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VECTOR VALUED DOUBLE SEQUENCE SPACES DEFINED BY ORLICZ FUNCTION

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ABSTRACT. In this article we introduce some vector valued double sequence spaces defined by Orlicz function. We study some of their properties like solidness, symmetricity, completeness etc. and prove some inclusion results.

1. Introduction

Throughout the article ${}_2w(q)$, ${}_2\ell_\infty(q)$, ${}_2c(q)$, ${}_2c_0(q)$, ${}_2c^R(q)$, ${}_2c_0^R(q)$ denote the spaces of *all*, *bounded*, *convergent in Pringsheim's sense*, *null in Pringsheim's sense*, *regularly convergent* and *regularly null* double sequences, defined over a seminormed space (X,q), seminormed by q. For $X = C$, the field of complex numbers, these represent the corresponding scalar sequence spaces. The zero element of X is denoted by θ, that of a single sequence by $\bar{\theta} = (\theta, \theta, \dots)$ and the zero double sequence is denoted by ${}_2\bar{\theta}$, a double infinite array of θ's.

An *Orlicz function* M is a mapping $M\colon [0,\infty) \to [0,\infty)$ such that it is *continuous*, *non-decreasing* and *convex* with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to construct the sequence space,

$$\ell^M = \left\{ (x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \ \text{for some } \rho > 0 \right\},$$

which is a Banach space normed by

$$\|(x_k)\| = \inf\left\{ \rho > 0 : \ \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}.$$

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The space ℓ^M is closely related to the space ℓ^p, which is an Orlicz sequence space with $M(x) = |x|^p$, for $1 \leq p < \infty$.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u, if there exists a constant $K > 0$, such that $M(2u) \leq K(Mu)$, $u \geq 0$.

The Orlicz sequence space was further investigated from sequence space point of view and related with summability theory by Tripathy [12], Tripathy and Mahanta [13], Altin *et.al.* [1] and many others.

Remark 1. Let $0 < \lambda < 1$, then $M(\lambda x) \leq M(x)$, for all $x \geq 0$.

2. Definitions and preliminaries

Throughout a double sequence is denoted by $A = \langle a_{nk} \rangle$, a double infinite array of elements $a_{nk} \in X$ for all $n, k \in \mathbb{N}$.

The initial works on double sequences is found in Bromwich [3]. Later on it was studied by Hardy [5], Moricz [8], Moricz and Rhoades [9], Tripathy [11], Basarir and Sonalcan [2], Colak and Turkmenoglu [4], Turkmenoglu [14] and many others. Hardy [5] introduced the notion of regular convergence for double sequences.

Definition 1. A double sequence space E is said to be *solid* if $\langle \alpha_{nk} a_{nk} \rangle \in E$ whenever $\langle a_{nk} \rangle \in E$ for all double sequences $\langle \alpha_{nk} \rangle$ of scalars with $|\alpha_{nk}| \leq 1$ for all $n, k \in \mathbb{N}$.

Definition 2. A double sequence space E is said to be *symmetric* if $\langle a_{nk} \rangle \in E$ implies $\langle a_{\pi(n)\pi(k)} \rangle \in E$, where π is a permutations of $\mathbb{N}$.

Definition 3. A double sequence space E is said to be *monotone* if it contains the canonical preimages of all its step spaces.

Definition 4. A double sequence space E is said to be *convergence free* if $\langle b_{nk} \rangle \in E$ whenever $\langle a_{nk} \rangle \in E$ and $b_{nk} = \theta$, whenever $a_{nk} = \theta$, where θ is the zero element of X.

From the definitions of solid space and monotone space we have the following remark.

Remark 2. A sequence space E is solid implies E is monotone.

Let M be an Orlicz function. Now we introduce the following double sequence spaces:

$$ {}_2\ell_\infty(M,q)=\left\{\langle a_{nk}\rangle\in {}_2w(q):\ \sup_{n,k} M\left(q\left(\frac{a_{nk}}{\rho}\right)\right)<\infty,\ \text{ for some }\ \rho>0\right\}, $$

$$ {}_2c(M,q)=\Bigg\{\langle a_{nk}\rangle\in {}_2w(q):\ M\left(q\left(\frac{a_{nk}-L}{\rho}\right)\right)\to 0 \quad \text{as }\ n,k\to\infty \quad \text{for some }\ \rho>0\Bigg\}. $$

