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ON THE LIÉNARD SYSTEM WITH TWO ISOCLINES

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ABSTRACT. In this paper, the non-existence of limit cycles of a Liénard system $\dot{x} = y - F(x)$, $\dot{y} = -g(x)$ is discussed. By using the transformation $y = z + \varphi(x)$ to the system, the new system has two special isoclines. We call the curves Vertical isocline or Horizontal isocline, respectively. It shall be shown that the existence of these isoclines play an important role in the non-existence of limit cycles of the system. The results are applied to many examples, and the known results are improved in certain cases.

1. Motivation

In this paper, the non-existence of the limit cycles of a Liénard system is discussed by changing the system to a new system. Usually, a Liénard system in the form of Type (I):

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x), \end{cases} \tag{L1}$$

or Type (II):

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x)y - g(x), \end{cases} \tag{L2}$$

is used for the existence and the non-existence of limit cycles, where $F(x) = \int f(x)\,\mathrm{d}x$. In [3], it was shown that the existence of the algebraic invariant curves for the Type (II) was deeply concerned with the non-existence of the limit cycles. It shall be given in Corollary 2.1 that the algebraic invariant curve

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is made by the isoclines of the system. In [9], the curve $y = F(x)$ for the Type (I) was used for proving the existence of the periodic solutions and the oscillation theory.

It shall be seen from our results in §2 that the existence of the proper isocline for the system plays an important role for investigating exactly the qualitative property of the orbits of the system. In Type (I) or (II), there exists a vertical isocline $y = F(x)$ or $y = 0$, respectively.

The key for our proofs is to use the transformation $y = z + \varphi(x)$ for Type (I). Two types of isoclines, namely, "Vertical Isocline $z = E(x)$" and "Horizontal Isocline $z = H(x)$" are created when the transformation is applied to some of the isoclines of Liénard system in Type (I). The main idea of the proof is that all solution orbits of the system remain in the domain $\{(x, z) : \ x \geq 0, \ H(x) \leq z \leq E(x)\}$. This is occured when the two types of isoclines are used.

The proposing method in this paper is applied to many examples (for instance [11]), the results in certain cases([1]) show that the condition of parameters are generalized.

Throughout this paper we assume the following conditions for System (L1): $F(x)$ and $g(x)$ satisfy smoothness conditions for the uniqueness of the solutions of the initial value problem and

[C1] there exists a number $p_1 \leq 0$ such that $xg(x) \geq 0$ for $x \in [p_1, +\infty)$,

[C2] there exists a number $a_1 < 0$ such that $xF(x) \geq 0$ for $x \in [a_1, +\infty)$.

Then we note that System (L1) has two equilibrium points O $(0, 0)$ and A $(p_1, F(p_1))$ and the Poincaré indices of these points are $\mathrm{Ind}(O) = +1$ and $\mathrm{Ind}(A) = -1$, respectively. See the textbook of [12] for the Poincaré index.

The following tools have been known as the methods to prove the non-existence of the limit cycles of a Liénard system.

[I] the Bendixson-Dulac theorem ([12] or [3])

[II] the existence of algebraic invariant curves ([4])

[III] the use of a plane curve $(F(x), G(x))$ ([10] or [5]),

where $G(x) = \int_0^x g(\xi)\, \mathrm{d}\xi$.

For an example, we shall consider applying these methods to the following system.

Example. (Motivation system)

$$F(x) = x^4 + \beta x^3 \quad \text{and} \quad g(x) = x^2 + \alpha x, \tag{M}$$

where $\alpha > 0$ and $\beta > 0$. The equilibrium points of System (M) are O $(0, 0)$ and A $(-\alpha, \alpha^4 - \beta\alpha^3)$, and the indices are $\mathrm{Ind}(O) = +1$ and $\mathrm{Ind}(A) = -1$, respectively. The mentioned methods above have not powers for the non-existence of the limit cycles of System (M).

We shall show this fact below.

[I] We must decide the function $B(x, y) \in C^1$ and the domain G including the origin such that $\partial\{B(x, y)(y - F(x))\}/\partial x - \partial\{B(x, y)g(x)\}/\partial y \neq 0$ in G. It is very difficult to construct the function $B(x, y)$ and the domain D.

[II] It follows from [8] that there doesn't exist the algebraic invariant curve for System (M). Thus, we cannot apply the given method in [4] to the system.

