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OSCILLATION OF NEUTRAL DELAY DIFFERENCE EQUATIONS OF SECOND ORDER WITH POSITIVE AND NEGATIVE COEFFICIENTS

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ABSTRACT. This paper is concerned with a class of neutral difference equations of second order with positive and negative coefficients of the forms

$$\Delta^2(x_n \pm c_n x_{n-\tau}) + p_n x_{n-\delta} - q_n x_{n-\sigma} = 0$$

where τ, δ and σ are nonnegative integers and $\{p_n\}$, $\{q_n\}$ and $\{c_n\}$ are nonnegative real sequences. Sufficient conditions for oscillation of the equations are obtained.

1. Introduction

In this paper, we consider the oscillation and asymptotic property of nonoscillatory solutions of the second order linear neutral delay difference equations of the forms

(E_1) $\Delta^2(x_n + c_n x_{n-\tau}) + p_n x_{n-\delta} - q_n x_{n-\sigma} = 0$

and

(E_2) $\Delta^2(x_n - c_n x_{n-\tau}) + p_n x_{n-\delta} - q_n x_{n-\sigma} = 0$

where $n \geq n_0 > 0$, τ, δ and σ are nonnegative integers such that $\delta \geq \sigma + 1$, $\{p_n\}$, $\{q_n\}$ and $\{c_n\}$ are nonnegative real sequences for $n \geq n_0$.

By a solution of (E_1) (or (E_2)), we mean a real sequence $\{x_n\}$ which is defined for $n \geq n_0 - \mu$ and satisfy (E_1) (or (E_2)) where $\mu = \max\{\delta, \tau\}$. A solution $\{x_n\}$ of

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(E_1) (or (E_2)) is said to be nonoscillatory if it is eventually positive or eventually negative; otherwise it is called oscillatory.

Sufficient conditions for oscillation of solutions of first order neutral difference equations with positive and negative coefficients have been investigated by many authors, see ([5], [11], [13], [10]) and the references cited therein. Although many authors (see [3], [9], [12]) studied oscillation and nonoscillation of second and higher order neutral difference equations of the forms

$$\Delta^m(x_n \pm c_n x_{n-\tau}) + p_n x_{n-\delta} = 0, \qquad m \geq 2,$$

it seems that no work has been done on the oscillation and asymptotic behaviour of nonoscillatory solutions of second order neutral difference equations of the forms (E_1) (or (E_2)). In this paper, an attempt has been made to study the behaviour of solutions of (E_1) (or (E_2)).

This work is organized as follows: Section 1 is introductory where as sufficient conditions for oscillation of (E_1) (or (E_2)) is studied in Section 2. Section 3 deals with the oscillation of (E_1) (or (E_2)) with forcing terms.

2. Oscillatory behaviour of solutions of (E_1) and (E_2)

In this section, we obtain the following oscillation criteria of (E_1) and (E_2). Examples are given to illustrate the results.

Theorem 2.1. *Assume that*

(H_1) $p_n - q_{n-\delta+\sigma} \geq k > 0$, $n \geq \delta - \sigma$

(H_2) $0 \leq c_n \leq c$, *c is a constant.*

hold. If

(H_3) $\sum\limits_{i=n_0}^{\infty} \sum\limits_{j=i-\delta+\sigma}^{i-1} q_j \leq 1,$

then every solution of (E_1) is oscillatory.

Proof. Suppose that $\{x_n\}$ is a nonoscillatory solution of (E_1). Without any loss of generality, we may assume that x_n is eventually positive. Let $n_1 \geq n_0 + \mu$ be such that $x_n > 0$ for $n \geq n_1$. Hence $x_{n-\tau} > 0, x_{n-\delta} > 0$ and $x_{n-\sigma} > 0$ for some $n \geq n_2 \geq n_1$. Define

$$z_n = x_n + c_n x_{n-\tau} - \sum_{i=n_0}^{n-1} \sum_{j=i-\delta+\sigma}^{i-1} q_j x_{j-\sigma}. \tag{2.1}$$

Then (E_1) gives, using (H_1)

$$\Delta^2 z_n \leq -k x_{n-\delta}, \qquad n \geq n_2. \tag{2.2}$$

Hence Δz_n is eventually nondecreasing. Then we have that $\Delta z_n > 0$ or $\Delta z_n < 0$ for $n \geq n_3 \geq n_2$.

Let $\Delta z_n < 0$ for $n \geq n_3$. Then the inequality $\Delta z_n \leq \Delta z_{n_3}$ implies that $z_n < 0$ for large n and $\lim_{n\to\infty} z_n = -\infty$. We claim that x_n is bounded from above. If not, then there exists a $n_4 > n_3$ such that $z_{n_4} < 0$ and $\max_{n_3 \leq n \leq n_4} x_n = x_{n_4}$. Then from (2.1), we obtain for $n = n_4$

$$\begin{aligned} 0 > z_{n_4} &= x_{n_4} + c_{n_4} x_{n_4-\tau} - \sum_{i=n_0}^{n_4-1} \sum_{j=i-\delta+\sigma}^{i-1} q_j x_{j-\sigma} \\ &\geq \left[1 - \sum_{i=n_0}^{n_4-1} \sum_{j=i-\delta+\sigma}^{i-1} q_j\right] x_{n_4} \\ &\geq \left[1 - \sum_{i=n_0}^{\infty} \sum_{j=i-\delta+\sigma}^{i-1} q_j\right] x_{n_4} \geq 0, \end{aligned}$$

