Lemma 1, we state and prove the following theorem:

Theorem 2. *Let $\alpha > 0$, $\eta < 2$, $\alpha+\delta > -2$, $\eta(\alpha+\delta) \le 3\alpha$. If $f(z)$ defined by (4) is in the class $W(a,b,c,\gamma,\beta)$, then*

$$|I_{0,z}^{\alpha,\eta,\delta} f(z)| \ge \frac{\Gamma(2-\eta+\delta)|z|^{1-\eta}}{\Gamma(2-\eta)\Gamma(2+\alpha+\delta)} \left(1 - \frac{2c\beta(1-\gamma)(2-\eta+\delta)}{(2-\eta)(2+\alpha+\delta)(1+\beta(1-2\gamma))ab}|z|\right) \tag{14}$$

and

$$|I_{0,z}^{\alpha,\eta,\delta}f(z)| \leq \frac{\Gamma(2-\eta+\delta)|z|^{1-\eta}}{\Gamma(2-\eta)\Gamma(2+\alpha+\delta)}\left(1+\frac{2c\beta(1-\gamma)(2-\eta+\delta)}{(2-\eta)(2+\alpha+\delta)(1+\beta(1-2\gamma))ab}|z|\right), \tag{15}$$

for $z \in U_0$, *where*

$$U_0 = \begin{cases} U & n \leq 1 \\ U-\{0\} & n > 1. \end{cases}$$

The result is sharp and is given by

$$f(z) = z - \frac{\beta(1-\gamma)}{ab(1+\beta(1-2\gamma))}z^2. \tag{16}$$

P r o o f. By using Lemma 1, we have

$$\begin{aligned} I_{0,z}^{\alpha,\eta,\delta}f(z) = & \frac{\Gamma(2-\eta+\delta)}{\Gamma(2-\eta)\Gamma(2+\alpha+\delta)}z^{1-\eta} \\ & -\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(n-\eta+\delta+1)}{\Gamma(n-\eta+1)\Gamma(n+\alpha+\delta+1)}a_n z^{n-\eta}. \end{aligned} \tag{17}$$

Setting

$$Q(z) = \frac{\Gamma(2-\eta)\Gamma(2+\alpha+\delta)}{\Gamma(2-\eta+\delta)}z^{\eta}I_{0,z}^{\alpha,\eta,\delta}f(z) = z - \sum_{n=2}^{\infty}q(n)a_n z^n,$$

where

$$q(n) = \frac{(2-\eta+\delta)_{n-1}(1)_n}{(2-\eta)_{n-1}(2+\alpha+\delta)_{n-1}} \qquad (n \geq 2). \tag{18}$$

It is easily verified that $q(n)$ is non-increasing for $n \geq 2$, and thus we have

$$0 < q(n) \leq q(2) = \frac{(2-\eta+\delta)2}{(2-\eta)(2+\alpha+\delta)}. \tag{19}$$

Now, by the application of Theorem 1 and (19), we obtain

$$\begin{aligned} |Q(z)| & \geq |z| - q(2)|z|^2\sum_{n=2}^{\infty}a_n \\ & \geq |z| - \frac{2c\beta(1-\gamma)(2-\eta+\delta)}{(2-\eta)(2+\alpha+\delta)(1+\beta(1-2\gamma))ab}|z|^2, \end{aligned}$$

which proves (14), and for (15) we can find that

$$\begin{aligned} |Q(z)| & \leq |z| + q(2)|z|^2\sum_{n=2}^{\infty}a_n \\ & \leq |z| + \frac{2c\beta(1-\gamma)(2-\eta+\delta)}{(2-\eta)(2+\alpha+\delta)(1+\beta(1-2\gamma))ab}|z|^2, \end{aligned}$$

and the proof is complete. □

Taking $\eta = -\alpha = -\lambda$ and $\eta = -\alpha = \lambda$ in Theorem 2, we get two separate corollaries, which are contained in:

Corollary 1. *Let the function f defined by* (4) *be in the class $W(a,b,c,\gamma,\beta)$. Then we have*

$$|D_z^{-\lambda} f(z)| \geq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left(1 - \frac{2c\beta(1-\gamma)}{(2+\lambda)(1+\beta(1-2\gamma))ab} |z| \right) \tag{20}$$

and

$$|D_z^{-\lambda} f(z)| \leq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left(1 + \frac{2c\beta(1-\gamma)}{(2+\lambda)(1+\beta(1-2\gamma))ab} |z| \right). \tag{21}$$

Corollary 2. *Let the function f defined by* (4) *be in the class $W(a,b,c,\gamma,\beta)$. Then we have*

$$|D_z^{\lambda} f(z)| \geq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} \left(1 - \frac{2c\beta(1-\gamma)}{(2-\lambda)(1+\beta(1-2\gamma))ab} |z| \right) \tag{22}$$

and

$$|D_z^{\lambda} f(z)| \leq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} \left(1 + \frac{2c\beta(1-\gamma)}{(2-\lambda)(1+\beta(1-2\gamma))ab} |z| \right). \tag{23}$$

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