[III] If $0 \leq \alpha \leq \beta$, we can apply this method to the system. In fact, the curve $(F(x), G(x))$ has an intersecting point with itself for $x = u_1 < -\beta$ and $x = u_2 > 0$. Thus, the closed orbit of the system does not exist on the domain $D = \{(x, y) : \ x \geq -\alpha\}$. (This fact is easily proved by the method in [4], too.). On the other hand, if the closed orbit C of the system meets the line $x = -\alpha$, then it follows from the vector fields on the line $x = -\alpha$ that the equilibrium point A must be in the inside of C. Thus the index of C equals to zero. This contradicts the Poincaré index theorem. Thus, it follows that if $0 \leq \alpha \leq \beta$, System (M) has no limit cycles. But, if $\alpha > \beta$, then the method [III] has not power. In fact, the curve $(F(x), G(x))$ has an intersecting point with itself for the case of either

(i) $x = u_1 \leq -\alpha$, $x = u_2 > 0$ or

(ii) $x = u_1 > -\alpha$, $x = u_2 > 0$.

In the case (i), the non-existence of the closed orbit is proved by the same discussion as the case $\alpha \leq \beta$. However, in the case (ii), we cannot prove the non-existence of the limit cycle of System (M) by using the above methods.

It shall be shown in §4 that the introduced method in §2 is valid for System (M).

2. Main theorems

We set $E(x) = F(x) - \varphi(x)$ and define the function $H(x)$ as

$$\varphi'(x)H(x) = \varphi'(x)E(x) - g(x).$$

Using the transformation $y = z + \varphi(x)$ to System (L1), we have a new Liénard system in the form of Type (III):

$$\begin{cases} \dot{x} = z - E(x) \\ \dot{y} = -\varphi'(x)\{z - H(x)\}, \end{cases} \tag{N}$$

where the function $\varphi(x)$ satisfies the following condition

[C3] there exists a C^1-function $\varphi(x)$ such that $\varphi(0) = 0$ and $\varphi'(x) > 0$ for $x > 0$.

Let $a_2 > 0$ be a number satisfying $G(a_1) = G(a_2)$ and set

$$P_{a_2} = -\min_{x \in [0, a_2]} \{-\varphi(x) - \sqrt{2(G(a_2) - G(x))}\}.$$

Our main results are the following

Theorem A. *Suppose that the condition*
[C4] $\varphi'(a_2)F(a_2) \geq g(a_2)$ *and* $P_{a_2} \leq \inf_{x\in[a_2,+\infty)} H(x)$.
Then System (L1) *has no limit cycles.*

Theorem B. *Suppose that the condition*
[C5] $\varphi'(a_2)F(a_2) < g(a_2)$ *and* $P_{a_2} \leq \inf_{x\in[0,+\infty)} H(x)$.
Then System (L1) *has no limit cycles.*

The following results are given by the same reasons as are shown in the proofs of the above theorems.

Corollary 2.1. *In stead of* [C3], *suppose that the condition*
[C6] *there exist a function* $\varphi'(x)$ *such that* $\varphi(0) = 0$ *and* $H(x) = 0$.
Then System (L1) *has no limit cycles.*

We must notice that the condition [C6] is equivalent to the condition given in [H4]. Namely, if the condition [C6] is satisfied, it follows that System (L1) has the algebraic invariant curves in the form $y = \varphi(x)$ containing the equilibrium point O.

Corollary 2.2. *Suppose the condition* [C4] *or* [C5] *in the above theorems. Then System* (L1) *has no heteroclinic orbits.*

3. Proofs

We set $W(x, z) = (1/2)(z + \varphi(x))^2 + G(x)$. Consider a closed plane curve Γ defined by the equation $W(x, z) = G(a_2)$. Γ is a closed curve that surrounds the origin. Let Ω be the region surrouded by Γ. Let us prove that the inside of Ω is an invariant set of System (N).

Lemma 3.1. *A solution orbit of System* (N) *starting from inside* Ω *cannot cross* Γ. *In other words, the set* Ω *is a negative invariant set of the system.*

P r o o f. We investigate the direction of the field of velocity vectors defined by System (N) on Γ. Let $(x(t), z(t))$ be a solution of the system. Then we have

$$\begin{aligned}&\frac{\mathrm{d}W(x(t), z(t))}{\mathrm{d}t}\\ &= (z(t) + \varphi(x))\{-\varphi'(x(t))\{z(t) - H(x(t))\}\} + g(x(t))\{z(t) - E(x(t))\}\\ &= -\,g(x(t))F(x(t)) \leq 0,\end{aligned}$$

if $a_1 \leq x(t) \leq a_2$. So the velocity vector on Γ points inwards. Therefore, no solution orbit of the system starting from inside Ω crosses Γ. $\square$

Lemma 3.2. *No non-trivial closed orbit of System* (N) *exists in the strip domain* $\Delta = \{(x, z) :\ a_1 < x < a_2,\ \ z \in \mathbb{R}\}$.