$A=\langle a_{nk}\rangle\in {}_2c^R(M,q)$ i.e. *regularly convergent* if $\langle a_{nk}\rangle\in {}_2c(M,q)$ and the following limits hold:
There exists $L_k\in X$, such that

$$ M\left(q\left(\frac{a_{nk}-L_k}{\rho}\right)\right)\to 0, \quad \text{as }\ n\to\infty, \qquad \text{for some }\quad \rho>0 \quad \text{and all}\quad k\in\mathbb{N}. $$

and exists $J_n\in X$, such that

$$ M\left(q\left(\frac{a_{nk}-J_n}{\rho}\right)\right)\to 0, \quad \text{as }\ k\to\infty, \qquad \text{for some }\quad \rho>0 \quad \text{and all}\quad n\in\mathbb{N}. $$

The definition of ${}_2c_0(M,q)$ and ${}_2c_0^R(M,q)$ follows from the above definition on taking $L=L_k=J_n=\theta$, for all $n,k\in\mathbb{N}$.

Remark 3. The space ${}_2c_0^R(M,q)$ has the following definition too.

$$ {}_2c_0^R(M,q)=\Bigg\{\langle a_{nk}\rangle\in {}_2w(q):\ M\left(q\left(\frac{a_{nk}}{\rho}\right)\right)\to 0,\ \text{ as }\ \max\{n,k\}\to\infty, \quad \text{for some }\ \rho>0\Bigg\}. $$

We also define

$$ {}_2c^B(M,q)={}_2c(M,q)\cap {}_2\ell_\infty(M,q) $$

and

$$ {}_2c_0^B(M,q)={}_2c_0(M,q)\cap {}_2\ell_\infty(M,q). $$

3. Main results

In this section we establish the results of this paper.

Theorem 1. *The classes $Z(M,q)$ for $Z={}_2\ell_\infty$, ${}_2c$, ${}_2c_0$, ${}_2c^B$, ${}_2c_0^B$, ${}_2c^R$ and ${}_2c_0^R$ of double sequences are linear spaces.*

P r o o f. We establish it for ${}_2\ell_\infty(M,q)$ and the other cases can be established following similar techniques.

Let $\langle a_{nk}\rangle$, $\langle b_{nk}\rangle \in {}_2\ell_\infty(M,q)$. Then we have

$$\sup_{n,k} M\left(q\left(\frac{a_{nk}}{\rho_1}\right)\right) < \infty, \qquad \text{for some} \quad \rho_1 > 0. \tag{1}$$

$$\sup_{n,k} M\left(q\left(\frac{b_{nk}}{\rho_2}\right)\right) < \infty, \qquad \text{for some} \quad \rho_2 > 0. \tag{2}$$

Let $\alpha, \beta \in C$ be scalars and $\rho = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Then

$$\begin{aligned}&\sup_{n,k} M\left(q\left(\frac{\alpha a_{nk} + \beta b_{nk}}{\rho}\right)\right)\\ &\le \frac{1}{2}\sup_{n,k} M\left(q\left(\frac{a_{nk}}{\rho_1}\right)\right) + \frac{1}{2}\sup_{n,k} M\left(q\left(\frac{b_{nk}}{\rho_2}\right)\right) < \infty.\end{aligned}$$

Hence $\langle \alpha a_{nk} + \beta b_{nk}\rangle \in {}_2\ell_\infty(M,q)$.

Thus ${}_2\ell_\infty(M,q)$ is a linear space. □

Theorem 2. *The spaces $Z(M,q)$ for $Z = {}_2\ell_\infty$, ${}_2c^B$, ${}_2c_0^B$, ${}_2c^R$ and ${}_2c_0^R$ are seminormed spaces, seminormed by*

$$f(\langle a_{nk}\rangle) = \inf\left\{\rho > 0 : \sup_{n,k} M\left(q\left(\frac{a_{nk}}{\rho}\right)\right) \le 1\right\}. \tag{3}$$

P r o o f. Clearly $f({}_2\bar{\theta}) = 0$ and $f(-\langle a_{nk}\rangle) = f(\langle a_{nk}\rangle)$, for all $\langle a_{nk}\rangle \in Z$.