a contradiction. Hence x_n must be bounded from above. So there exists a constant $L > 0$ such that $x_n \leq L$ for $n \geq n_3$. Accordingly, we have

$$\begin{aligned} z_n &\geq -L \sum_{i=n_0}^{n-1} \sum_{j=i-\delta+\sigma}^{i-1} q_j \\ &\geq -L \sum_{i=n_0}^{\infty} \sum_{j=i-\delta+\sigma}^{i-1} q_j \\ &\geq -L > -\infty, \qquad n \geq n_3, \end{aligned}$$

which contradicts the fact that $z_n \to -\infty$ as $n \to \infty$. We therefore have $\Delta z_n \geq 0$ for $n \geq n_3$. Now, the summation of (2.2) from n_3 to $n-1$ gives

$$\infty > \Delta z_{n_3} \geq -\Delta z_n + \Delta z_{n_3} \geq k \sum_{j=n_3}^{n-1} x_{j-\delta}$$

and therefore

$$\sum_{j=n_3}^{\infty} x_j < \infty. \tag{2.3}$$

If we set

$$y_n = x_n + c_n x_{n-\tau} \tag{2.4}$$

then from (2.3) and (H_2), it follows that

$$\sum_{j=n_0}^{\infty} y_j < \infty. \tag{2.5}$$

On the other hand, from (2.1) we have

$$\Delta y_n = \Delta z_n + \sum_{j=n-\delta+\sigma}^{n-1} q_j x_{j-\sigma} \geq 0, \qquad n \geq n_3$$

so that y_n is a nondecreasing sequence. Therefore $y_n > 0$ for $n \geq n_3$ and $y_n \geq y_{n_3}$ for $n \geq n_3$ implies that $\sum\limits_{j=n_0}^{\infty} y_j = \infty$, a contradiction to (2.5). Hence every solution of (E_1) oscillates. This completes the proof of the theorem. □

Example 2.2. Consider

$$\Delta^2[x_n + 2x_{n-1}] + (n+2)x_{n-3} - \mathrm{e}^{-n} x_{n-1} = 0, \qquad n \geq 3. \tag{2.6}$$

All the conditions of Theorem2.1 are satisfied. Hence every solution of (2.6) oscillates.

Theorem 2.3. *Let* (H_1) *and*

(H_4) $0 \leq c_n \leq c < 1$

hold. If

(H_5) $c + \sum\limits_{i=n_0}^{n-1} \sum\limits_{j=i-\delta+\sigma}^{i-1} q_j \leq 1,$

then every solution of (E_2) *is oscillatory or tend to zero as* $n \to \infty$.

Proof. Let x_n be a nonoscillatory solution of (E_2) such that $x_n > 0$ and $x_{n-\mu} > 0$ for $n \geq n_1 \geq n_0 + \mu$. Setting

$$w_n = x_n - c_n x_{n-\tau} - \sum_{i=n_0}^{n-1} \sum_{j=i-\delta+\sigma}^{i-1} q_j x_{j-\sigma}, \tag{2.7}$$

we obtain, from (E_2) using (H_1)

$$\Delta^2 w_n \leq -k x_{n-\delta}, \qquad n \geq n_1. \tag{2.8}$$

Hence $\Delta w_n \geq 0$ or $\Delta w_n < 0$ for $n \geq n_2 \geq n_1$. First suppose that $\Delta w_n < 0$ for $n \geq n_2$. Then $w_n < 0$ for large n and $\lim\limits_{n\to\infty} w_n = -\infty$. We claim that x_n is bounded from above. If it is not the case, there exists a number $n_3 \geq n_2$ such that $w_{n_3} < 0$ and $\max\limits_{n_2 \leq n \leq n_3} x_n = x_{n_3}$ and we have

$$\begin{aligned} 0 > w_{n_3} &= x_{n_3} - c_{n_3} x_{n_3-\tau} - \sum_{i=n_0}^{n_3-1} \sum_{j=i-\delta+\sigma}^{i-1} q_j x_{j-\sigma} \\ &\geq \left[1 - c - \sum_{i=n_0}^{\infty} \sum_{j=i-\delta+\sigma}^{i-1} q_j\right] x_{n_3} \\ &\geq 0. \end{aligned}$$

This contradiction shows that x_n is bounded from above. Thus, there exists a constant $L > 0$ such that $x_n < L$ for $n \geq n_2$. Then it follows from (2.7) that

$$w_n \geq -L\Big\{c + \sum_{i=n_0}^{\infty} \sum_{j=i-\delta+\sigma}^{i-1} q_j\Big\} \geq -L > -\infty,$$

which contradicts the fact that $w_n \to -\infty$ as $n \to \infty$. Hence $\Delta w_n \geq 0$ for $n \geq n_2$. Now summing (2.8) from n_2 to n and letting $n \to \infty$, we obtain (2.3). Then $x_n \to 0$ as $n \to \infty$. The proof of the theorem is complete. □

We have the following corollary from Theorem 2.3:

Corollary 2.4. *Let $p_n \geq k > 0$ for $n \geq n_0$. Then every solution of*

$$\Delta^2[x_n - x_{n-\tau}] + p_n x_{n-\delta} = 0, \qquad n \geq n_0 \tag{2.9}$$

oscillates or tend to zero as $n \to \infty$.

Example 2.5. By Theorem 2.3, every solution of

$$\Delta^2\Big[x_n - \frac{1}{\mathrm{e}} x_{n-1}\Big] + (n+2)x_{n-3} - \mathrm{e}^{-n} x_{n-1} = 0, \qquad n \geq 3 \tag{2.10}$$

oscillates or tend to zero as $n \to \infty$.