P r o o f. Suppose that a non-trivial closed orbit C of System (N) exists in the strip domain Δ. Let $(x(t), z(t))$ be the periodic solution corresponding to C and let T be its period. We have

$$\oint_C \mathrm{d}W = \int_0^T \frac{\mathrm{d}W(x(t), z(t))}{\mathrm{d}t}\,\mathrm{d}t = [W(x(t), z(t))]_0^T = 0.$$

On the other hand, $\dfrac{\mathrm{d}W(x(t), z(t))}{\mathrm{d}t} = -g(x(t))F(x(t)) < 0$ if $a_1 < x(t) < a_2$. Hence

$$\oint_C \mathrm{d}W = \int_0^T \frac{\mathrm{d}W(x(t), z(t))}{\mathrm{d}t}\,\mathrm{d}t < 0.$$

This is in contradiction to the first equality. □

Lemma 3.3. *If System* (N) *has a non-trivial closed orbit C, it cannot meet the line $x = p_1$.*

P r o o f. If System (N) has a non-trivial closed orbit C, by the Poincaré index theorem, the inside of C must contain the origin O. Let us express C as $C = \{(x(t), z(t)) :\ 0 \le t \le T\}$, where the pair $(x(t), z(t))$ is a solution of System (N) and T is the smallest period of the solution. Suppose that C meets the line $x = p_1$. The line $x = p_1$ consists of two open half lines $m^+ = \{(p_1, z) :\ z > 0\}$, $m^- = \{(p_1, z) :\ z < 0\}$ and the equilibrium point A. C does not meet the point A. If C meets m^+ at a point $(x(t_0), z(t_0))$ with $0 \le t_0 < T$, then $x(t_0) = 2\sqrt{-\mu_1}$, $z(t_0) > 0$ and $x'(t_0) = z(t_0) - E(p_1) > 0$. Hence the point $(x(t), z(t))$ crosses the line m^+ from left to right at $t = t_0$. From this fact it follows that the point $(x(t), z(t))$ cannot cross m^+ more than once in the time interval $0 \le t < T$ without crossing the lower half line m^-. Similarly we see that $(x(t), z(t))$ cannot cross m^- more than once without crossing m^+. From these facts we conclude that, if C meets the line $x = p_1$, it crosses the two half lines m^+ and m^- once each in the time interval $0 \le t < T$. It follows from this fact that, if C meets the line $x = p_1$, the saddle point A must be in the inside of C. This contradicts the Poincaré index theorem. □

From the above lemmas, if System (N) has a non-trivial closed orbit C, we see that it must be outside the domain Ω and cannot meet the line $x = p_1$.

We denote the three domains Z_1, Z_2 and Z_3 as follows.

$$Z_1 = \{(x,z): \ x > 0 \ \text{ and } \ z > E(x)\},$$
$$Z_2 = \{(x,z): \ x > 0 \ \text{ and } \ H(x) < z < E(x)\},$$
$$Z_3 = \{(x,z): \ x > 0 \ \text{ and } \ z < H(x)\}.$$

Lemma 3.4. *Solution orbits of System* (N) *meeting the curve $z = E(x)$* (*or $z = H(x)$*) *must cross vertically* (*or horizontally*) *the curve from the domain Z_1* (*or Z_3*) *to Z_2* (*or Z_2*), *respectively.*

We can check these facts by investigating the velocity vector on these curves. We call the curve $z = E(x)$ or $z = H(x)$ to "Vertical isocline" or "Horizontal isocline", respectively. Similarly, we can easily check the following:

Lemma 3.5. *Let the pair $(x(t), z(t))$ be a solution of System* (N). *If the point $(x(t_0), z(t_0))$ with $t = t_0$ is in the domain Z_2* (*or Z_3*), *then $\dot{x}(t_0) < 0$ and $\dot{z}(t_0) < 0$* (*or $\dot{z}(t_0) > 0$*), *respectively.*

With the above preliminaries, we shall prove the Theorems. Suppose that there exists a non-trivial closed orbit C of System (N). From Lemma 3.1 and Lemma 3.2, C must contain inside the domain Ω. If $a_1 \leq p_1$, this contradicts Lemma 3.3. So we shall consider the case of $a_1 > p_1$. It follows from the above lemmas that C must cross the open interval (p_1, a_1) and $(a_2, +\infty)$ on the x-axis.