Let $\lambda \in C$, then we have

$$f(\langle a_{nk}\rangle) = \inf\left\{r > 0 : \sup_{n,k} M\left(q\left(\frac{a_{nk}}{r}\right)\right) \le 1\right\}$$

and

$$\begin{aligned}f(\lambda\langle a_{nk}\rangle) &= \inf\left\{\rho > 0 : \sup_{n,k} M\left(q\left(\frac{\lambda a_{nk}}{\rho}\right)\right) \le 1\right\}\\ &= |\lambda| \inf\left\{r > 0 : \sup_{n,k} M\left(q\left(\frac{a_{nk}}{r}\right)\right) \le 1\right\},\\ &\qquad \text{where} \quad r = \frac{\rho}{|\lambda|} = |\lambda| f(\langle a_{nk}\rangle).\end{aligned}$$

Let $\langle a_{nk}\rangle$, $\langle b_{nk}\rangle \in {}_2\ell_\infty(M,q)$. Then for some $\rho_1 > 0$, $r_1 > 0$,

$$f(\langle a_{nk}\rangle) = \inf\left\{\rho_1 > 0 : \sup_{n,k} M\left(q\left(\frac{a_{nk}}{\rho_1}\right)\right) \le 1\right\}, \tag{4}$$

$$f(\langle b_{nk}\rangle) = \inf\left\{r_1 > 0 : \sup_{n,k} M\left(q\left(\frac{b_{nk}}{r_1}\right)\right) \le 1\right\}. \tag{5}$$

Let $\rho = \rho_1 + r_1$, then we have,

$$\begin{aligned}&\sup_{n,k} M\left(q\left(\frac{a_{nk}+b_{nk}}{\rho}\right)\right)\\ &\le \frac{\rho_1}{\rho_1+r_1}\sup_{n,k} M\left(q\left(\frac{a_{nk}}{\rho_1}\right)\right) + \frac{r_1}{\rho_1+r_1}\sup_{n,k} M\left(q\left(\frac{b_{nk}}{r_1}\right)\right) \le 1.\end{aligned}$$

Since ρ_1 and r_1 are non-negative, so we have

$$\begin{aligned}f(\langle a_{nk}\rangle + \langle b_{nk}\rangle) &= \inf\left\{\rho = \rho_1 + r_1 > 0 :\ M\left(q\left(\frac{a_{nk}+b_{nk}}{\rho}\right)\right) \le 1\right\}\\ &\le \inf\left\{\rho_1 > 0 :\ \sup_{n,k} M\left(q\left(\frac{a_{nk}}{\rho_1}\right)\right) \le 1\right\}\\ &\quad + \inf\left\{r_1 > 0 :\ \sup_{n,k} M\left(q\left(\frac{b_{nk}}{r_1}\right)\right) \le 1\right\}\\ &= f(\langle a_{nk}\rangle) + f(\langle b_{nk}\rangle).\end{aligned} \tag{6}$$

Hence f is a seminorm on $Z(M,q)$ for $Z = {}_2\ell_\infty$, ${}_2c^B$, ${}_2c_0^B$, ${}_2c^R$ and ${}_2c_0^R$. □

Theorem 3. *The spaces ${}_2\ell_\infty(M,q)$ and ${}_2c_0^R(M,q)$ are symmetric where as the spaces $Z(M,q)$, for $Z = {}_2c$, ${}_2c_0$, ${}_2c^B$, ${}_2c_0^B$, ${}_2c^R$ are not symmetric.*

Proof. The space ${}_2\ell_\infty(M,q)$ is symmetric is obvious. We prove it for ${}_2c_0^R(M,q)$.

Let $\langle a_{nk}\rangle \in {}_2c_0^R(M,q)$. Then for a given $\varepsilon > 0$, there exists positive integers k_1, k_2, k_3 such that,

$$M\left(q\left(\frac{a_{nk}}{\rho}\right)\right) < \varepsilon, \qquad \text{for all} \quad n \ge k_1 \quad \text{and all} \quad k \in \mathbb{N}.$$

$$M\left(q\left(\frac{a_{nk}}{\rho}\right)\right) < \varepsilon, \qquad \text{for all} \quad k \ge k_2 \quad \text{and all} \quad n \in \mathbb{N}.$$

and $M\left(q\left(\frac{a_{nk}}{\rho}\right)\right) < \varepsilon$, for all $n \ge k_3$ and $k \ge k_3$.

It is possible to get the same ρ for the three above expressions.

Let $k_0 = \max\{k_1, k_2, k_3\}$. Let $\langle b_{nk}\rangle$ be a rearrangement of $\langle a_{nk}\rangle$. Then we have $a_{ij} = b_{m_i n_j}$ for all $i, j \in \mathbb{N}$.