Remark 2.6. Parhi and Tripathy [9] proved that if

(H_6) $\sum\limits_{n=n_0}^{\infty} p_n = \infty$

holds, then every solution of (2.9) oscillates (see [9, Theorems 2.6, 2.7]). However, (H_6) cannot be regarded as a sufficient condition for the oscillation of (2.9). This is evident from the following example.

Example 2.7. Consider

$$\Delta^2[x_n - x_{n-2}] + \frac{3}{16} x_{n-2} = 0, \qquad n \geq 2. \tag{2.11}$$

Clearly, $x_n = \frac{1}{2^n}$ is a nonoscillatory solution of (2.11) which tends to zero as $n \to \infty$, although (H_6) is satisfied. By Corollary 2.4 we come to the right conclusion.

Remark 2.8. One may observe from the proof of [9, Theorems 2.6, 2.7] that the authors have proved $\lim\limits_{n\to\infty} y(n) = 0$ when $z(n) < 0$ and m is even. The same has also been proved in the theorem when $z(n) > 0$ and m is even.

Thus the statement of [9, Theorems 2.6, 2.7] should be stated as:

Theorem 2.9. *Let $-\infty < c_1 \leq c_n \leq -1$. If (H_6) holds, then every solution of*

$$\Delta^2[x_n - c_n x_{n-\tau}] + p_n x_{n-\delta} = 0$$

oscillates or tends to zero as $n \to \infty$.

Theorem 2.10. *Let $-\alpha < c_1 \le c_n \le c_3 \le -1$. If (H_6) holds, then the conclusion of* Theorem 2.9 *holds.*

Theorem 2.11. *Let*

(H_7) $h_n = p_n - q_{n-\delta+\sigma} \ge 0$, $n \ge n_0$

and

(H_8) $c + \sum\limits_{i=n_0}^{\infty} \sum\limits_{j=i-\delta+\sigma}^{i-1} q_j < 1$

hold. Set

$$P_n = nh_n. \tag{2.12}$$

Assume that $P_n < 2$ for $n \ge n_1 \ge n_0$ and

(H_9) $\sum\limits_{n=n_0}^{\infty} \left\{ \frac{2^n h_n \cdot c_{n-\delta}}{\prod\limits_{j=1}^{n} (2-P_j)} \right\} = \infty,$

holds, then every solution of (E_2) is either oscillatory or tend to zero as $n \to \infty$.

P r o o f. Let x_n be a nonoscillatory solution of (E_2). Assume that $x_n > 0$ for $n \ge n_1 \ge n_0$. Then there exist a $n_2 \ge n_1$ such that $x_{n-\mu} > 0$ for $n \ge n_2$. Setting w_n as in (2.7), we obtain

$$\Delta^2 w_n + h_n x_{n-\delta} = 0, \qquad n \ge n_2. \tag{2.13}$$

Thus $w_n > 0$ or $w_n < 0$ for some $n \ge n_3 \ge n_2$. Let $w_n < 0$ for $n \ge n_3$. Then since (H_8) holds, then x_n is bounded. Indeed, if, x_n is unbounded, then there exists a sequence $\{N_\alpha\}$, $N_\alpha > n_3$, for each α, such that $N_\alpha \to \infty$ as $\alpha \to \infty$ and $\max\limits_{n_3 \le n \le N_\alpha} x_n = x_{N_\alpha}$ and $\lim\limits_{\alpha\to\infty} x_{N_\alpha} = \infty$. Then from (2.7) we obtain

$$\begin{aligned} 0 > w_{N_\alpha} &= x_{N_\alpha} - c_{N_\alpha} x_{N_\alpha - \tau} - \sum_{i=n_0}^{N_\alpha - 1} \sum_{j=i-\delta+\sigma}^{i-1} q_j x_{j-\sigma} \\ &\ge \left[1 - c - \sum_{i=n_0}^{\infty} \sum_{j=i-\delta+\sigma}^{i-1} q_j \right] x_{N_\alpha} \to \infty \end{aligned}$$

as $\alpha \to \infty$, a contradiction to the fact that $w_n < 0$ for $n \ge n_3$. Hence x_n is bounded. Suppose that $\limsup\limits_{n\to\infty} x_n = L > 0$. Then there exist a sequence $\{N_\xi\}$, $N_\xi > n_3$, for each ξ, such that $N_\xi \to \infty$ as $\xi \to \infty$ and $\limsup\limits_{n\to\infty} x_n = \lim\limits_{\xi\to\infty} x_{N_\xi} = L$. Since $\limsup\limits_{\xi\to\infty} x_{N_{\xi-\tau}} \le L$, then $w_n < 0$ for $n \ge n_3$ yields that

$$0 > w_{n_\mu} \ge L \left\{ 1 - c_{N_\mu} - \sum_{i=n_0}^{N_\mu - 1} \sum_{j=i-\delta+\sigma}^{i-1} q_j \right\} > 0,$$

a contradiction. Hence $\limsup_{n\to\infty} x_n = 0$. This in turn implies that $\lim_{n\to\infty} x_n = 0$.

