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# MORE ON THE *APLE* STATISTIC

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*It is dedicated to Andrej Pázman on the occasion of his 70th birthday*

(*Communicated by Gejza Wimmer*)

ABSTRACT. The exact distribution of the *APLE* statistic for testing spatial dependencies is derived. Examples are given to compare with the more common Moran's $\mathcal{I}$ statistic.



## 1. Introduction

Spatial dependence is commonly measured by the so called Moran's $\mathcal{I}$ statistic (cf. [11]). However, if it can be safely assumed that the data stems from a particular spatial process its use may be inefficient. This is particularly so, if a so called spatial autoregressive (SAR) process generates the data. Then it is well known that $\mathcal{I}$ can not be used as an estimator for the spatial autocorrelation parameter. Therefore Li et al. [10] have suggested the use of the *APLE* statistic, which is based on a least squares estimator of this parameter.

In their paper they provide a wealth of simulations to back up their choice, however they do not compute moments or exact distributions. Also although they introduce the measure as a rival of $\mathcal{I}$ they do not give comparisons to its use.

In this paper, we will derive the exact distribution of the *APLE* statistics following a corresponding derivation for the Moran's $\mathcal{I}$ given in [18] (note that an alternative derivation technique as in [7] could be applied as well), which is presented in Section 1. Section 2 will then give some comparisons with $\mathcal{I}$ for particular example situations.

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Note that indicators as the above can also be used in the experimental design context. G u m p r e c h t et al. [5] for instance employ them for defining a criterion to optimally select sample points. Since the classical optimal design conditions are violated adaptations of alternative techniques such as given in [13] might be useful.

## 2. Theoretical considerations

### 2.1. Moran's $\mathcal{I}$

The Moran's $\mathcal{I}$ statistic is commonly used as an indicator of spatial dependence in a data set. It is defined as a ratio of quadratic forms in the normally distributed regression residuals $\hat{\boldsymbol{\varepsilon}} = (\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T)y = \boldsymbol{M}y$ from a regression of $y$ on $\boldsymbol{X}$, and thus of the same structure as the Durbin-Watson statistic for serial autocorrelation, i.e.

$$\mathcal{I} = \frac{\hat{\boldsymbol{\varepsilon}}^T \cdot \frac{1}{2} \cdot \left(\boldsymbol{V} + \boldsymbol{V}^T\right) \cdot \hat{\boldsymbol{\varepsilon}}}{\hat{\boldsymbol{\varepsilon}}^T \cdot \hat{\boldsymbol{\varepsilon}}}, \tag{1}$$

where $\boldsymbol{V}$ denotes the (usually row-standardized) spatial link matrix.

One of the most commonly used methods to test for spatial independence is to employ the asymptotic normal distribution of $\mathcal{I}$ (derived by [3]) as an approximation, and test the standardized value of the statistic

$$z(\mathcal{I}) = \frac{\mathcal{I}_0 - E[\mathcal{I}]}{\sqrt{V[\mathcal{I}]}} \sim \mathcal{N}(0,1) \tag{2}$$

against a standard normal. It turns out, however, that the normal approximation is not very suitable for small lattices, so that it is advisable to compute the exact distribution of $\mathcal{I}$ under the null, which was derived independently in [18] and [7]. A comprehensive exposition of these derivations and corresponding R-code (which is now implemented in the package spdep, see [1]) can be found in the master thesis by R e d e r [15].

### 2.2. The *APLE*

The Moran's $\mathcal{I}$ test statistic measures the intensity of spatial autocorrelation in a spatial process but not directly the spatial autocorrelation level $\rho$. When we assume that a Gaussian simultaneous spatial autoregressive (SAR) process, i.e. $\boldsymbol{\varepsilon} = \rho\boldsymbol{V} \cdot \boldsymbol{\varepsilon} + \boldsymbol{\eta}$ with $\boldsymbol{\eta} \sim \mathcal{N}(\boldsymbol{0}, \sigma^2 \cdot \boldsymbol{I})$, which yields $\boldsymbol{\Omega}^{\frac{1}{2}} = \left(\boldsymbol{I} - \rho\boldsymbol{V}^T\right)^{-1}$, without loss of generality, generates the data, O r d [12] suggested to use the least squares estimator $\hat{\rho}$ instead, albeit it being biased and inconsistent. Using it as

a basis Li et al. [10] have further suggested the approximate profile-likelihood estimator $APLE$

$$APLE = \frac{\hat{\boldsymbol{\varepsilon}}^T \cdot \frac{1}{2} \cdot \left( \boldsymbol{V} + \boldsymbol{V}^T \right) \cdot \hat{\boldsymbol{\varepsilon}}}{\hat{\boldsymbol{\varepsilon}}^T ( \boldsymbol{V}^T \boldsymbol{V} + v^T v \boldsymbol{I}/n) \cdot \hat{\boldsymbol{\varepsilon}}}, \tag{3}$$

where $v$ is the vector of eigenvalues of $\boldsymbol{V}$ as a competitor for $\mathcal{I}$ in tests for spatial independence.