Let $k_4 = \max\{n_1, n_2, n_3, n_4, \ldots, n_{k_0}, m_1, m_2, m_3, m_4, \ldots, m_{k_0}\}$.

Then we have

$$M\left(q\left(\frac{b_{nk}}{\rho}\right)\right) < \varepsilon, \qquad \text{for all} \quad n \geq k_4 \quad \text{and all} \quad k \in \mathbb{N}.$$

$$M\left(q\left(\frac{b_{nk}}{\rho}\right)\right) < \varepsilon, \qquad \text{for all} \quad k \geq k_4 \quad \text{and all} \quad n \in \mathbb{N}.$$

and

$$M\left(q\left(\frac{b_{nk}}{\rho}\right)\right) < \varepsilon, \qquad \text{for all} \quad n \geq k_4 \quad \text{and} \quad k \geq k_4.$$

Thus $\langle b_{nk}\rangle \in {}_2c_0^R(M,q)$. Hence ${}_2c_0^R(M,q)$ is symmetric space. To show that ${}_2c^R(M,q)$ is not symmetric, consider the following example. □

Example 1. Let $X = C$, $M(x) = x^2$ and define $\langle a_{nk}\rangle$ by

$$a_{nk} = \begin{cases} 1, & \text{for all } k \in \mathbb{N} \text{ and } n = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Then $\langle a_{nk}\rangle \in {}_2c^R(M,q)$. Now consider the rearranged sequence $\langle b_{nk}\rangle$ defined by

$$b_{nk} = \begin{cases} 1, & \text{for all } n = k; \\ 0, & \text{otherwise.} \end{cases}$$

Then $\langle b_{nk}\rangle \notin {}_2c^R(M,q)$. Hence ${}_2c^R(M,q)$ is not symmetric. From this example, it is also clear that the other spaces are not symmetric.

Theorem 4. *The spaces ${}_2c_0(M,q)$, ${}_2c_0^B(M,q)$, ${}_2c_0^R(M,q)$ and ${}_2\ell_\infty(M,q)$ are solid, but the spaces ${}_2c(M,q)$, ${}_2c^B(M,q)$ and ${}_2c^R(M,q)$ are not solid.*

Proof. The spaces $Z(M,q)$, for $Z = {}_2c_0$, ${}_2c_0^B$, ${}_2c_0^R$ and ${}_2\ell_\infty$ are solid follows from the following inequality:

$$M\left(q\left(\frac{\alpha_{nk}a_{nk}}{\rho}\right)\right) \leq M\left(q\left(\frac{a_{nk}}{\rho}\right)\right),$$

for all $n,k \in \mathbb{N}$ and scalars $\langle \alpha_{nk}\rangle$ with $|\alpha_{nk}| \leq 1$, for all $n,k \in \mathbb{N}$. □

To show that ${}_2c(M,q)$, ${}_2c^B(M,q)$ and ${}_2c^R(M,q)$ are not solid, consider the following example.

Example 2. Let $X = C$, $M(x) = x$, for all $x \in [0,\infty)$ and $q(x) = |x|$. Define the sequence $\langle a_{nk}\rangle$ by $a_{nk} = 1$, for all $n,k \in \mathbb{N}$. Consider the sequence $\langle \alpha_{nk}\rangle$ of scalars defined by $\alpha_{nk} = (-1)^{n+k}$, for all $n,k \in \mathbb{N}$. Then $\langle a_{nk}\rangle \in Z(M,q)$, but $\langle \alpha_{nk}a_{nk}\rangle \notin Z(M,q)$, for $Z = {}_2c$, ${}_2c^B$ and ${}_2c^R$.

Hence the spaces $Z(M,q)$ are not solid for $Z = {}_2c$, ${}_2c^B$ and ${}_2c^R$.

Theorem 5. *The spaces $Z(M,q)$ are monotone for $Z = {}_2c_0$, ${}_2c_0^B$, ${}_2c_0^R$ and ${}_2\ell_\infty$ but the spaces $Z(M,q)$ are not monotone for $Z = {}_2c$, ${}_2c^B$ and ${}_2c^R$.*

Proof. The first part is a consequence of the Remark 2 and Theorem 4. For the second part, consider the following example. □

Example 3. Let $X = C$, $M(x) = x$ and $q(x) = |x|$. Consider the sequence $\langle a_{nk}\rangle$, defined by $a_{nk} = 1$, for all $n, k \in \mathbb{N}$.