Next, suppose that $w_n > 0$ for $n \geq n_3$. Thus there exists a $n_4 \geq n_3$ such that $\Delta w_n > 0$ for $n \geq n_4$. Then multiplying (2.13) by n and summing obtained equation from n_4 to n we conclude that there exists a $n_5 \geq n_4$ such that

$$w_{n-\delta} \geq \frac{n}{2}\Delta w_{n-\delta}, \qquad 4 \geq n_5. \tag{2.14}$$

From (2.7), it follows that $x_n - c_n x_{n-\tau} > w_n$ which using the nondecreasing nature of w_n yields that there exists a real $\theta > 0$ such that $x_n > \theta c_n + w_n$. Thus there exists a $n_6 \geq n_5$ such that

$$x_{n-\delta} > \theta c_{n-\delta} + w_{n-\delta}. \tag{2.15}$$

Hence from (2.13), (2.14) and (2.15), we obtain

$$\Delta^2 w_n + \frac{n h_n}{2}\Delta w_{n-\delta} + \theta h_n c_{n-\delta} \leq 0, \qquad n \geq n_6. \tag{2.16}$$

Let $r_n = \frac{1}{\prod_{j=1}^{n-1}(1-\frac{P_j}{2})}$. Multiplying (2.16) by r_{n+1}, we obtain by using the decreasing nature of Δw_n

$$\Delta(r_n \Delta w_n) + \theta\left\{\frac{2^n h_n \cdot c_{n-\delta}}{\prod_{j=1}^{n}(2-P_j)}\right\} \leq 0, \qquad n \geq n_6.$$

Summing the above difference inequality from n_6 to n and letting $n \to \infty$ we obtain

$$\sum_{n=n_0}^{\infty}\left\{\frac{2^n h_n \cdot c_{n-\delta}}{\prod_{j=1}^{n}(2-P_j)}\right\} < \infty,$$

a contradiction to (H_9). Thus the theorem is proved. □

We note that (H_7) is weaker than (H_1). When $P_n \geq 2$, where P_n is defined in (2.12), we have the following result:

Theorem 2.12. *Assume that $P_n \geq 2$. Let (H_7) and (H_8) hold. If*

(H_{10}) $\sum_{n=n_0}^{\infty} 2^n h_n c_{n-\delta} = \infty,$

then the conclusion of Theorem 2.11 *holds.*

Proof. Let x_n be a positive nonoscillatory solution of (E_2). Then proceeding as in the proof of Theorem 2.11, one may show that $\lim_{n\to\infty} x_n = 0$ when $w_n < 0$ for large n. Next, suppose that $w_n > 0$ for large n, say for $n \geq n_3$. Then

$\Delta w_n > 0$ for some $n \geq n_4 \geq n_3$. Then from (2.16), $P_n \geq 2$ and the decreasing nature of Δw_n, we get

$$\Delta^2 w_n + \frac{1}{2}\Delta w_n + \theta h_n c_{n-\delta} \leq 0, \qquad n \geq n_6 \geq n_3.$$

The above inequality can be written in the form

$$\Delta(2^{n-1}\Delta w_n) + \theta 2^n h_n c_{n-\delta} \leq 0, \qquad n \geq n_6.$$

Summing the above inequality from n_6 to $n-1$ and letting $n \to \infty$, we obtain a contradiction. Thus the theorem is proved. □

The following lemma due to Gyori and Ladas [4, pp. 183] is needed for our use in the sequel.

Lemma 2.13. *If*

$$\liminf_{n\to\infty} \sum_{i=n-k}^{n-1} R_i > (k/k+1)^{k+1},$$

then $\Delta u_n + R_n u_{n-k} \leq 0$ *has no eventually positive solution and* $\Delta u_n + R_n u_{n-k} \geq 0$ *has no eventually negative solution.*

Using Lemma 2.13 we have the following theorem.

Theorem 2.14. *Let* (H_7) *and* (H_8) *hold.If*

$$\liminf_{n\to\infty} \sum_{i=n-\delta}^{n-1} P_i > 2(\delta/\delta+1)^{\delta+1}, \tag{2.17}$$

holds, then the conclusion of Theorem 2.11 *hold, where* P_n *is defined as in* (2.12).

Proof. Let x_n be an eventually nonoscillatory solution of (E_2). One may proceed as in the proof of Theorem 2.11 to show that $x_n \to 0$ as $n \to \infty$ when $\Delta w_n < 0$ for large n. Next suppose that $\Delta w_n > 0$ for large n. As in the proof of Theorem 2.11 it is easy to obtain (2.16) from which we see that Δw_n is a positive solution of

$$\Delta^2 w_n + \frac{P_n}{2}\Delta w_{n-\delta} \leq 0$$

for large n which is again a contradiction due to Lemma 2.13. Hence the theorem is proved. □

From the proof of the above theorems, it seems that the assumption $\delta \geq \sigma + 1$ leads to the conclusion that: *every solution of* (E_2) *oscillates or tend to zero as* $n \to \infty$. Thus in our next theorem, we make the assumption that $\sigma \geq \delta + 1$ which will lead us to the conclusion that every solution of (E_2) oscillates.

Theorem 2.15. *Let* $\sigma \geq \delta + 1$, $\delta \geq \tau + 1$ *and*

(H_{11}) $0 \leq c_n \leq 1$.