### 2.3. Normal approximation

Since the term $v^T v \boldsymbol{I}/n$ plays no role asymptotically and $\boldsymbol{V}$ is fixed, one may expect to safely employ the normal approximation under the usual conditions (cf. [17]). However, for this purpose one needs to evaluate the first two moments of the $APLE$ under the assumption of spatial independence, which is not as simple as those for the $\mathcal{I}$ (as first given in [6]).

Here the derivation is similar as to the moments of Moran's $\mathcal{I}$ under the influence of a spatial process (cf. [16]) and we may employ his formulae with minor adaptations, since the regression residuals $\hat{\boldsymbol{\varepsilon}}$ are here normally distributed with covariance matrix $\sigma^2 \boldsymbol{M}$. The expected values of the random errors $\boldsymbol{\varepsilon}$ and the expectation of the regression residuals $\hat{\boldsymbol{\varepsilon}}$ are zero, which is eventually important as it leads to central $\chi^2$-distributed variables. Let us then define

$$\begin{aligned} \boldsymbol{A} &\equiv \boldsymbol{M} \cdot \frac{1}{2} \cdot \left( \boldsymbol{V} + \boldsymbol{V}^T \right) \cdot \boldsymbol{M} \\ \boldsymbol{B} &\equiv \boldsymbol{M} \cdot \left( \boldsymbol{V}^T \boldsymbol{V} + v^T v \boldsymbol{I}/n \right) \cdot \boldsymbol{M} \end{aligned}$$

with $\beta_i$ being the eigenvalues of $\boldsymbol{B}$ and $\boldsymbol{P}$ a $n \times n$ matrix whose columns are the normalized eigenvectors of $\boldsymbol{B}$. The expectation of the $APLE$ under the assumption of spatial independence is then given by

$$E\left[APLE\right] = \int_0^\infty \prod_{i=1}^{n-k} \left(1 + 2 \cdot \beta_i \cdot t\right)^{-\frac{1}{2}} \cdot \sum_{i=1}^{n-k} \frac{h_{ii}}{1 + 2 \cdot \beta_i \cdot t} \, \mathrm{d}t$$

and the second moment of the $APLE$ can be written as

$$E\left[APLE^2\right] = \int_0^\infty \prod_{i=1}^{n-k} \left(1 + 2 \cdot \beta_i \cdot t\right)^{-\frac{1}{2}} \cdot \sum_{i=1}^{n-k} \sum_{j=1}^{n-k} \frac{(h_{ii} \cdot h_{jj} + 2 \cdot h_{ij}^2) \cdot t}{(1 + 2 \cdot \beta_i \cdot t) \cdot (1 + 2 \cdot \beta_j \cdot t)} \, \mathrm{d}t$$

where the $h_{ij}$ are the elements of the matrix $\boldsymbol{P}^T \cdot \boldsymbol{A} \cdot \boldsymbol{P}$.

### 2.4. The exact distribution

For the derivation of the distribution of the $APLE$ a similar train of thought can be followed and for a specific observed value $APLE_0$ it can be written as

$$\begin{aligned} &F\left(APLE_0\right) = \\ =&P\left(\frac{\boldsymbol{\varepsilon}^T\cdot\boldsymbol{M}\cdot\frac{1}{2}\cdot\left(\boldsymbol{V}+\boldsymbol{V}^T\right)\cdot\boldsymbol{M}\cdot\boldsymbol{\varepsilon}}{\boldsymbol{\varepsilon}^T\cdot\boldsymbol{M}\cdot\left(\boldsymbol{V}^T\boldsymbol{V}+v^Tv\boldsymbol{I}/n\right)\cdot\boldsymbol{M}\cdot\boldsymbol{\varepsilon}}\leq APLE_0\right) \qquad (4)\\ =&P\left(\boldsymbol{\varepsilon}^T\cdot\boldsymbol{M}\left[\frac{1}{2}\cdot\left(\boldsymbol{V}+\boldsymbol{V}^T\right)-APLE_0\left(\boldsymbol{V}^T\boldsymbol{V}+\frac{v^Tv\boldsymbol{I}}{n}\right)\right]\boldsymbol{M}\cdot\boldsymbol{\varepsilon}\leq 0\right). \qquad (5)\end{aligned}$$