Now consider its pre-image on the step space E defined by $\langle b_{nk}\rangle \in E$ implies $b_{nk} = 0$, for k even and all $n \in \mathbb{N}$. Then the pre-image of $\langle a_{nk}\rangle \notin Z(M,q)$, but $\langle a_{nk}\rangle \in Z(M,q)$, for $Z = {}_2c$, ${}_2c^B$ and ${}_2c^R$. Hence $Z(M,q)$ is not monotone for $Z = {}_2c$, ${}_2c^B$ and ${}_2c^R$.

Theorem 6. *Let X be a complete seminormed space, then the spaces $Z(M,q)$ for $Z = {}_2\ell_\infty$, ${}_2c^B$, ${}_2c_0^B$, ${}_2c^R$ and ${}_2c_0^R$ are complete seminormed spaces under the seminorm f defined by* (3).

Proof. We prove it for the case ${}_2\ell_\infty(M,q)$ and the other cases can be established following similar techniques.

Let $A^i = \langle a^i_{nk}\rangle$ be a Cauchy sequence in ${}_2\ell_\infty(M,q)$. Let $\varepsilon > 0$ be given and for $r > 0$, choose x_0 fixed such that $M\left(\frac{rx_0}{2}\right) \geq 1$ and there exists $m_0 \in \mathbb{N}$ such that

$$f(\langle a^i_{nk} - a^j_{nk}\rangle) < \frac{\varepsilon}{rx_0}, \qquad \text{for all} \quad i,j \geq m_0.$$

By the definition of seminorm we have

$$M\left(q\left(\frac{a^i_{nk} - a^j_{nk}}{f(a^i_{nk} - a^j_{nk})}\right)\right) \leq 1 \leq M\left(\frac{rx_0}{2}\right), \qquad \text{for all} \quad i,j \geq m_0 \tag{7}$$

$$\implies q(a^i_{nk} - a^j_{nk}) \leq \frac{rx_0}{2} \cdot \frac{\varepsilon}{rx_0} = \frac{\varepsilon}{2} \qquad \text{for all} \quad i,j \geq m_0$$

$\implies \langle a^i_{nk}\rangle$ is a Cauchy sequence in X.

Since X is complete, there exists $a_{nk} \in X$ such that $\lim\limits_{i\to\infty} a^i_{nk} = a_{nk}$, for all $n, k \in \mathbb{N}$.

Since M is continuous, so for all $i \geq m_0$, on taking limit as $j \to \infty$, we have from (7),

$$M\left(q\left(\frac{a^i_{nk} - \lim\limits_{j\to\infty} a^j_{nk}}{\rho}\right)\right) \leq 1 \implies M\left(q\left(\frac{a^i_{nk} - a_{nk}}{\rho}\right)\right) \leq 1.$$

On taking the infimum of such ρ's, we have

$$\inf\left\{\rho > 0: \ M\left(q\left(\frac{a^i_{nk} - a_{nk}}{\rho}\right)\right) \leq 1\right\} < \varepsilon, \qquad \text{for all} \quad i \geq m_0$$

$$\implies \langle a^i_{nk} - a_{nk}\rangle \in {}_2\ell_\infty(M,q), \qquad \text{for all} \quad i \geq m_0.$$

Since ${}_2\ell_\infty(M,q)$ is a linear space, we have for all $i \geq m_0$,

$$\langle a_{nk}\rangle = \langle a^i_{nk}\rangle - \langle a^i_{nk} - a_{nk}\rangle \in {}_2\ell_\infty(M,q).$$

Thus ${}_2\ell_\infty(M,q)$ is a complete space. □

The proof of the following result is a routine work, in view of the techniques used for establishing the above result.

Theorem 7. *The spaces $Z(M, q)$, for $Z = {}_2\ell_\infty$, ${}_2c^B$, ${}_2c_0^B$, ${}_2c^R$ and ${}_2c_0^R$ are K-spaces.*

Since the inclusions $Z(M, q) \subset {}_2\ell_\infty(M, q)$, for $Z = {}_2c^B$, ${}_2c_0^B$, ${}_2c^R$ and ${}_2c_0^R$ are proper, the following result is a consequence of Theorem 6.