Further suppose that (H_6) *and* $p_n \geq 2q_{n-\delta+\sigma}$ *hold for* $n \geq n_0$*. If*

$$\limsup_{n\to\infty} \sum_{i=n-\delta+\tau-1}^{n-1} \sum_{j=i}^{n-1} \frac{p_j - q_{j-\delta+\sigma}}{c_{j-\delta+\sigma}} > 1, \tag{2.18}$$

then every solution of (E_2) *is oscillatory.*

P r o o f. Let x_n be a nonoscillatory solution of (E_2) such that $x_n > 0$ and $x_{n-\mu} > 0$ for some $n \geq n_1 \geq n_0$. Setting

$$y_n = x_n - c_n x_{n-\tau} + \sum_{i=n_0}^{n-1} \sum_{j=i}^{i-\delta+\sigma-1} q_j x_{j-\sigma}, \tag{2.19}$$

we see from (E_2) that

$$\Delta^2 y_n + (p_n - q_{n-\delta+\sigma}) x_{n-\delta} = 0. \tag{2.20}$$

Then $\Delta^2 y_n < 0$ for $n \geq n_1$. This in turn implies that $y_n > 0$ or $y_n < 0$ for some $n \geq n_2 \geq n_1$. First suppose that $y_n < 0$ for $n \geq n_2$. If $\Delta y_n < 0$ for large n, then $y_n < -\lambda$ for some $n \geq N \geq n_2$ and $\lambda > 0$. Since $x_N < y_N + c_N x_{N-\tau}$, then

$$x_{N+\tau} < y_{N+\tau} + x_N < -\lambda + x_N \tag{2.21}$$

and therefore,

$$x_N < -\lambda + x_{N-\tau}. \tag{2.22}$$

By combining (2.21) and (2.22) we get

$$x_{N+\tau} < -2\lambda + x_{N-\tau}, \tag{2.23}$$

and if we continue with this procedure we can prove that

$$x_{N+m\tau} < -(m+1)\lambda + x_{N-\tau} \tag{2.24}$$

for any integer $m > 1$. If we let $m \to \infty$ in (2.24) we come to a contradiction. Hence $\Delta y_n > 0$ for large n, say for $n \geq n_3 \geq n_2$. Then we have from $x_{n-\delta} > -\frac{y_{n-\delta+\sigma}}{c_{n-\delta+\sigma}}$ and (2.20) implies

$$\Delta^2 y_n - \frac{p_n - q_{n-\delta+\sigma}}{c_{n-\delta+\tau}} y_{n-\delta+\tau} \leq 0$$

for $n \geq n_3$. Summing the above inequality from s to $n-1$, we have

$$-\Delta y_s \leq \sum_{i=s}^{n-1} \frac{p_i - q_{i-\delta+\sigma}}{c_{i-\delta+\tau}} y_{i-\delta+\tau}$$

Again summing the above inequality from $n-\delta+\tau-1$ to $n-1$, we have

$$\begin{aligned} y_{n-\delta+\tau-1} &\le \sum_{i=n-\delta+\tau-1}^{n-1} \sum_{j=i}^{n-1} \frac{p_j - q_{j-\delta+\sigma}}{c_{j-\delta+\tau}} y_{i-\delta+\tau} \\ &\le y_{n-\delta+\tau-1} \sum_{j=n-\delta+\tau-1}^{n-1} \sum_{j=i}^{n-1} \frac{p_j - q_{j-\delta+\sigma}}{c_{j-\delta+\tau}}. \end{aligned}$$

Consequently, we have that $\sum\limits_{i=n-\delta+\tau-1}^{n-1} \sum\limits_{j=i}^{n-1} \frac{p_j - q_{j-\delta+\sigma}}{c_{j-\delta+\sigma}} < 1$, a contradiction to the assumption of the theorem.

Hence $y_n > 0$ for $n \ge n_2$. In this case $\Delta y_n > 0$ for large n, say for $n \ge n_4 \ge n_2$. First, notice from (2.19) we have

$$x_n \ge y_n - \sum_{i=n_0}^{n-1} \sum_{j=i}^{i-\delta+\sigma-1} q_j x_{j-\sigma}. \tag{2.25}$$

Moreover, using $p_{j+\delta-\sigma} - q_j \ge q_j$, $j \ge n_0$, we get

$$\begin{aligned} \sum_{i=n_0}^{n-1} \sum_{j=i}^{i-\delta+\sigma-1} q_j x_{j-\sigma} &\le -\sum_{i=n_0}^{n-1} \sum_{j=i}^{i-\delta+\sigma-1} \frac{q_j}{p_{j+\delta-\sigma} - q_j} \Delta^2 y_{j+\delta-\sigma} \\ &\le -\sum_{i=n_0}^{n-1} \sum_{j=i}^{i-\delta+\sigma-1} \Delta^2 y_{j+\delta-\sigma} \\ &\le y_n - k, \end{aligned}$$

that is,

$$\sum_{i=n_0}^{n-1} \sum_{j=i}^{i-\delta+\sigma-1} q_j x_{j-\sigma} \le y_n - k, \tag{2.26}$$

where $k = y_{n_0+\delta-\sigma}$. By combining (2.25) and (2.26) it follows that $x_n > k$ for $n \ge n_4$. Hence $x_{n-\delta} > k$ for $n \ge n_5 \ge n_4$. Then summing (2.20) from n_5 to $n-1$, we get

$$\sum_{k=n_5}^{\infty} q_{k-\delta+\sigma} < \infty,$$

contradicting (H_6). Hence every solution of (E_2) oscillates. This completes the proof of the theorem. □

For $q_n \equiv 0$ and $c_n \equiv 1$, we have the following corollary from Theorem 2.15:

Corollary 2.16. *Let $\delta > \tau, (H_6)$ and*

$$\limsup_{n\to\infty} \sum_{i=n-\delta+\tau-1}^{n-1} \sum_{j=i}^{n-1} p_j > 1$$

then every solution of (2.9) *is oscillatory.*

3. Oscillatory behaviour of solutions of equations (E_1) and (E_2) with forcing terms

This section deals with the oscillation and asymptotic behavior of nonoscillatory solutions of

(E_3) $\Delta^2(x_n + c_n x_{n-\tau}) + p_n x_{n-\delta} - q_n x_{n-\sigma} = f_n$

and

(E_4) $\Delta^2(x_n - c_n x_{n-\tau}) + p_n x_{n-\delta} - q_n x_{n-\sigma} = f_n$

where $n \geq n_0 > 0$, τ, δ and σ are defined as before and $\{f_n\}$ is a real sequence defined for $n \geq n_0$.