By the Spectral Decomposition Theorem

$$\boldsymbol{L}\equiv\boldsymbol{M}\cdot\left[\frac{1}{2}\cdot\left(\boldsymbol{V}+\boldsymbol{V}^T\right)-APLE_0(\boldsymbol{V}^T\boldsymbol{V}+v^Tv\boldsymbol{I}/n)\right]\cdot\boldsymbol{M} \qquad (6)$$

(note that $\boldsymbol{L}$ is symmetric) can be written as $\boldsymbol{L}=\boldsymbol{H}^T\cdot\boldsymbol{\Lambda}\cdot\boldsymbol{H}$, where $\boldsymbol{H}$ is the matrix of the normalized eigenvectors and $\boldsymbol{\Lambda}=\operatorname{diag}(\lambda_1,\ldots\lambda_n)$ is the diagonal eigenvalue matrix of $\boldsymbol{L}$ given in equation (6).

From matrix algebra it is well known that here is an orthogonal matrix $\boldsymbol{K}$ such that

$$\boldsymbol{K}^T\cdot\boldsymbol{M}\cdot\boldsymbol{K}=\begin{pmatrix}\boldsymbol{I}_{n-k} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}_{k\times k}\end{pmatrix}.$$

Now that because the projection matrix $\boldsymbol{M}$ is not of full rank only $n-k$ eigenvalues of $\boldsymbol{B}$ are non zero, which allow us some simplification for quicker calculation. Recall $\operatorname{rank}(\boldsymbol{M})=n-k$ and $\operatorname{rank}(\boldsymbol{X})=k$. Defining $\boldsymbol{A}\equiv\frac{1}{2}\cdot\left(\boldsymbol{V}+\boldsymbol{V}^T\right)$ we apply the orthogonal matrix $\boldsymbol{K}$ simultaneously onto the kernel of the numerator of the APLE in (3). Additionally applying $\boldsymbol{I}=\boldsymbol{K}\cdot\boldsymbol{K}^T$ we get

$$\begin{aligned}\boldsymbol{K}^T\cdot\boldsymbol{M}\cdot\boldsymbol{A}\cdot\boldsymbol{M}\cdot\boldsymbol{K}&=\boldsymbol{K}^T\cdot\boldsymbol{M}\cdot\boldsymbol{K}\cdot\boldsymbol{K}^T\cdot\boldsymbol{A}\cdot\boldsymbol{K}\cdot\boldsymbol{K}^T\cdot\boldsymbol{M}\cdot\boldsymbol{K}\\ &=\begin{pmatrix}\boldsymbol{I}_{n-k} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}_{k\times k}\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{B}_1 & \boldsymbol{B}_2\\ \boldsymbol{B}_3 & \boldsymbol{B}_4\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{I}_{n-k} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}_{k\times k}\end{pmatrix}\\ &=\begin{pmatrix}\boldsymbol{B}_1 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}_{k\times k}\end{pmatrix}\end{aligned}$$

where $\begin{pmatrix}\boldsymbol{B}_1 & \boldsymbol{B}_2\\ \boldsymbol{B}_3 & \boldsymbol{B}_4\end{pmatrix}$ is the appropriate partition of the symmetric matrix $\boldsymbol{K}^T\cdot\boldsymbol{A}\cdot\boldsymbol{K}$.

Let $\boldsymbol{N}_1$ be an orthogonal matrix diagonalizing $\boldsymbol{B}_1$

$$\boldsymbol{N}_1^T\cdot\boldsymbol{B}_1\cdot\boldsymbol{N}_1=\begin{pmatrix}\gamma_1 & & & \boldsymbol{0}\\ & \gamma_2 & & \\ & & \ddots & \\ \boldsymbol{0} & & & \gamma_{n-k}\end{pmatrix}$$

the matrix $\boldsymbol{N} = \begin{pmatrix} \boldsymbol{N}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_k \end{pmatrix}$ is also orthogonal, so that the product $\boldsymbol{H} \equiv \boldsymbol{K} \cdot \boldsymbol{N}$ is again orthogonal because $\boldsymbol{H} \cdot \boldsymbol{H}^T = \boldsymbol{K} \cdot \boldsymbol{N} \cdot \boldsymbol{N}^T \cdot \boldsymbol{K}^T = \boldsymbol{K} \cdot \boldsymbol{I} \cdot \boldsymbol{K}^T = \boldsymbol{I}$.