Proofs of Theorem A and B. From Lemma 3.4 a closed orbit C starting from Z_1 crosses vertically the isocline $z = E(x)$ from Z_1 to Z_2. If $\varphi'(a_2)F(a_2) \geq g(a_2)$, we have $-\varphi(a_2) \leq H(a_2)$ from the condition [C3]. Namely, the point $\mathrm{R}(a_2, -\varphi(a_2))$ on the closed curve Γ is in the domain Z_3. Thus, if System (N) satisfies the condition [C4], then it follows that the orbit starting from Z_2 must stay in the domain Ω by Lemma 3.4 and 3.5, or after it crossed horizontally the isocline $z = H(x)$ from Z_2 to Z_3, it must stay in Ω by Lemma 3.5. This contradicts Lemma 3.2.

Similarly, if System (N) satisfies the condition [C5], then the point R is in the domain Z_2. Thus, we see from the second condition in [C5] that the closed orbit C must stay in the domain Ω by the same reason as the proof of Theorem A. This also contradicts Lemma 3.2. Thus the proofs of the Theorems are now completed. □

Remark 3.1. This method is also applied to the system with the case ($p_1 > 0$ and $a_1 > 0$).

Remark 3.2. In [13], the special case of which $\varphi(x)$ is an even function has been considered. The function $\varphi(x)$ in the Theorems is not necessarily even. In virture of this, their results are improved by our method (see [7]). For instance, consider the Van der Pol system:

$$F(x) = \frac{\lambda}{3}(x^3 - 3x) \quad \text{and} \quad g(x) = x,$$

where λ is some positive parameter.

It follows that the unique closed orbit of the system crosses the open intervals $(-\beta, -\alpha)$ and (α, β) on the x-axis, where $\alpha = \sqrt{3(1 + a/\lambda)}$, $\beta = \sqrt{\alpha + 3/(a\lambda)}$ and a is a positive number.

4. Examples

We will present in this section the phase portraits of the Liénard system for Type (III) as an example illustrating the application of the Theorems. It shall be shown that our methods have powers for these systems.

Example 4.1. (The motivation system in §1) Consider System (M) with $\alpha = 3$ and $\beta = 2$. Then we have $a_1 = -2$ and $p_1 = -3$. Taking $\varphi(x) = 2x^3$, we get System (N) with

$$E(x) = x^4 \qquad \text{and} \qquad H(x) = x^4 - \frac{x+3}{6x} \quad (x \neq 0).$$

We have that there exists a unique number $a_2 \in (1, 2)$ such that $G(a_1) = G(a_2)$, and $H(a_2) > -\varphi(a_2)$. Since $H'(x) = 4x^3 + \dfrac{1}{2x^2} > 0$ for $x > 0$, $\lim_{x \to +0} H(x) = -\infty$ and $\lim_{x \to +\infty} H(x) = +\infty$, we see that the condition [C4] in Theorem A is satisfied. Thus, it follows that this system (M) has no limit cycles. Moreover, from Corollary 2.2, it has no heteroclinic orbits as is shown in Figure 4.1 below.

Example 4.2. ([11, Example 2]) Consider the Lipschitz dynamical system (L) with

$$F(x) = \begin{cases} 2x - (3/2) & (x \geq 1) \\ (1/2)x & (0 < x < 1) \\ (1/4)x(x+2) & (x \leq 0). \end{cases} \quad \text{and} \quad g(x) = x.$$

We have $a_1 = -2$ and $p_1 = 0$. Taking $\varphi(x) = x$, we get System (N) with

$$E(x) = \begin{cases} x - (3/2) & (x \geq 1) \\ -(1/2)x & (0 < x < 1) \\ (1/4)x(x-2) & (x \leq 0) \end{cases} \quad \text{and}$$

$$H(x) = \begin{cases} -3/2 & (x \geq 1) \\ -(3/2)x & (0 < x < 1) \\ (1/4)x(x-6) & (x \leq 0). \end{cases}$$

Setting $a_2 = 2$, we have $G(a_1) = G(a_2)$ and $H(a_2) = -\varphi(a_2)$. Moreover, we notice that $P_{a_2} = -\varphi(2) = -2 < H(x) = -\dfrac{3}{2}$ $(x \geq 1)$. Thus, the condition in Theorem A is satisfied. Therefore, this system has no limit cycles. We remark that the horizontal isocline $z = H(x) = -\dfrac{3}{2}$ coincides with an algebraic invariant curve of this system. See Figure 4.2 below.