Theorem 3.1. *Let* (H_1), (H_2) *and* (H_3) *hold. Further, assume that*

(H_{11}) *There exists a sequence* $\{F_n\}_{n=n_0}^{\infty}$ *such that* $\Delta^2 F_n = f_n$ *and* $\lim\limits_{n\to\infty} F_n = 0$.

Then every solution of (E_3) *is oscillatory or tend to zero as* $n \to \infty$.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of (E_3) such that $x_n > 0$ and $x_{n-\mu} > 0$ for $n \geq n_1 \geq n_0$. Define

$$u_n = z_n - F_n \tag{3.1}$$

where z_n is defined by (2.1). Then from (E_3) and (H_1) we obtain

$$\Delta^2 u_n \leq -k x_{n-\delta}, \qquad n \geq n_1. \tag{3.2}$$

Thus Δu_n is eventually a nonincreasing function and $\Delta u_n \geq 0$ or $\Delta u_n < 0$ for some $n \geq n_2 \geq n_1$. First suppose that $\Delta u_n < 0$ for $n \geq n_2$. Then $u_n < 0$ for some $n \geq n_3 \geq n_2$ and $\lim\limits_{n\to\infty} u_n = -\infty$. We claim that x_n is bounded from above. If not, then there exists a sequence $\{N_\alpha\}_{\alpha=1}^{\infty}$, $N_\alpha \geq n_3$, such that

$$\lim_{\alpha\to\infty} N_\alpha = \infty, \quad \lim_{\alpha\to\infty} u_{N_\alpha} = -\infty, \quad \lim_{\alpha\to\infty} F_{N_\alpha} = 0, \quad \lim_{\alpha\to\infty} x_{N_\alpha} = -\infty$$

and $\max\limits_{n_3 \leq n \leq N_\alpha} x_n = x_{N_\alpha}$. Then, we have from (3.1)

$$\begin{aligned} 0 > u_{N_\alpha} &= x_{N_\alpha} + c_{N_\alpha} x_{N_\alpha - \tau} - \sum_{i=n_0}^{N_\alpha - 1} \sum_{j=i-\delta+\sigma}^{i-1} q_j x_{j-\sigma} - F_{N_\alpha} \\ &\geq \left\{ 1 - \sum_{i=n_0}^{\infty} \sum_{j=i-\delta+\sigma}^{i-1} q_j \right\} x_{N_\alpha} - F_{N_\alpha}. \end{aligned}$$

Taking limit as $\alpha \to \infty$, we see that

$$\lim_{\alpha\to\infty} u_{N_\alpha} \geq \left\{1 - \sum_{i=n_0}^{\infty} \sum_{j=i-\delta+\sigma}^{i-1} q_j\right\} \lim_{\alpha\to\infty} x_{N_\alpha} = \infty,$$

a contradiction. Hence x_n is bounded from above. Thus there exists a constant $L > 0$ such that $x_n \leq L$ for $n \geq n_3$. Hence from (3.1)

$$u_n \geq -L \sum_{i=n_0}^{n-1} \sum_{j=i-\delta+\sigma}^{i-1} q_j \geq -L \sum_{i=n_0}^{\infty} \sum_{j=i-\delta+\sigma}^{i-1} q_j \geq -l > -\infty,$$

a contradiction.

Therefore, $\Delta u_n \geq 0$ for $n \geq n_2$. Then summing (3.2) from n_2 to ∞, we obtain (2.3). This proves that $x_n \to 0$ as $n \to \infty$.Thus the theorem is proved. □

Example 3.2. By Theorem 3.1, every solution of

$$\Delta^2\Big[x_n + \frac{1}{2}x_{n-1}\Big] + 2x_{n-3} - \mathrm{e}^{-n}x_{n-1} = (-1)^n\mathrm{e}^{-n}, \qquad n \geq 3, \tag{3.3}$$

is oscillatory or tends to zero as $n \to \infty$. In particular, $x_n = (-1)^n$ is an oscillatory solution of the equation (3.3). In this case, $F_n = \frac{(-1)^n\mathrm{e}^{-n}}{(1+\frac{1}{e})^2} \to 0$ as $n \to \infty$ and $\Delta^2 F_n = f_n = (-1)^n\mathrm{e}^{-n}$.

One may proceed as in the proof of Theorem 3.1 to prove the following result.

Theorem 3.3. *Let* (H_1), (H_4), (H_5) *and* (H_{11}) *hold. Then every solution of* (E_4) *is oscillatory or tends to zero as* $n \to \infty$.

Example 3.4. Consider

$$\Delta^2[x_n - \mathrm{e}^{-n}x_{n-1}] + 4x_{n-3} - \frac{1}{\mathrm{e}}\Big(1+\frac{1}{\mathrm{e}}\Big)^2\mathrm{e}^{-n}x_{n-1} = \Big(1+\frac{1}{\mathrm{e}}\Big)^3\mathrm{e}^{-n}(-1)^n, \qquad n \geq 3. \tag{3.4}$$

All the conditions of Theorem 3.3 are satisfied. $x_n = (-1)^n$, $n \geq 3$, is an oscillatory solution of (3.4). In this case, $F_n = (1+1/\mathrm{e})\mathrm{e}^{-n}(-1)^n$ and $\Delta^2 F_n = f_n$ and $\lim\limits_{n\to\infty} F_n = 0$.