Now applying $\boldsymbol{H}$ simultaneously onto the kernel of the numerator of the APLE we get

$$\begin{aligned} \boldsymbol{\Gamma}_A &= \boldsymbol{H}^T \cdot \boldsymbol{M} \cdot \boldsymbol{A} \cdot \boldsymbol{M} \cdot \boldsymbol{H} \\ &= \boldsymbol{H}^T \cdot \boldsymbol{M} \cdot \boldsymbol{K} \cdot \boldsymbol{K}^T \cdot \boldsymbol{A} \cdot \boldsymbol{K} \cdot \boldsymbol{K}^T \cdot \boldsymbol{M} \cdot \boldsymbol{H} \\ &= \boldsymbol{N}^T \cdot \begin{pmatrix} \boldsymbol{I}_{n-k} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0}_{k\times k} \end{pmatrix} \cdot \boldsymbol{N} \\ &= \begin{pmatrix} \gamma_{A1} & & & & \boldsymbol{0} \\ & \gamma_{A2} & & & \\ & & \ddots & & \\ & & & \gamma_{An-k} & \\ \boldsymbol{0} & & & & \boldsymbol{I}_{k\times k} \end{pmatrix} \end{aligned}$$

and analogously for the denominator $\boldsymbol{\Gamma}_B = \boldsymbol{H}^T \cdot \boldsymbol{M} \cdot \boldsymbol{B} \cdot \boldsymbol{M} \cdot \boldsymbol{H}$.

So by applying the orthogonal transformation $\boldsymbol{\varepsilon} = \boldsymbol{H} \cdot \boldsymbol{\eta}$, the APLE can be written as

$$APLE = \frac{\boldsymbol{\eta}^T \cdot \boldsymbol{H}^T \cdot \boldsymbol{M} \cdot \frac{1}{2} \cdot \left(\boldsymbol{V} + \boldsymbol{V}^T\right) \cdot \boldsymbol{M} \cdot \boldsymbol{H} \cdot \boldsymbol{\eta}}{\boldsymbol{\eta}^T \cdot \boldsymbol{H}^T \cdot \boldsymbol{M} \cdot (\boldsymbol{V}^T \boldsymbol{V} + v^T v \boldsymbol{I}/n) \cdot \boldsymbol{M} \cdot \boldsymbol{H} \cdot \boldsymbol{\eta}} = \frac{\boldsymbol{\eta}^T \cdot \boldsymbol{\Gamma}_A \cdot \boldsymbol{\eta}}{\boldsymbol{\eta}^T \cdot \boldsymbol{\Gamma}_B \cdot \boldsymbol{\eta}}$$

hence

$$APLE = \frac{\sum_{i=1}^{n-k} \gamma_{Ai} \cdot \eta_i^2}{\sum_{i=1}^{n-k} \gamma_{Bi} \cdot \eta_i^2} \tag{7}$$

where $\boldsymbol{\Gamma}_A = \mathrm{diag}(\gamma_1, \ldots, \gamma_{n-k}, 0, \ldots, 0)$ is the diagonal matrix of the eigenvalues of the matrix $\boldsymbol{M} \cdot \frac{1}{2} \cdot \left(\boldsymbol{V} + \boldsymbol{V}^T\right) \cdot \boldsymbol{M}$ and $\boldsymbol{\Gamma}_B$ respectively.

Thus the distribution of the APLE under spatial independence is given by

$$F\left(APLE_0\right) = P\left(\frac{\sum_{i=1}^{n-k} \gamma_{Ai} \cdot \eta_i^2}{\sum_{i=1}^{n-k} \gamma_{Bi} \cdot \eta_i^2} \leq APLE_0\right) \tag{8}$$

$$= P\left(\sum_{i=1}^{n-k} \left(\gamma_{Ai} - APLE_0 \gamma_{Bi}\right) \cdot \eta_i^2 \leq 0\right). \tag{9}$$

Because the random error vector $\boldsymbol{\varepsilon}$ belongs to the class of the spherically symmetric distributions, the orthogonal transformation $\boldsymbol{\eta} \equiv \boldsymbol{H} \cdot \boldsymbol{\varepsilon}$ is again independent normally distributed with $\boldsymbol{\eta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I})$. Thus we are able to use I m h o f's formula (see below) since $\sum_{i=1}^{n-k} (\gamma_i - APLE_0 \gamma_{Bi}) \cdot \eta_i^2$ is a weighted sum of $\chi_1^2$-distributed variables.