Theorem 8. *The spaces $Z(M, q)$, for $Z = {}_2c^B$, ${}_2c_0^B$, ${}_2c^R$ and ${}_2c_0^R$ are nowhere dense subsets of ${}_2\ell_\infty(M, q)$.*

Theorem 9. *Let M_1 and M_2 be Orlicz functions. Then we have*

(i) $Z(M_1, q) \subseteq Z(M_2 \diamond M_1, q)$, *for* $Z = {}_2\ell_\infty$, ${}_2c$, ${}_2c_0$, ${}_2c^B$, ${}_2c_0^B$, ${}_2c^R$ *and* ${}_2c_0^R$.

(ii) $Z(M_1, q) \cap Z(M_2, q) \subseteq Z(M_1 + M_2, q)$, *for* $Z = {}_2\ell_\infty$, ${}_2c$, ${}_2c_0$, ${}_2c^B$, ${}_2c_0^B$, ${}_2c^R$ *and* ${}_2c_0^R$.

(iii) $Z(M_1, q_1) \cap Z(M_1, q_2) \subseteq Z(M_1, q_1 + q_2)$, *for* $Z = {}_2\ell_\infty$, ${}_2c$, ${}_2c_0$, ${}_2c^B$, ${}_2c_0^B$, ${}_2c^R$ *and* ${}_2c_0^R$ *where* q_1, q_2 *are two seminorms on* X.

(iv) *If* q_1 *is stronger than* q_2, *then* $Z(M_1, q_1) \subseteq Z(M_1, q_2)$, *for* $Z = {}_2\ell_\infty$, ${}_2c$, ${}_2c_0$, ${}_2c^B$, ${}_2c_0^B$, ${}_2c^R$ *and* ${}_2c_0^R$.

Proof.

(i) We prove this result for the space ${}_2c^R(M_1, q)$ and the proofs of the other spaces can be established similarly. Let $\langle a_{nk} \rangle \in {}_2c^R(M_1, q)$. Then for a given $\varepsilon > 0$ with $0 < \frac{\varepsilon}{M_2(1)} < 1$, there exists $\rho > 0$ and $n_0, k_0 \in \mathbb{N}$ such that

$$M_1\left(q\left(\frac{a_{nk} - L}{\rho}\right)\right) < \frac{\varepsilon}{M_2(1)}, \qquad \text{for all} \quad n \geq n_0 \quad \text{and} \quad k \geq k_0. \tag{8}$$

$$M_1\left(q\left(\frac{a_{nk} - L_k}{\rho}\right)\right) < \frac{\varepsilon}{M_2(1)}, \qquad \text{for all} \quad n \geq n_0 \quad \text{and all} \quad k \in \mathbb{N}. \tag{9}$$

$$M_1\left(q\left(\frac{a_{nk} - J_n}{\rho}\right)\right) < \frac{\varepsilon}{M_2(1)}, \qquad \text{for all} \quad k \geq k_0 \quad \text{and all} \quad n \in \mathbb{N}. \tag{10}$$

Since $\frac{\varepsilon}{M_2(1)} < 1$, so by Remark 1 and (8), (9), (10) we have,

$$(M_2 \diamond M_1)\left(q\left(\frac{a_{nk} - L}{\rho}\right)\right) < \varepsilon, \qquad \text{for all} \quad n \geq n_0 \quad \text{and} \quad k \geq k_0.$$

$$(M_2 \diamond M_1)\left(q\left(\frac{a_{nk} - L_k}{\rho}\right)\right) < \varepsilon, \qquad \text{for all} \quad n \geq n_0 \quad \text{and all} \quad k \in \mathbb{N}.$$

$$(M_2 \diamond M_1)\left(q\left(\frac{a_{nk} - J_n}{\rho}\right)\right) < \varepsilon, \qquad \text{for all} \quad k \geq k_0 \quad \text{and all} \quad n \in \mathbb{N}.$$

Hence $\langle a_{nk} \rangle \in {}_2c^R(M_2 \diamond M_1, q)$. Thus ${}_2c^R(M_1, q) \subseteq {}_2c^R(M_2 \diamond M_1, q)$.

(ii) We prove this result for the space ${}_2\ell_\infty(M,q)$. The other cases will follow similarly.