Remark 3.5. From Theorem 3.1 and Theorem 3.3, it seems that the behaviour of F_n forces all nonoscillatory solutions of (E_3) and (E_4) tend to zero as $n \to \infty$. In the following, we do not insist that $F_n \to 0$ as $n \to \infty$. Instead, we assume that F_n changes sign with $\Delta^2 F_n = f_n$. This enables us to show that every solution of (E_3) and (E_4) oscillates. However, these results do not hold good for the corresponding unforced equations (E_1) and (E_2) respectively.

The following conditions are needed for our use in the sequel.

(H_{12}) There exists a real valued function F_n, $n \geq n_0$, which changes sign and $\Delta^2 F_n = f_n$.

(H_{13}) $\sum_{n=n_0+\mu}^{\infty} h_n^* F_{n-\delta}^{\pm} = \infty$ where $F_n^+ = \max\{F_n, 0\}$ and $F_n^- = \max\{-F_n, 0\}$, and $h_n^* = \min\{h_n, h_{n-\tau}\}$.

(H_{14}) $-\infty < \liminf_{n\to\infty} F_n < 0 < \limsup_{n\to\infty} F_n < \infty$.

(H_{15}) $\liminf_{n\to\infty} \frac{F_n}{n} = -\infty$ and $\limsup_{n\to\infty} \frac{F_n}{n} = \infty$.

Theorem 3.6. *Let (H_3), (H_7), (H_{12}) and (H_{15}) hold and $c_n \geq 0$. Then every solution of (E_3) oscillates.*

P r o o f. Let x_n be a nonoscillatory solution of (E_3) such that $x_n > 0$ and $x_{n-\mu} > 0$ for $n \geq n_1 \geq n_0 + \mu$. Setting z_n as in (2.1) and u_n as in (3.1), we obtain

$$\Delta^2 u_n + h_n x_{n-\delta} = 0, \qquad n \geq n_1. \tag{3.5}$$

Then for $n \geq n_2 \geq n_1$,

$$\Delta u_n \leq \Delta u_{n_2}.$$

This in turn implies that

$$z_n \leq F_n + u_{n_2} + (n - n_2)\Delta u_{n_2}.$$

Hence

$$\frac{z_n}{n} \leq \frac{F_n}{n} + \frac{u_{n_2}}{n} + \left\{1 - \frac{n_2}{n}\right\}\Delta u_{n_2}.$$

Taking limit as $n \to \infty$ both sides in the above inequality, we obtain $\liminf_{n\to\infty} \frac{z_n}{n} = -\infty$. This in turn implies that $\liminf_{n\to\infty} z_n = -\infty$ and

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=n_0}^{n-1} \sum_{j=i-\delta+\sigma}^{i-1} q_j x_{j-\sigma} = \infty$$

and hence $\lim_{n\to\infty} x_n = \infty$. Thus there exists an increasing sequence $\{N_\alpha\}_{\alpha=1}^{\infty}$, $N_\alpha \geq n_2$ and $N_\alpha \to \infty$ as $\alpha \to \infty$ such that $\lim_{\alpha\to\infty} z_{N_\alpha} = -\infty$, $\max_{n_2 \leq n \leq N_\alpha} x_n = x_{N_\alpha}$ and $\lim_{\alpha\to\infty} x_{N_\alpha} = \infty$. Then from (2.1)

$$\begin{aligned} z_{N_\alpha} &> x_{N_\alpha} - \sum_{i=n_0}^{N_\alpha - 1} \sum_{j=i-\delta+\sigma}^{i-1} q_j x_{j-\sigma} \\ &> \left\{1 - \sum_{i=n_0}^{\infty} \sum_{j=i-\delta+\sigma}^{i-1} q_j\right\} x_{N_\alpha}. \end{aligned}$$

Now, taking $\liminf_{\alpha\to\infty}$ both sides in the above inequality, we see that

$$-\infty = \liminf_{n\to\infty} z_n \geq \left\{1 - \sum_{i=n_0}^{\infty} \sum_{j=i-\delta+\sigma}^{i-1} q_j\right\} \liminf_{\alpha\to\infty} x_{N_\alpha} \geq 0,$$

a contradiction. Hence every solution of (E_3) is oscillatory. This completes the proof of the theorem. □

Proceeding as in the lines of proof of Theorem 3.6, one may obtain the following theorem.

Theorem 3.7. *Let (H_5), (H_8), (H_{12}) and (H_{15}) hold. Then every solution of (E_4) oscillates.*

Theorem 3.8. *Let (H_7), (H_{12}), (H_{13}) and (H_{14}) hold. Then every solution of (E_3) oscillates provided that (H_{10}) and $\liminf_{n\to\infty} F_n^- = 0$ hold.*