**Imhof's Formula** ([8]):
*The distribution function $F(y)$ of the weighted sum of independent central $\chi^2$-distributed variables is given by*

$$F(y) = P(Y \leq y) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{1}{u} \cdot \sin\left(\boldsymbol{\Theta}(u)\right) \cdot \xi(u) \, \mathrm{d}u.$$

*Where $X_1, X_2, \ldots, X_n$ are independent $\chi_1^2$-distributed random variables, with the weights $\lambda_1, \lambda_2 \ldots, \lambda_n \in \mathbb{R}$. Thus the weighted sum $Y = \lambda_1 \cdot X_1 + \lambda_2 \cdot X_2 + \cdots + \lambda_n \cdot X_n$.*

*The two functions $\boldsymbol{\Theta}(u)$ and $\xi(u)$ are given by*

$$\boldsymbol{\Theta}(u) = \frac{1}{2} \sum_{j=1}^{n} \arctan(u\lambda_j) - \frac{1}{2} u \, y$$

$$\xi(u) = \prod_{j=1}^{n} (1 + u^2 \lambda_j^2)^{-\frac{1}{4}}.$$

Note that all zero eigenvalues can be ignored, and because of $y = 0$, the term $-\frac{1}{2} u \cdot y$ in Imhof's formula is irrelevant for our purposes.

Another way is the direct evaluation of the complex-valued characteristic function of a weighted sum of $\chi^2$-distributed variables. It has not succeeded in practice, because the calculation is not easy to implement and the approach above with real-valued integration is much easier to handle. Here, the solution of the integral in Imhof's formula can be approximated by numerical integration. The behavior of the improper integral at $u = 0$ and at $u \to \infty$ have to be considered especially, yielding starting and truncation values respectively.

As a side issue it would be quite useful to know more about the feasible range of the distribution of the APLE. The ratio of the quadratic form

$$\frac{\boldsymbol{\varepsilon}^T \cdot \boldsymbol{A} \cdot \boldsymbol{\varepsilon}}{\boldsymbol{\varepsilon}^T \cdot \boldsymbol{B} \cdot \boldsymbol{\varepsilon}}$$

for a matrix $\boldsymbol{B}$ with full rank and $\varepsilon_i$ i.i.d. normal distributed is bound by the minimum and maximum eigenvalues of $\boldsymbol{B}^{-1} \cdot \boldsymbol{A}$, see [9]. However, it might be useful to find a characterization of these bounds directly in terms of $\gamma_{Ai}$ and $\gamma_{Bi}$ along the lines given in [4], but this is a future issue.

## 3. Examples

To back up their suggestion of APLE as a spatial dependence measure L i et al. [10] provide several simulations, contrasting its performance with the theoretically superior, but computationally costly maximum likelihood estimator of $\rho$. Although they have motivated their paper in providing an alternative to $\mathcal{I}$, however, they do not provide any direct comparisons. Also they never investigated robustness issues, neither with respect to other data generating processes nor with respect to other designs than regular grids. In this section we attempt to fill these gaps: all our simulations are based on 2000 replications.

We have first calculated the performance of the $\mathcal{I}$, the $APLE$ and the ML-estimators for a spatial autoregressive and a spatial lag process respectively. The latter was for robustness comparison purposes, but did not yield essentially different results and is thus not reported here. From Figure 1, which displays the resulting distributions assuming a regular $4\times4$ and $30\times30$ grid respectively, it is clear that under independence, $\mathcal{I}$ has the smallest variance. Also its distribution is much closer to the normal also for small numbers of observations.

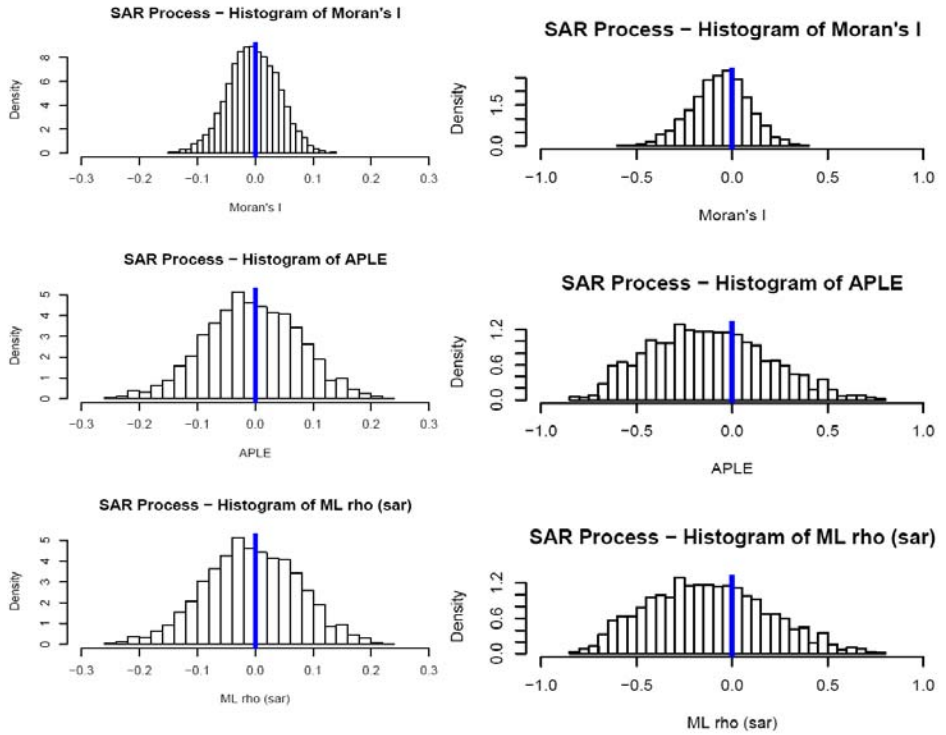


FIGURE 1: Various distributions for a $4\times4$ grid (left column) and a $30\times30$ grid (right column) with $\rho=0$