Example 4.3. (The example in [1] or [2]) Consider System (L) with

$$F(x) = ax^2 + bx \quad \text{and} \quad g(x) = x^2 + x. \tag{CD}$$

According to the result in [1] or [2], it has been known that the system has no limit cycles if $b(2b - a) \geq 0$. By using our methods, the following result is given.

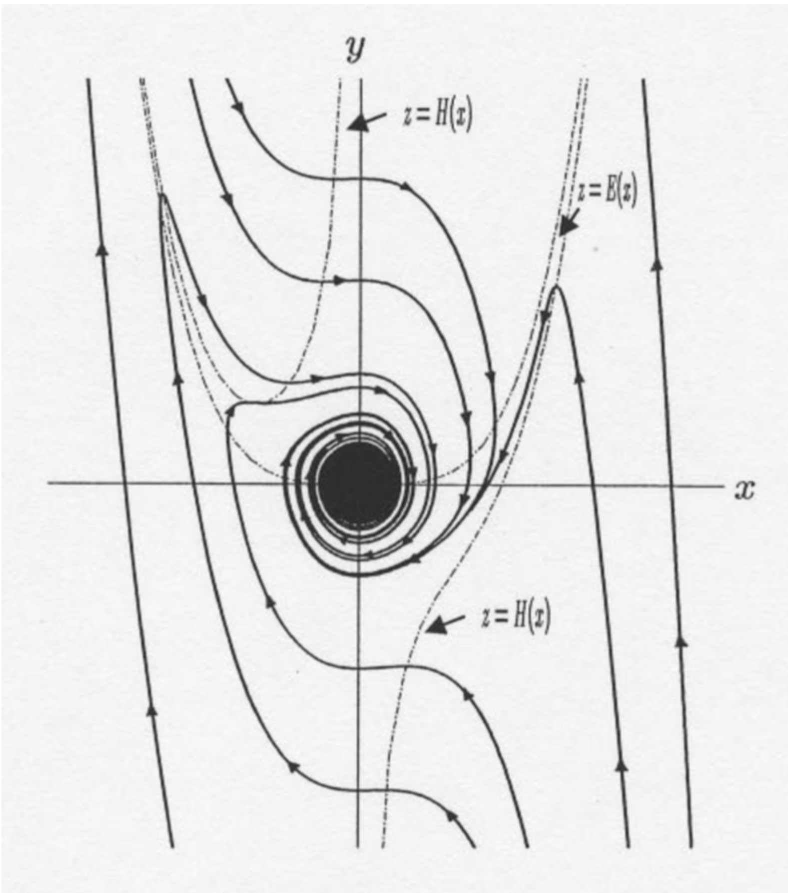

FIGURE 4.1

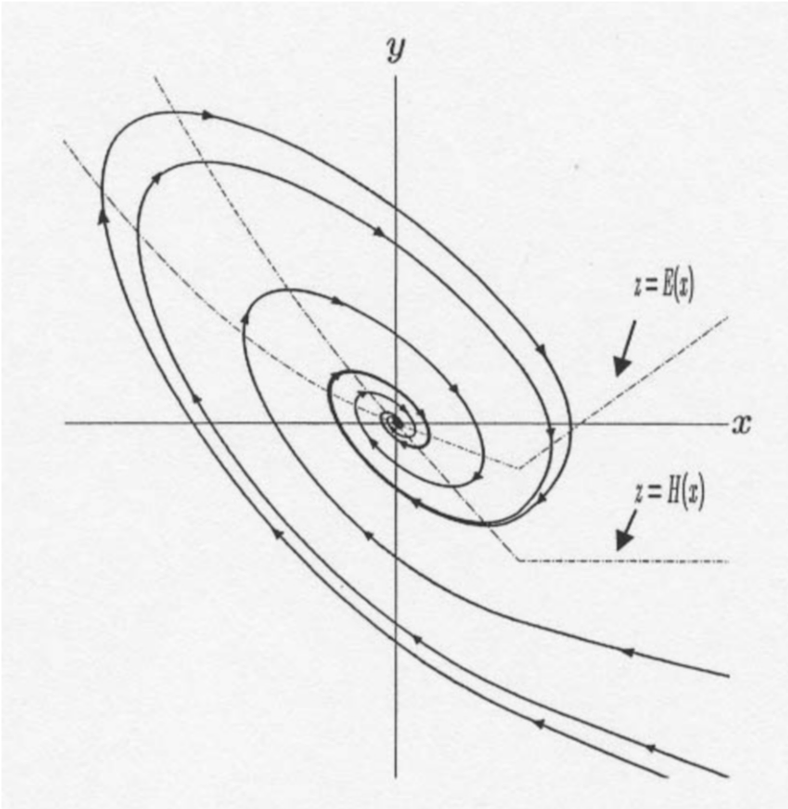

FIGURE 4.2

COROLLARY 4.1. *If $ab > 0$, $|b| \geq 2$ and $a^2 - ab + 1 > 0$ are satisfied, then the system* (CD) *has no limit cycles and heteroclinic orbits.*

Without loss of generality, we can set $a > 0$ and $b > 0$. In fact, if $a < 0$ and $b < 0$, by taking the variables t and y to $-t$ and $-y$, respectively, the previous system is reduced. Let $\varphi(x) = \dfrac{b - \sqrt{b^2 - 4}}{2}x$ for $|b| \geq 4$. Then, we have that $E(x) \geq 0$ and $H(x) \geq 0$ if $a^2 - ab + 1 > 0$. Thus, it follows from Corollaries 2.1 and 2.2 that System (CD) has no limit cycles and heteroclinic orbits. This fact is shown in Figure 4.3 of the case $a = 5$ and $b = 4$.

From the mentioned facts above, we get the following

THEOREM C. *Let the set $D = \{(a, b) : \ (|a| \leq 2|b| \vee |b| \geq 2) \wedge ab > 0\}$. If the pair (a, b) belongs to D, then System* (CD) *has no limit cycles and heteroclinic orbits.*

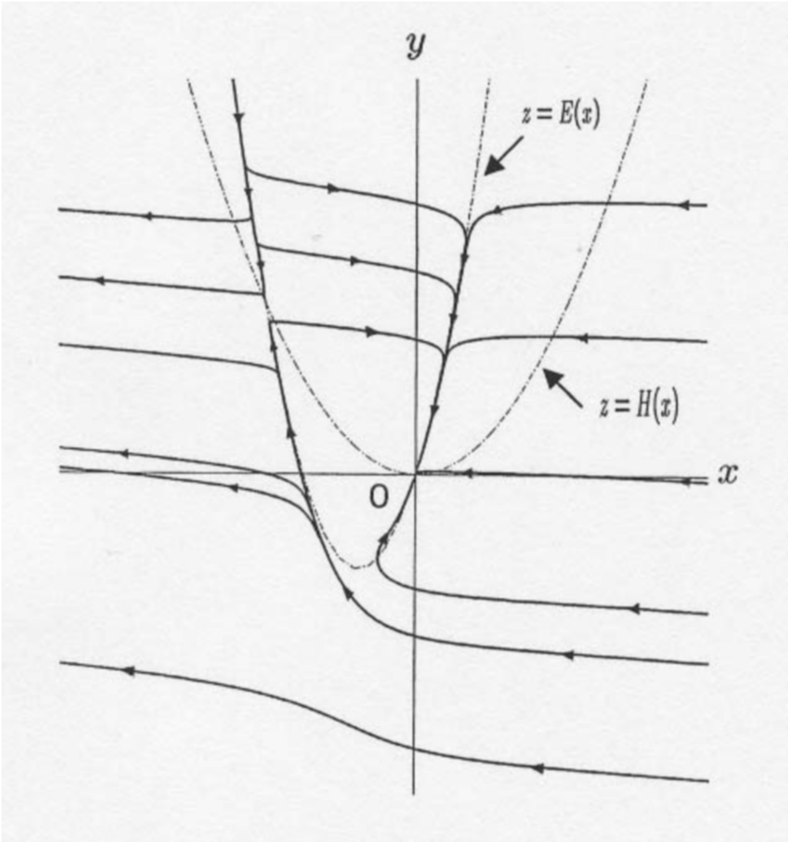

FIGURE 4.3

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