Proof. Let x_n be a nonoscillatory solution of (E_3) such that $x_n > 0$ and $x_{n-\mu} > 0$ for $n \geq n_1 \geq n_0 + \mu$. Setting z_n as in (2.1) and u_n as in (3.1), we obtain (3.5). Hence $\Delta^2 u_n \leq 0$ for $n \geq n_2 \geq n_1$. Thus there exists a $n_3 \geq n_2$ such that $u_n > 0$ or $u_n < 0$ for $n \geq n_3$. Let $u_n > 0$ for $n \geq n_3$. Then $\Delta u_n \geq 0$ for $n \geq n_4 \geq n_3$. Further, $u_n > 0$ for $n \geq n_3$ and $0 \leq c_n \leq 1$ implies that $x_n + x_{n-\tau} \geq F_n^+$ for $n \geq n_3$. From (3.5) we obtain

$$\begin{aligned} 0 &= \Delta^2 u_n + h_n x_{n-\delta} + \Delta^2 u_{n-\tau} + h_{n-\tau} x_{n-\delta-\tau} \\ &\geq \Delta^2 u_n + \Delta^2 u_{n-\tau} + h_n^*[x_{n-\delta} + x_{n-\delta-\tau}] \\ &\geq \Delta^2 u_n + \Delta^2 u_{n-\tau} + h_n^* F_{n-\delta}^+, \end{aligned}$$

that is,

$$0 \geq \Delta^2 u_n + \Delta^2 u_{n-\tau} + h_n^* F_{n-\delta}^+. \tag{3.6}$$

Summing the above inequality from n_4 to $n-1$ and letting $n \to \infty$, we obtain

$$\sum_{n=n_4}^{\infty} h_n^* F_{n-\delta}^+ < \infty,$$

a contradiction to (H_{13}). Hence $u_n < 0$ for $n \geq n_3$. There are two cases in hand, $\Delta u_n \geq 0$ and $\Delta u_n < 0$ for some $n \geq n_5 \geq n_3$. First suppose that $\Delta u_n < 0$ for $n \geq n_5 \geq n_3$. Then $u_n \to -\infty$ as $n \to \infty$. If x_n is bounded from above, then from (H_{14}) and (3.1) it follows that u_n is bounded, a contradiction. Hence x_n must be unbounded. Thus there exists an increasing sequence $\{N_\alpha\}_{\alpha=1}^{\infty}$, $N_\alpha \geq n_5$, and $N_\alpha \to \infty$ as $\alpha \to \infty$ such that $u_{N_\alpha} \to -\infty$ as $\alpha \to \infty$, $\max_{n_5 \leq n \leq N_\alpha} x_n = x_{N_\alpha}$ and $\lim_{\alpha\to\infty} x_{N_\alpha} = \infty$. Hence

$$\begin{aligned} u_{N_\alpha} &= x_{N_\alpha} + c_{N_\alpha} x_{N_{\alpha-\tau}} - \sum_{i=n_0}^{N_\alpha - 1} \sum_{j=i-\delta+\sigma}^{i-1} q_j x_{j-\sigma} - F_{N_\alpha}, \\ &\geq \left\{ 1 - \sum_{i=n_0}^{\infty} \sum_{j=i-\delta+\sigma}^{i-1} q_j \right\} x_{N_\alpha} - F_{N_\alpha}. \end{aligned}$$

Letting $\alpha \to \infty$, we obtain a contradiction. Next, suppose that $\Delta u_n \geq 0$ for $n \geq n_5$. Then from (3.6) we have

$$\sum_{i=n_3}^{\infty} h_n^*(x_n + x_{n-\tau}) < \infty.$$

Using (H_{13}) we obtain

$$\liminf_{n\to\infty} \frac{x_n + x_{n-\tau}}{F_{n-\delta}^-} = 0. \tag{3.7}$$

Set

$$v_n = x_n + c_n x_{n-\tau} - F_n. \tag{3.8}$$

Then $\Delta v_n = \Delta u_n + \sum_{j=n-\delta+\sigma}^{n-1} q_j x_{j-\sigma} > 0$ and $v_n > 0$ for $n \geq n_6 \geq n_4$. Hence $\lim_{n\to\infty} v_n = \beta$, $0 < \beta \leq \infty$. From (3.7), there exists an increasing sequence $\{N_\alpha\}_{\alpha=1}^{\infty}$, $N_\alpha \geq n_6$, and a real $\lambda \in (0,1)$ such that

$$x_{N_\alpha} + x_{N_\alpha-\tau} < \lambda F_{N_\alpha-\delta}^-.$$

Thus using (3.8) we see that

$$\begin{aligned} v_{N_\alpha} &= x_{N_\alpha} + c_{N_\alpha} x_{N_\alpha-\tau} - F_{N_\alpha} \\ &< x_{N_\alpha} + x_{N_\alpha-\tau} - F_{N_\alpha} \\ &< \lambda F_{N_\alpha-\delta}^- - F_{N_\alpha} \\ &< \infty. \end{aligned}$$

Hence $0 < \beta < \infty$, that is v_n is bounded. Clearly x_n is bounded, because $u_n < 0$. Then from (3.7), $\liminf_{n\to\infty} x_n = 0$. Thus

$$\begin{aligned} 0 < \beta = \liminf_{n\to\infty} v_n &\leq \liminf_{n\to\infty} [x_n + x_{n-\tau} + F_n^-] \\ &\leq \liminf_{n\to\infty} F_n^- = 0, \end{aligned}$$

a contradiction. Hence every solution of (E_3) oscillates. The proof is complete. □

Let $q_n \equiv 0, n \geq n_0$. Then it is easy to prove the following result:

Theorem 3.9. *Let (H_7), (H_{10}), (H_{12}) and (H_{13}) hold. Then every solution of*

$$\Delta^2[x_n - c_n x_{n-\tau}] + p_n x_{n-\delta} = f_n \tag{3.9}$$

oscillates.

From Theorems 3.8 and 3.9, it seems that the presence of q_n in (E_3) forces us to assume some additional conditions in Theorem 3.8, these are (H_{14}) and $\liminf_{n\to\infty} F_n^- = 0$. Hence an improvement of Theorem 3.8 is necessary.

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