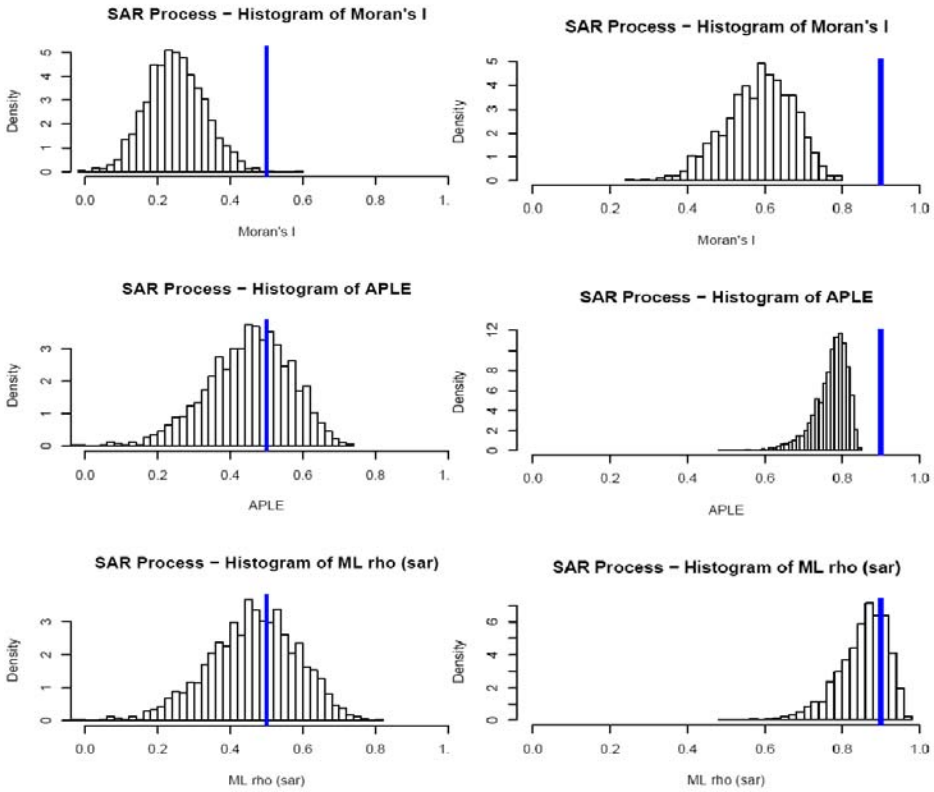


FIGURE 2: Various distributions for a $10 \times 10$ grid with $\rho = 0.5$ (left) and $\rho = 0.9$ (right)

The left panels of Figure 2 now show the behaviour of the statistics, when a spatial SAR process with $\rho = 0.5$ is generating the data. Evidently, here the $\mathcal{I}$ is strongly biased and not very useful compared with its competitors. Clearly, this is where L i et al. [10] draw their support for the APLE from. However, looking at the right panel of Figure 2, showing the situation for $\rho = 0.9$ the APLE is quite off the mark as well, whereas the ML-estimator fares well. Nonrobustness with respect to the wrong choice of the model seems to be the only issue there (but only in the extreme case).

Let us now investigate the situation, when we do not have a regular grid (as is quite usual in practice). As a somewhat extreme showcase example consider a set of fourteen maximally connected planar spatial structures called the B-series with a fixed number $n = 8$ nodes and an overall connectivity $D = 36$, introduced in [2]. We will look at the shape of the distribution of Moran's $\mathcal{I}$ and the $APLE$ for a subset, namely the *B07* and *B14-Structure* shown in Figures 3 and 4 using the C-coding scheme.