Let $\langle a_{nk}\rangle \in {}_2\ell_\infty(M_1,q) \cap {}_2\ell_\infty(M_2,q)$. Then there exists $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\sup_{n,k} M_1\left(q\left(\frac{a_{nk}}{\rho_1}\right)\right) < \infty \qquad \text{and} \qquad \sup_{n,k} M_2\left(q\left(\frac{a_{nk}}{\rho_2}\right)\right) < \infty.$$

Let $\rho = \max\{\rho_1, \rho_2\}$. Then we have

$$\sup_{n,k}(M_1+M_2)\left(q\left(\frac{a_{nk}}{\rho}\right)\right) \le \sup_{n,k} M_1\left(q\left(\frac{a_{nk}}{\rho_1}\right)\right) + \sup_{n,k} M_2\left(q\left(\frac{a_{nk}}{\rho_2}\right)\right) < \infty.$$

Hence $\langle a_{nk}\rangle \in (M_1+M_2,q)$.

Thus ${}_2\ell_\infty(M_1,q) \cap {}_2\ell_\infty(M_2,q) \subseteq {}_2\ell_\infty(M_1+M_2,q)$.

(iii) Proof is easy, so omitted.

(iv) Proof is easy, so omitted. □

The following result is a consequence of Theorem 9(i).

Theorem 10. *Let M be an Orlicz function, then $Z(q) \subseteq Z(M,q)$, for $Z = {}_2\ell_\infty$, ${}_2c$, ${}_2c_0$, ${}_2c^B$, ${}_2c_0^B$, ${}_2c^R$ and ${}_2c_0^R$.*

4. Particular cases

If we take X to be a normed linear space, instead of a seminormed space, then all the results of section 3 will follow directly. The spaces $Z(M, \|\cdot\|)$, for $Z = {}_2\ell_\infty$, ${}_2c^B$, ${}_2c_0^B$, ${}_2c^R$ and ${}_2c_0^R$ will be normed linear spaces, normed by

$$h(\langle a_{nk}\rangle) = \inf\left\{\rho > 0 : \sup_{n,k} M\left(\left\|\frac{a_{nk}}{\rho}\right\|\right) \le 1\right\}.$$

When X will be a Banach space, these spaces will be Banach spaces by the norm h.

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REFERENCES

[1] ALTIN, Y.—ET, M.—TRIPATHY, B. C.: *The sequence space $|\bar{N}_p|(M,r,q,s)$ on seminormed spaces*, Appl. Math. Comput. **154** (2004), 423–430.

[2] BASARIR, M.—SONALCAN, O.: *On some double sequence spaces*, J. Indian Acad. Math. **21** (1999), 193–200.

[3] BROWMICH, T. J. I: *An Introduction to the Theory of Infinite Series*, MacMillan & Co. Ltd. New York, 1965.

[4] COLAK, R.—TURKMENOGLU, A.: *The double sequence spaces $\ell^2_\infty(p)$, $c^2_0(p)$ and $c^2(p)$* (To appear).
[5] HARDY, G. H.: *On the convergence of certain multiple series*, Math. Proc. Cambrigde Philos. Soc. **19** (1917) 86–95.
[6] KRASNOSELKII, M. A.—RUTITSKY, Y. B.: *Convex Function and Orlicz Spaces*, Groningen, Netherlands, 1961.
[7] LINDENSTRAUSS, J.—TZAFRIRI, L.: *On Orlicz sequence spaces*, Israel J. Math. **10** (1971), 379–390.
[8] MORICZ, F.: *Extension of the spaces c and c_0 from single to double sequences*, Acta. Math. Hungar. **57** (1991), 129–136.
[9] MORICZ, F.—RHOADES, B. E.: *Almost convergence of double sequences and strong regularity of summability matrics*, Math. Proc. Cambridge Philos. Soc. **104** (1988), 283–294.
[10] PATTERSON, R. F.: *Analogues of some fundamental theorems of summability theory*, Int. J. Math. Math. Sci. **23** (2000), 1–9.
[11] TRIPATHY B. C.: *Statistically convergent double sequence spaces*, Tamkang J. Math. **34** (2003), 231–237.
[12] TRIPATHY, B. C.: *Generalized difference paranormed statistically convergent sequences defined by Orlicz functions in a locally convex space*, Soochow J. Math. **30** (2004), 431–446.
[13] TRIPATHY, B. C.—MAHANTA, S.: *On a class of generalized lacunary difference sequence spaces defined by Orlicz functions*, Acta Math. Appl. Sinica (English Ser.) **20** (2004), 231–238.
[14] TURKMENOGLU, A.: *Matrix transformation between some classes of double sequences*, J. Inst. Math. Comput. Sci. Math. Ser. **12** (1999), 23–31.

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