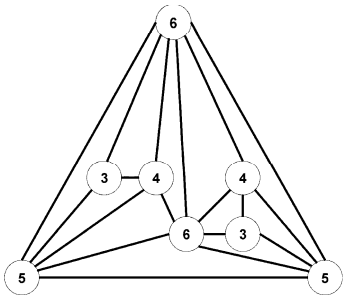


FIGURE 3: B07-Structure

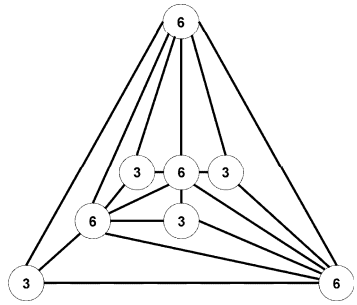


FIGURE 4: B14-Structure

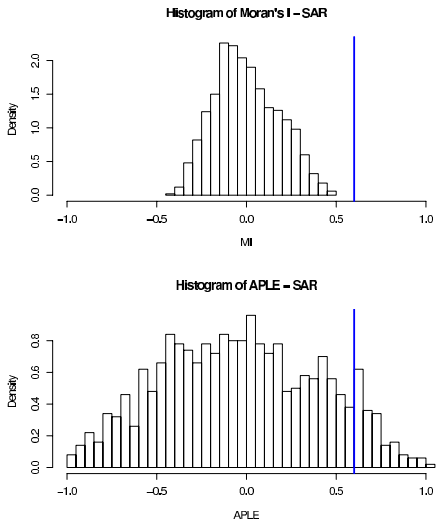


FIGURE 5: B07-Structure

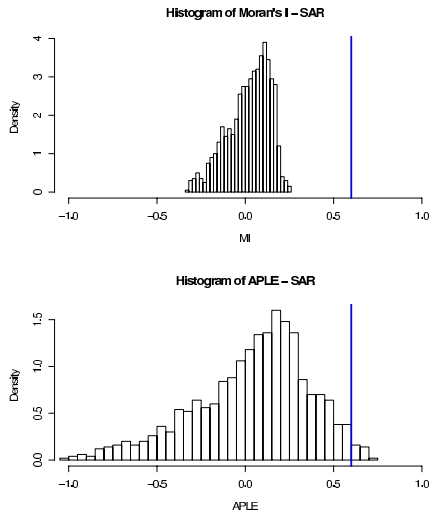


FIGURE 6: B14-Structure

Deriving the eigenvalues lead to $\{-0.444, -0.379, -0.340, -0.111, -0.077, -0.065, 0.418\}$ for the *B07-Structure* and to $\{-0.360, -0.360, -0.360, -0.333, 0.137, 0.137, 0.137\}$ for the *B14-Structure*. The corresponding histograms for $\rho = 0.6$ can be found in Figures 5 and 6, from which it is evident, that the $APLE$ looses its advantage completely over the $\mathcal{I}$. Not only is it also biased, but exhibits higher variance. We thus do not consider the case strong enough for replacing a well established test statistic.

## Acknowledgments

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## Appendix: R-code

The R-scripts for examples are given below. Further code can be found in [15].

```
rm(list=ls(all=TRUE)) source("routines\\DA Packages.R")
source("routines\\DA Structures.R") source("routines\\DA Maps.R")

# Structures +++++++++++++++++++++++++++++++++++++++++++++++++++++
    st="W"
    area.w <- nb2listw(cell2nb(4, 4, torus=TRUE), style=st);
        title="4 x 4 - grid"
    area.w <- nb2listw(cell2nb(10, 10, torus=TRUE), style=st);
        title="10 x 10 - grid"
    area.w <- nb2listw(cell2nb(30, 30, torus=FALSE), style=st);
        title="30 x 30 - grid"
    area.w <- matb2listw(B07.mat,style=st);title="B07 Structure"
    area.w <- matb2listw(B14.mat,style=st);title="B14 Structure"

# SAR und MA-Process simulieren
++++++++++++++++++++++++++++++++++++
    r=0.9
    lim=c(0,1.0)
    sim=2000
    par(mfrow=c(4,2))
    time1=proc.time()[3]
    n=length(area.w$neighbour)
    V <- listw2mat(area.w)
    lambda=eigen(V)$values
    N=diag(as.real(t(lambda)%*%lambda/n),n)
    MI1=c();MI2=c()
    APLE1=c();APLE2=c()
    MLsar1=c();MLsar2=c()
    MLlag1=c();MLlag2=c()
    for(i in 1:sim){
        y = matrix(rnorm(n))
```

```
    # SAR-Process
        WOmega1=invIrW(area.w, rho=r)
        x1=WOmega1 %*% lm(y~1)$residuals
    # Moran's I
        MI1 <- c(MI1,n/sum(V)*(t(x1)%*%(0.5*(V+t(V)))%*%x1)/
            (t(x1)%*%x1))
    # APLE1
        APLE1 <- c(APLE1,(t(x1)%*%(0.5*(V+t(V)))%*%x1)/
            (t(x1)%*%(t(V)%*%V+N)%*%x1))
    # ML-SAR
        MLsar1=c(MLsar1,errorsarlm(x1~1,listw=area.w,
            na.action=na.exclude,zero.policy=TRUE)$lambda)
    # ML-Lag
        MLlag1=c(MLlag1,lagsarlm(x1~1,listw=area.w,
            na.action=na.exclude,zero.policy=TRUE)$rho)
    # MA-Process
        WOmega2=diag(n)+r*V
        x2=WOmega2 %*% lm(y~1)$residuals
    # Moran's I
        MI2 <- c(MI2,n/sum(V)*(t(x2)%*%(0.5*(V+t(V)))%*%x2)/
            (t(x2)%*%x2))
    # APLE
        APLE2 <- c(APLE2,(t(x2)%*%(0.5*(V+t(V)))%*%x2)/
            (t(x2)%*%(t(V)%*%V+N)%*%x2))
    # ML-SAR
        MLsar2=c(MLsar2,errorsarlm(x2~1,listw=area.w,
            na.action=na.exclude,zero.policy=TRUE)$lambda)
    # ML-Lag
        MLlag2=c(MLlag2,lagsarlm(x2~1,listw=area.w,
            na.action=na.exclude,zero.policy=TRUE)$rho)
    }
time2=proc.time()[3]
time2-time1
b=30
hist(MI1,breaks=b,xlim=lim,freq=FALSE,xlab="Moran's I",
    main="SAR Process - Histogram of Moran's I");
    abline(v=r,col="blue",lty=1,lwd=3)
hist(MI2,breaks=b,xlim=lim,freq=FALSE,xlab="Moran's I",
    main="MA Process - Histogram of Moran's I");
    abline(v=r,col="blue",lty=1,lwd=3)
hist(APLE1,breaks=b,xlim=lim,freq=FALSE,xlab="APLE",
```

```
        main="SAR Process - Histogram of APLE");
        abline(v=r,col="blue",lty=1,lwd=3)
    hist(APLE2,breaks=b,xlim=lim,freq=FALSE,xlab="APLE",
        main="MA Process - Histogram of APLE");
        abline(v=r,col="blue",lty=1,lwd=3)
    hist(MLsar1,breaks=b,xlim=lim,freq=FALSE,xlab="ML rho (sar)",
        main="SAR Process - Histogram of ML rho (sar)");
        abline(v=r,col="blue",lty=1,lwd=3)
    hist(MLsar2,breaks=b,xlim=lim,freq=FALSE,xlab="ML rho (sar)",
        main="MA Process - Histogram of ML rho (sar)");
        abline(v=r,col="blue",lty=1,lwd=3)
    hist(MLlag1,breaks=b,xlim=lim,freq=FALSE,xlab="ML rho (lag)",
        main="SAR Process - Histogram of ML rho (lag)");
        abline(v=r,col="blue",lty=1,lwd=3)
    hist(MLlag2,breaks=b,xlim=lim,freq=FALSE,xlab="ML rho (lag)",
        main="MA Process - Histogram of ML rho (ag)");
        abline(v=r,col="blue",lty=1,lwd=3)
    savePlot(file=paste("10x10_Hist_rho_0_",r*10,sep=""),"pdf")

# SAR-Process simulieren ++++++++++++++++++++++++++++++++++++++
    r=0.6
    lim=c(-1,1)
    sim=1000
    par(mfrow=c(2,1))
    time1=proc.time()[3]
    n=length(area.w$neighbour)
    V <- listw2mat(area.w)
    lambda=eigen(V)$values
    N=diag(as.real(t(lambda)%*%lambda/n),n)
    MI=c()
    APLE=c()
    for(i in 1:sim){
        y = matrix(rnorm(n))
        # SAR-Process
            WOmega=invIrW(area.w, rho=r)
            x=WOmega %*% lm(y~1)$residuals
        # Moran's I
            MI <- c(MI,n/sum(V)*(t(x)%*%(0.5*(V+t(V)))%*%x)/
            (t(x)%*%x))
        # APLE
            APLE <- c(APLE,(t(x)%*%(0.5*(V+t(V)))%*%x)/
```

```
        (t(x)%*%(t(V)%*%V+N)%*%x))
    }
time2=proc.time()[3]
time2-time1
b=30
hist(MI,breaks=b,xlim=lim,freq=FALSE,
    main="Histogram of Moran's I - SAR");
    abline(v=r,col="blue",lty=1,lwd=3)
hist(APLE,breaks=b,xlim=lim,freq=FALSE,
    main="Histogram of APLE - SAR");
    abline(v=r,col="blue",lty=1,lwd=3)
savePlot(file=paste("B14_SAR_Hist_rho_0_",r*10,sep=""),"pdf")
```

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